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Adaptive sliding mode synchronization for a class of fractional-order chaotic systems with disturbance

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Abstract This paper studies the fractional-order disturbance observer (FODO)-based adaptive sliding mode synchronization control for a class of fractional-order chaotic systems with unknown bounded disturbances. To handle unknown disturbances, the nonlinear FODO is explored for the fractional-order chaotic system. By choosing the appropriate control gain parameter, the disturbance observer can approximate the disturbance well. On the basis of the sliding mode control technique, a simple sliding mode surface is defined. A synchronization control scheme incorporating the introduced sliding mode surface and the designed disturbance observer is then developed. Under the control of the synchronization scheme, a good synchronization performance is realized between two identical fractional-order chaotic systems with different initial conditions. Finally, the numerical simulation results illustrate the effectiveness of the developed synchronization control scheme for fractional-order chaotic systems in the presence of external disturbances.

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1 Introduction

Over the past decades, fractional calculus has attracted increasing concerns of researchers, which has been widely applied in the fields of engineering and physics, such as system control [1], electromechanics [2] and signal processing [3]. So far, integer-order nonlinear systems have been studied extensively [4–7]. Since the mathematical model of a real plant can be accurately described via the fractional-order differential method [8,9], many systems can be expressed as fractional differential equations, for example fractional-order economic system [10], fractional-order biological population model [11], fractional-order financial system [12] and fractional-order chaotic and hyperchaotic systems [13–19]. Recently, the synchronization of fractional-order chaotic systems has been extensively investigated because of the potential applications in electrical engineering and secure communication. Therefore, it is significant to develop the synchronization control of fractional-order chaotic systems based on fractional calculus.

The chaotic synchronization is that the synchronization errors asymptotically approach zero for the trajectories of drive system and response system. Since the synchronization was firstly realized between two identical chaotic systems by Pecora and Carroll [20, 21], the chaotic synchronization has been developed quickly and many synchronization control schemes for fractional-order chaotic systems have been proposed including impulsive control [22], active control [23], adaptive control [24], generalized projective control [25] and passive control [26]. In addition, it is well known that sliding mode control is an effective robust control scheme and the sliding mode control scheme has the features of fast global convergence and high robustness to external disturbances [27]. In recent years, sliding mode control has been investigated for linear and nonlinear systems [28–31] and many important results have been reported for the synchronization of fractional-order chaotic systems by using the sliding mode control strategy. In [32,33], from the stability theory of fractional-order systems and active sliding mode control method, the synchronization was achieved for two fractional-order chaotic systems. The sliding mode synchronization control was realized for uncertain fractional-order Duffing–Holmes systems in [34]. In [35], the stabilization and synchronization were investigated for a class of chaotic fractional-order systems via a novel fractional-order sliding mode method. A robust fractional-order sliding mode scheme was proposed, and the synchronization was realized for uncertain fractional-order chaotic systems in [36]. In [37], a new three-dimensional fractional-order chaotic system was presented and its adaptive sliding mode synchronization was studied. The synchronization was studied for a class of fractional-order arbitrary dimensional hyperchaotic systems based on the sliding mode control method in [38]. The above-mentioned works focused on synchronization of fractional-order chaotic systems via sliding mode control approach. In practice, many real physical systems are subjected to exogenous disturbance and the disturbance may lead to oscillations and even increase instability of systems; it is significant to investigate the synchronization of fractional-order chaotic systems with external disturbance. According to the conclusion above, the bounded assumption for fractional derivative of disturbances was introduced [34]. In [36,38], unknown disturbances in fractional-order systems were tackled by adaptive estimation method. However, the nonlinear FODO has rarely been considered in synchronization control of fractional-order chaotic systems in the literature.

Since the nonlinear disturbance observer can approximate unknown disturbance well, it can be employed to restrain the interference of external disturbance. In the past decades, many design techniques of integer-order disturbance observer have been reported. In [39], a disturbance observer-based control was proposed for nonlinear systems with disturbances. The nonlinear disturbance observer was developed for robot manipulators in [40]. In [41], an adaptive fuzzy tracking control scheme was explored based on disturbance observer for multi-input and multi-output nonlinear systems. By using the terminal sliding mode technique, a disturbance observer-based adaptive sliding mode control scheme was proposed for near-space vehicles (NSV) in [42]. In [43], a Nussbaum disturbance observer was designed for NSV. On the basis of the terminal sliding mode technique and the disturbance observer method, an anti-disturbance control scheme was presented for NSV in [44]. With such experience of the applications of disturbance observers, it is necessary to design nonlinear FODO to compensate for the effects caused by unknown disturbances.

Inspired by the above discussions, we develop a synchronization control scheme to synchronize fractional-order chaotic systems with unknown external disturbances based on a designed nonlinear FODO and a simple sliding mode surface. To illustrate the effectiveness of the given synchronization control method, a modified fractional-order Jerk system is analyzed by using the proposed synchronization control scheme.

The organization of the paper is as follows. Section 2 details the problem formulation. The nonlinear FODO is designed in Sect. 3. The sliding mode surface is constructed, and the sliding mode synchronization controller is proposed based on the developed nonlinear FODO in Sect. 4. A modified fractional-order Jerk system is presented, and the effectiveness of the proposed synchronization control method is demonstrated via numerical simulation in Sect. 5, followed by some concluding remarks in Sect. 6.

2 Problem statement and preliminaries

Fractional calculus is an extension to integer-order calculus. Several existing definitions of fractional derivatives are given in [45], where the Caputo definition is used in engineering applications extensively. We firstly introduce the following Caputo definition.

Definition 1 [45] For the function $g(t)$, the Caputo fractional derivative of fractional-order α is defined as follows:

$$D^{\alpha}g(t) = \frac{1}{\Gamma(m-\alpha)}\int_{t_0}^{t}\frac{g^{(m)}(\tau)}{(t-\tau)^{\alpha-m+1}}\mathrm{d}\tau. \tag{1}$$

where $m-1<\alpha<m$, $m=[\alpha]+1$, $[\alpha]$ denotes the integer part of α, and the $\Gamma(\cdot)$ is gamma function, which is defined as $\Gamma(m-\alpha)=\int_0^{\infty}t^{m-\alpha-1}e^{-t}\mathrm{d}t$. The main advantage of (1) is that Caputo derivative of a constant is equal to zero. In this paper, the fractional-order chaotic systems will be described by using Caputo definition with lower limit of integral $t_0=0$ and the order $0<\alpha<1$.

Definition 2 [46] The Mittag–Leffler function with two parameters is defined as

$$E_{\alpha_1,\alpha_2}(z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(k\alpha_1+\alpha_2)} \tag{2}$$

where $\alpha_1>0$, $\alpha_2>0$, z stands for set of complex numbers.

On the basis of the Caputo definition of fractional derivative, the fractional-order chaotic system will be introduced.

Consider the following fractional-order chaotic system as the drive system:

$$D^{\alpha}x(t) = Ax(t) + f(x(t)). \tag{3}$$

where $A\in R^{n\times n}$ denotes a constant matrix, $x(t)=(x_1(t),x_2(t),\ldots,x_n(t))^T\in R^n$ is a state vector, $f(x(t))=(f_1(x(t)),f_2(x(t)),\ldots,f_n(x(t)))^T\in R^n$ is the nonlinear function vector.

The response system is defined as follows:

$$D^{\alpha}y(t) = Ay(t) + f(y(t)) + d(t) + u(t). \tag{4}$$

where $y(t)=(y_1(t),y_2(t),\ldots,y_n(t))^T\in R^n$ is the state vector, $f(y(t))=(f_1(y(t)),f_2(y(t)),\ldots,f_n(y(t)))^T\in R^n$ is the nonlinear function vector, $d(t)=(d_1(t),d_1(t),\ldots,d_n(t))^T\in R^n$ is the unknown bounded disturbance, and $u(t)=(u_1(t),u_1(t),\ldots,u_n(t))^T\in R^n$ is the control input.

This paper aims at developing a FODO-based adaptive sliding mode synchronization control scheme, so that the synchronization is realized between two identical fractional-order chaotic systems in the presence of external unknown disturbances. On the basis of the designed sliding mode controller, the response system can well synchronize the drive system under the proper condition. In order to obtain the main results, the following lemmas, properties and assumption are introduced.

Lemma 1 [47] *Let $\chi(t)\in\Re$ be a continuous and derivable function. Then, for any time instant $t\geq t_0$, we have*

$$\frac{1}{2}D^{\alpha}\chi^2(t) \leq \chi(t)D^{\alpha}\chi(t). \tag{5}$$

where $0<\alpha<1$.

Lemma 2 [48] *Consider the following fractional-order system*

$$D^{\alpha}q(t) \leq -b_0q(t) + b_1 \tag{6}$$

then there exists a constant $t_1>0$ such that for all $t\in(t_1,\infty)$, we have

$$\|q(t)\| \leq \frac{2b_1}{b_0} \tag{7}$$

where $q(t)$ is the state variable and b_0 and b_1 are two positive constants.

Property 1 [49] *If g_1 is a constant and the order $\beta_2>0$, the Caputo fractional derivative satisfies the following condition:*

$$D^{\beta_2}g_1 = 0. \tag{8}$$

Property 2 [49] *The Caputo fractional derivative satisfies the following linear characteristic:*

$$D^{\alpha}\left[a_1g_2(t)+a_2g_3(t)\right] = a_1D^{\alpha}g_2(t) + a_2D^{\alpha}g_3(t). \tag{9}$$

where $g_2(t)$ and $g_3(t)$ are functions and a_1 and a_2 are constants.

Assumption 1 For the external disturbance $d_i(t)$ with $i=1,2,\ldots,n$, the Caputo fractional derivative of $d_i(t)$ is bounded, that is $|D^{\alpha}d_i(t)|\leq\zeta_i$ and $\zeta_i>0$ is an unknown positive constant.

3 Design of fractional-order disturbance observer

In this section, a nonlinear FODO will be designed to approximate the external disturbance in the response system (4). Without loss of generality, according to the response system (4), we have

$$D^{\alpha}y_i(t) = \theta_i + f_i(y(t)) + u_i(t) + d_i(t) \tag{10}$$

where θ_i is ith element of $Ay(t)$, $y_i(t)$ is the ith element of $y(t)$, $f_i(y(t))$ is the ith element of $f(y(t))$, $u_i(t)$ is the ith element of $u(t)$, $d_i(t)$ is the ith element of $d(t)$ and $i = 1, 2, \ldots, n$.

Since $d(t)$ in (4) is unknown, $d(t)$ cannot be applied to developing synchronization control for the drive system (3) and the response system (4). To overcome the above problem, a fractional-order nonlinear disturbance observer is designed to estimate disturbance.

To design the nonlinear FODO, an auxiliary variable is introduced based on the design technique of integer-order disturbance observer as follows [41]:

$$\phi_i(t) = d_i(t) - \sigma_i y_i(t) \tag{11}$$

where $\sigma_i > 0$ is a constant to be determined.

Combining (10) and (11), the Caputo fractional derivative of $\phi_i(t)$ can be written as

$$\begin{aligned} D^\alpha \phi_i(t) &= D^\alpha d_i(t) - \sigma_i D^\alpha y_i(t) \\ &= -\sigma_i(\theta_i + f_i(y(t)) + d_i(t)) \\ &\quad - \sigma_i u_i(t) + D^\alpha d_i(t) \\ &= -\sigma_i(\theta_i + f_i(y(t)) + \phi_i(t) + \sigma_i y_i(t)) \\ &\quad - \sigma_i u_i(t) + D^\alpha d_i(t) \end{aligned} \tag{12}$$

To obtain the disturbance estimate, the estimate of intermediate variable $\phi_i(t)$ is described as

$$\begin{aligned} D^\alpha \hat{\phi}_i(t) &= -\sigma_i(\theta_i + f_i(y(t)) + \sigma_i y_i(t)) \\ &\quad - \sigma_i \hat{\phi}_i(t) - \sigma_i u_i(t) \end{aligned} \tag{13}$$

where $\hat{\phi}_i$ is the estimate of ϕ_i.

According to (11), the disturbance $d_i(t)$ can be estimated as

$$\hat{d}_i(t) = \hat{\phi}_i(t) + \sigma_i y_i(t) \tag{14}$$

Define $\tilde{d}_i(t) = d_i(t) - \hat{d}_i(t)$. Considering (11) and (14), we have

$$\tilde{\phi}_i(t) = \phi_i(t) - \hat{\phi}_i(t) = d_i(t) - \hat{d}_i(t) = \tilde{d}_i(t) \tag{15}$$

Considering (12) and (13), the Caputo fractional derivative of $\tilde{\phi}_i(t)$ can be written as

$$D^\alpha \tilde{\phi}_i(t) = -\sigma_i \tilde{\phi}_i(t) + D^\alpha d_i(t) \tag{16}$$

On the basis of the above discussions, in order to analyze the convergence of disturbance estimate error $\tilde{d}_i(t)$, a Lyapunov function candidate can be chosen as

$$V_d = \frac{1}{2}\tilde{d}_i^2(t) = \frac{1}{2}\tilde{\phi}_i^2(t) \tag{17}$$

Invoking (17) and Lemma 1, the Caputo fractional derivative of V_d is described as

$$D^\alpha V_d(t) \le \tilde{\phi}_i(t) D^\alpha \tilde{\phi}_i(t) \tag{18}$$

Substituting (16) into (18), we obtain

$$D^\alpha V_d(t) \le \tilde{\phi}_i(t)(-\sigma_i \tilde{\phi}_i(t) + D^\alpha d_i(t)) \tag{19}$$

According to (19) and Assumption 1, we have

$$\begin{aligned} D^\alpha V_d(t) &\le -\sigma_i \tilde{\phi}_i^2(t) + \frac{1}{2}\tilde{\phi}_i^2(t) + \frac{1}{2}\zeta_i^2 \\ &= -\left(\sigma_i - \frac{1}{2}\right)\tilde{\phi}_i^2(t) + \frac{1}{2}\zeta_i^2 \\ &= -B_0 V_d(t) + B_1 \end{aligned} \tag{20}$$

where $B_0 = 2\sigma_i - 1$ and $B_1 = \frac{1}{2}\zeta_i^2$. To ensure the estimated error is bounded, the nonlinear FODO control gain σ_i should be chosen to make $\sigma_i > 0.5$. The conclusion that the signals $\tilde{\phi}_i(t)$ and $\tilde{d}_i(t)$ are bounded can be drawn from (20) and Lemma 2.

On the basis of Lemma 2 and (20), we obtain

$$|V_d(t)| \le \frac{\zeta_i^2}{2(\sigma_i - 0.5)} \tag{21}$$

which means

$$\left|\tilde{d}_i(t)\right| \le \sqrt{\frac{\zeta_i^2}{(\sigma_i - 0.5)}} \tag{22}$$

According to (22), the disturbance estimation error $\tilde{d}_i$ is upper bounded. For the external disturbance $d_i(t)$ with $i = 1, 2, \ldots, n$, the disturbance approximation error $\tilde{d}_i(t) = d_i(t) - \hat{d}_i(t)$ satisfies $\left|\tilde{d}_i(t)\right| \le \kappa_i$ and $\kappa_i > 0$ is an unknown positive constant. In actual application, the upper bound of $\left|\tilde{d}_i(t)\right|$ is difficult to determine; therefore, the estimated value $\hat{\kappa}_i$ of κ_i is introduced, where $i = 1, 2, \ldots, n$.

The above design procedure of nonlinear FODO can be summarized in the following theorem:

Theorem 1 *Consider the response system* (4) *and the nonlinear FODO is designed as* (13) *and* (14)*. The disturbance estimate error of the proposed nonlinear FODO is bounded.*

On the basis of the above-mentioned analyses, Theorem 1 can be easily proven.

4 Synchronization control of fractional-order chaotic systems

In this section, the nonlinear FODO-based adaptive sliding mode control scheme will be proposed to guarantee the trajectories of drive system (3) and response system (4) which are ultimately bounded synchronization. To design the synchronization control scheme, we firstly define error state $e(t) = y(t) - x(t)(e(t) = (e_1(t), e_2(t), \ldots, e_n(t))^T \in R^n)$. From (3) and (4), the corresponding synchronization error system is as follows:

$$D^\alpha e(t) = Ae(t) + f(y(t)) - f(x(t)) + d(t) + u(t) \tag{23}$$

To investigate the stabilization of fractional-order synchronization error system (23), a simple sliding mode surface is defined as

$$s_i(t) = e_i(t). \tag{24}$$

where $i = 1, 2, \ldots, n$

From (24), we have

$$\begin{aligned} D^\alpha s_i(t) &= D^\alpha e_i(t) \\ &= A_i e(t) + f_i(y(t)) - f_i(x(t)) \\ &\quad + d_i(t) + u_i(t) \end{aligned} \tag{25}$$

where A_i denotes the ith line of A and $f_i(x(t))$ denotes the ith element of $f(x(t))$.

Using the adaptive sliding control approach, the desired synchronization control input is designed as

$$\begin{aligned} u_i(t) = &-A_i e(t) - (f_i(y(t)) - f_i(x(t))) - \eta_i s_i(t) \\ &- \hat{\kappa}_i \operatorname{sign}(s_i(t)) - \hat{d}_i(t) \end{aligned} \tag{26}$$

where $\operatorname{sign}(\cdot)$ is the sign function and $\eta_i > 0$ is a design constant. Meanwhile, the estimated value $\hat{\kappa}_i$ is updated by

$$D^\alpha \hat{\kappa}_i = \gamma_i \left(|s_i(t)| - \hat{\kappa}_i\right) \tag{27}$$

where $\gamma_i > 0$ is a design constant.

If the synchronization control scheme is designed as (26) for fractional-order synchronization error system (23), the sliding mode surface satisfies that the sliding mode surface $s_i(t)$ is bounded stable in the form of

$$|s_i(t)| \le a \tag{28}$$

where $a > 0$ is a unknown constant.

From (24) and (28), one obtains

$$|e_i(t)| \le a \tag{29}$$

According to the above discussion, if the sliding surface $s_i(t)$ is bounded, then the synchronization error $e_i(t)$ is also bounded. Therefore, the nonlinear FODO-based adaptive sliding mode synchronization control scheme for fractional-order chaotic systems with unknown disturbances can be summarized in the following theorem and will be proved by using fractional-order Lyapunov method.

Theorem 2 *For the synchronization error system* (23) *with* $0 < \alpha < 1$*, the sliding mode surface is designed according to* (24)*. The external unknown bounded disturbance is estimated by using the designed nonlinear FODO* (13) *and* (14)*. Then, the synchronization error* $e(t)$ *is ultimately bounded stable under the adaptive sliding control scheme* (26) *and* (27).

Proof To analyze the convergence of synchronization error $e(t)$, we consider the Lyapunov candidate function as

$$\begin{aligned} V(t) = &\sum_{i=1}^n \frac{1}{2} s_i^2(t) + \sum_{i=1}^n \frac{1}{2} \tilde{d}_i^2(t) \\ &+ \sum_{i=1}^n \frac{1}{2} \left(\frac{1}{\sqrt{\gamma_i}} (\hat{\kappa}_i - \kappa_i) \right)^2 \end{aligned} \tag{30}$$

According to Property 2 and (30), we have

$$\begin{aligned} D^\alpha V(t) = &\sum_{i=1}^n \frac{1}{2} D^\alpha s_i^2(t) + \sum_{i=1}^n \frac{1}{2} D^\alpha \left(\frac{1}{\sqrt{\gamma_i}} \tilde{\kappa}_i \right)^2 \\ &+ \sum_{i=1}^n \frac{1}{2} D^\alpha \tilde{d}_i^2(t) \end{aligned} \tag{31}$$

where $\tilde{\kappa}_i = \hat{\kappa}_i - \kappa_i$.

From Lemma 1, (31) can be written as

$$\begin{aligned} D^\alpha V(t) \le &\sum_{i=1}^n s_i(t) D^\alpha s_i(t) + \sum_{i=1}^n \frac{1}{\sqrt{\gamma_i}} \tilde{\kappa}_i D^\alpha \left(\frac{1}{\sqrt{\gamma_i}} \tilde{\kappa}_i \right) \\ &+ \sum_{i=1}^n \frac{1}{2} D^\alpha \tilde{d}_i^2(t) \end{aligned} \tag{32}$$

Based on Property 2, (32) is equivalent to

$$\begin{aligned} D^\alpha V(t) \le &\sum_{i=1}^n s_i(t) D^\alpha s_i(t) + \sum_{i=1}^n \frac{1}{\gamma_i} \tilde{\kappa}_i D^\alpha \tilde{\kappa}_i \\ &+ \sum_{i=1}^n \frac{1}{2} D^\alpha \tilde{d}_i^2(t) \end{aligned} \tag{33}$$

On the basis of (25), one has

$$\begin{aligned} D^\alpha V(t) \le &\sum_{i=1}^n s_i(t) \left(A_i e(t) + f_i(y(t)) - f_i(x(t))\right) \\ &+ \sum_{i=1}^n s_i(t) \left(d_i(t) + u_i(t)\right) + \sum_{i=1}^n \frac{1}{\gamma_i} \tilde{\kappa}_i D^\alpha \tilde{\kappa}_i \\ &+ \sum_{i=1}^n \frac{1}{2} D^\alpha \tilde{d}_i^2(t) \end{aligned} \tag{34}$$

Referring to Property 1 and $\tilde{\kappa}_i = \hat{\kappa}_i - \kappa_i$, we obtain

$$D^\alpha \tilde{\kappa}_i = D^\alpha \hat{\kappa}_i \tag{35}$$

According to (27) and (35), we have

$$\begin{aligned}\sum_{i=1}^{n} \frac{1}{\gamma_i} \tilde{\kappa}_i D^\alpha \tilde{\kappa}_i &= \sum_{i=1}^{n} \tilde{\kappa}_i \left(|s_i(t)| - \hat{\kappa}_i \right) \\ &= \sum_{i=1}^{n} \tilde{\kappa}_i \, |s_i(t)| - \sum_{i=1}^{n} \tilde{\kappa}_i \hat{\kappa}_i \\ &\leq \sum_{i=1}^{n} \tilde{\kappa}_i \, |s_i(t)| - \sum_{i=1}^{n} \frac{1}{2} \tilde{\kappa}_i^2 + \sum_{i=1}^{n} \frac{1}{2} \kappa_i^2\end{aligned} \tag{36}$$

Invoking (36), we obtain

$$\begin{aligned}D^\alpha V(t) \leq &\sum_{i=1}^{n} s_i(t) \left(A_i e(t) + f_i(y(t)) - f_i(x(t)) \right) \\ &+ \sum_{i=1}^{n} s_i(t)(d_i(t) + u_i(t)) + \sum_{i=1}^{n} \tilde{\kappa}_i \, |s_i(t)| \\ &- \sum_{i=1}^{n} \frac{1}{2} \tilde{\kappa}_i^2 + \sum_{i=1}^{n} \frac{1}{2} \kappa_i^2 + \sum_{i=1}^{n} \frac{1}{2} D^\alpha \tilde{d}_i^2(t)\end{aligned} \tag{37}$$

Substituting (26) into (37) yields

$$\begin{aligned}D^\alpha V(t) \leq &\sum_{i=1}^{n} s_i(t) \Big(-\eta_i s_i(t) + \tilde{d}_i(t) - \hat{\kappa}_i \operatorname{sign}(s_i(t)) \Big) \\ &+ \sum_{i=1}^{n} \tilde{\kappa}_i \, |s_i(t)| - \sum_{i=1}^{n} \frac{1}{2} \tilde{\kappa}_i^2 + \sum_{i=1}^{n} \frac{1}{2} \kappa_i^2 \\ &+ \sum_{i=1}^{n} \frac{1}{2} D^\alpha \tilde{d}_i^2(t)\end{aligned} \tag{38}$$

Furthermore, (38) can be rewritten as

$$\begin{aligned}D^\alpha V(t) \leq &\sum_{i=1}^{n} -\eta_i s_i^2(t) + \sum_{i=1}^{n} |s_i(t)| \left| \tilde{d}_i(t) \right| \\ &+ \sum_{i=1}^{n} \tilde{\kappa}_i \, |s_i(t)| - \sum_{i=1}^{n} \frac{1}{2} \tilde{\kappa}_i^2 + \sum_{i=1}^{n} \frac{1}{2} \kappa_i^2 \\ &- \sum_{i=1}^{n} \hat{\kappa}_i \, |s_i(t)| + \sum_{i=1}^{n} \frac{1}{2} D^\alpha \tilde{d}_i^2(t)\end{aligned} \tag{39}$$

with

$$\sum_{i=1}^{n} \tilde{\kappa}_i \, |s_i(t)| - \sum_{i=1}^{n} \hat{\kappa}_i \, |s_i(t)| = -\sum_{i=1}^{n} \kappa_i \, |s_i(t)| \tag{40}$$

According to (40), it yields

$$\begin{aligned}D^\alpha V(t) \leq &\sum_{i=1}^{n} -\eta_i s_i^2(t) - \sum_{i=1}^{n} \frac{1}{2} \tilde{\kappa}_i^2 + \sum_{i=1}^{n} \frac{1}{2} \kappa_i^2 \\ &+ \sum_{i=1}^{n} \frac{1}{2} D^\alpha \tilde{d}_i^2(t)\end{aligned} \tag{41}$$

Considering (20) and (41), we have

$$\begin{aligned}D^\alpha V(t) \leq &\sum_{i=1}^{n} -\eta_i s_i^2(t) - \sum_{i=1}^{n} \frac{1}{2} \tilde{\kappa}_i^2 + \sum_{i=1}^{n} \frac{1}{2} \kappa_i^2 \\ &\sum_{i=1}^{n} -(\sigma_i - \frac{1}{2}) \tilde{d}_i^2(t) + \sum_{i=1}^{n} \frac{1}{2} \zeta_i^2 \\ \leq &-B_2 V(t) + B_3\end{aligned} \tag{42}$$

where $B_2 = \min(2\eta_i, 1, 2\sigma_i - 1)$ and $B_3 = \sum_{i=1}^{n} \frac{1}{2} \zeta_i^2 + \sum_{i=1}^{n} \frac{1}{2} \kappa_i^2$.

To ensure the synchronization error is bounded, the corresponding design parameters η_i and σ_i should be chosen to make $\eta_i > 0$ and $\sigma_i > 0.5$. According to (42) and Lemma 2, it may directly show that the signals $s(t)$, $e(t)$ and $\tilde{d}_i(t)$ are ultimately bounded. From Lemma 2 and (42), we obtain

$$|V(t)| \leq \frac{\sum_{i=1}^{n} \zeta_i^2 + \sum_{i=1}^{n} \kappa_i^2}{B_2} \tag{43}$$

which implies

$$\|s(t)\| \leq \sqrt{\frac{2 \left(\sum\limits_{i=1}^{n} \zeta_i^2 + \sum\limits_{i=1}^{n} \kappa_i^2 \right)}{B_2}} \tag{44}$$

From the inequality (44), the synchronization error $e(t)$ and $s(t)$ will be ultimately bounded as $t \to \infty$. Therefore, the synchronization error system (23) is bounded stable based on Lemma 2. The bounded synchronization of drive system (3) and response system (4) is achieved. This concludes the proof. □

Remark 1 Since the response system (4) is with the unknown time-varying disturbance, the nonlinear FODO is employed to estimate the disturbance in this paper. To develop the disturbance observer, Assumption 1 is introduced. This assumption means that the Caputo derivative of the disturbance is bounded. If the Caputo derivative of the disturbance is unbounded, the estimation performance of nonlinear FODO could be poor. Thus, Assumption 1 is necessary for the disturbance.

Remark 2 As for the proposed nonlinear FODO, we can see that the estimated error with suitable transient performance can be obtained by appropriately adjusting design parameter σ_i. For example, the approximation error could be decreased by increasing the value of σ_i. Therefore, appropriate parameter should be chosen based on the performance of whole systems.

5 Simulation example

5.1 Modified fractional-order Jerk system

In [50], a new chaotic generator was investigated by constructing a three-segment piecewise-linear odd function with variable break point. From the differential equation of chaotic generator in [50], the modified fractional-order Jerk system is given as follows:

$$\begin{aligned} D^{\alpha}x_1(t) &= x_2(t) \\ D^{\alpha}x_2(t) &= x_3(t) \\ D^{\alpha}x_3(t) &= -\varepsilon_1 x_1(t) - x_2(t) - \varepsilon_2 x_3(t) - f_3(x(t)) \end{aligned} \tag{45}$$

where the parameters $\varepsilon_1 = 1.5$, $\varepsilon_2 = 0.35$ and $f_3(x(t))$ is a piecewise-linear function defined by

$$\begin{aligned} f_3(x(t)) = &\frac{1}{2}(\vartheta_0 - \vartheta_1)(|x_1(t)+1| - |x_1(t)-1|) \\ &+ \vartheta_1 x_1(t) \end{aligned} \tag{46}$$

where $\vartheta_0 < -1 < \vartheta_1 < 0$ and $\vartheta_0 = -2.5$, $\vartheta_1 = -0.5$.

According to the system (45) and the piecewise-linear function (46), the three equilibrium points of the modified fractional-order Jerk system are given in Table 1. The Jacobian matrix for system (45) can be written as

$$J = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1.5 - \frac{\partial f_3(x(t))}{\partial x_1(t)} & -1 & -0.35 \end{bmatrix}. \tag{47}$$

Table 1 Equilibrium points of the modified fractional-order Jerk system

Linear region	$f_3(x(t))$	Equilibrium points
$x_1(t) > 1$	$-0.5x_1(t) - 2$	$Q_+ = (2, 0, 0)$
$-1 \le x_1(t) \le 1$	$-2.5x_1(t)$	$Q_0 = (0, 0, 0)$
$x_1(t) < -1$	$-0.5x_1(t) + 2$	$Q_+ = (-2, 0, 0)$

On the basis of Table 1, the corresponding eigenvalues for equilibrium point Q_0 are $\lambda_1 = 0.6228$ and $\lambda_{2,3} = -0.4864 \pm 1.1701j$. And, for equilibrium points Q_+ and Q_-, the eigenvalues are $\lambda_1 = -0.7614$ and $\lambda_{2,3} = 0.2057 \pm 1.1274j$. When the fractional-order $\alpha = 0.98$ is chosen, we obtain the following characteristic equation of the equilibrium points Q_+ and Q_-:

$$\lambda^{294} + 0.35\lambda^{196} + \lambda^{98} + 1 = 0 \tag{48}$$

with unstable $\lambda_{1,2} = 1.0013 \pm 0.0142j$, and $|\arg(\lambda_{1,2})| = 0.0142 < \pi/2\vartheta = 0.0157$, in which $\vartheta = 100$ (ϑ is the lowest common multiple of fractional-order denominator). Thus, the modified fractional-order Jerk system (45) with chaotic dynamic behaviors is based on the theorem in [51]. When the initial values are chosen as $(1, 1, 1)^T$ and the fractional-order $\alpha = 0.98$, the fractional-order modified Jerk system exhibits chaotic behaviors as shown in Fig. 1.

5.2 Numerical simulation of synchronization control

In this section, to illustrate the effectiveness of the proposed synchronization controller, the synchronization of modified fractional-order Jerk system (45) is investigated. Consider the fractional-order chaotic system (45) as drive system. From (4), the response system is defined as follows:

$$\begin{aligned} D^{\alpha}y_1(t) &= y_2(t) + d_1(t) + u_1(t) \\ D^{\alpha}y_2(t) &= y_3(t) + d_2(t) + u_2(t) \\ D^{\alpha}y_3(t) &= -\varepsilon_1 y_1(t) - y_2(t) - \varepsilon_2 y_3(t) \\ &\quad - f_3(y(t)) + d_3(t) + u_3(t) \end{aligned} \tag{49}$$

where $d_1(t)$, $d_2(t)$ and $d_3(t)$ are unknown bounded disturbances. $u_1(t)$, $u_2(t)$ and $u_3(t)$ are designed synchronization control inputs. $f_3(y(t))$ is defined as

$$\begin{aligned} f_3(y(t)) = &\frac{1}{2}(\vartheta_0 - \vartheta_1)(|y_1(t)+1| - |y_1(t)-1|) \\ &+ \vartheta_1 y_1(t) \end{aligned} \tag{50}$$

According to (45) and (49), the synchronization error system can be written as

$$\begin{aligned} D^{\alpha}e_1(t) &= e_2(t) + d_1(t) + u_1(t) \\ D^{\alpha}e_2(t) &= e_3(t) + d_2(t) + u_2(t) \\ D^{\alpha}e_3(t) &= -\varepsilon_1 e_1(t) - e_2(t) - \varepsilon_2 e_3(t) \\ &\quad - f_3(y(t)) + f_3(x(t)) + d_3(t) + u_3(t) \end{aligned} \tag{51}$$

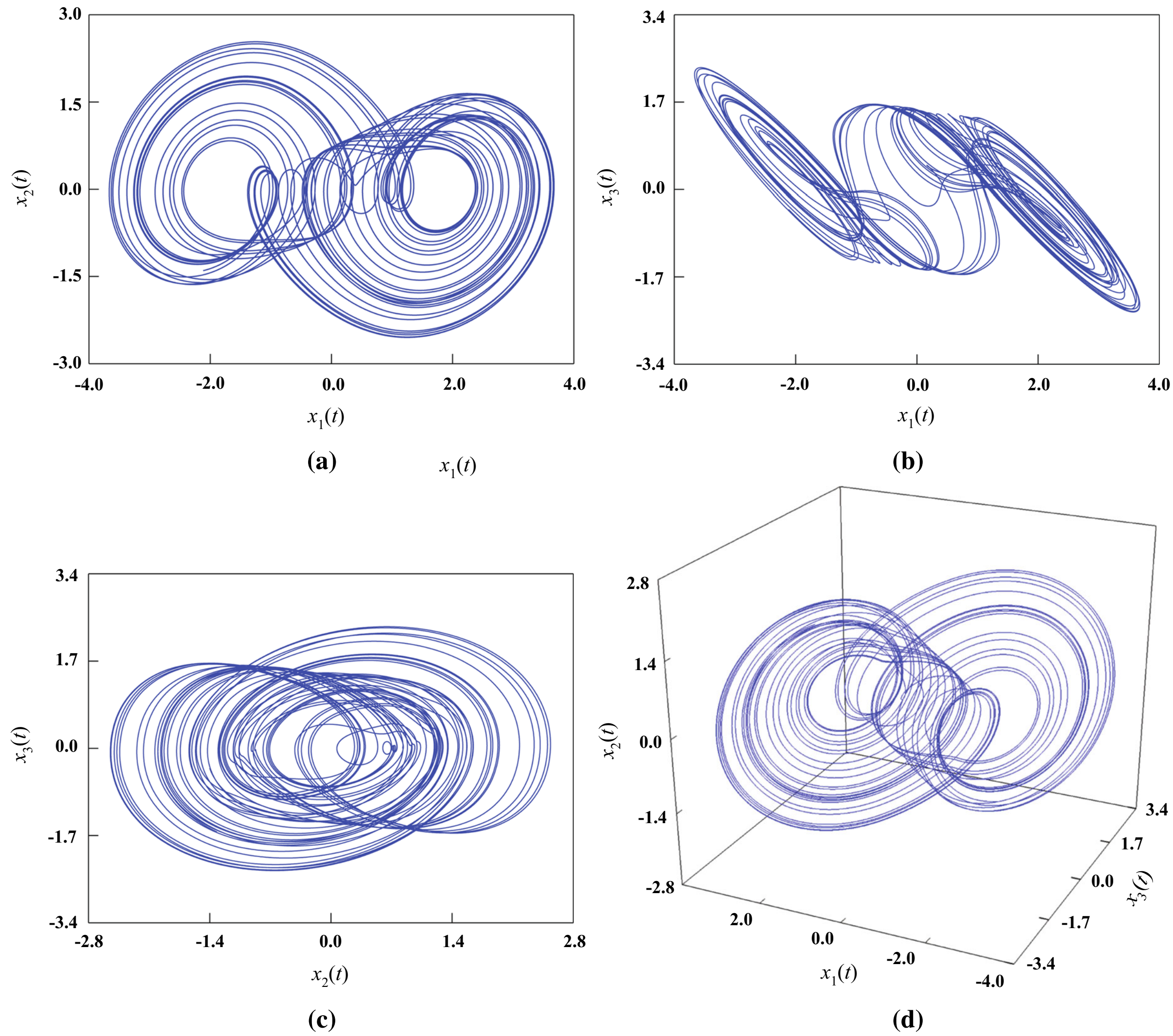

Fig. 1 Chaotic behaviors of modified fractional-order Jerk system. **a** $x_1(t) - x_2(t)$ plane, **b** $x_1(t) - x_3(t)$ plane, **c** $x_2(t) - x_3(t)$ plane, **d** $x_3(t) - x_1(t) - x_2(t)$ space

Referring to the designed controller (26), the synchronization controller can be written as

$$\begin{aligned} u_1(t) &= -e_2(t) - \eta_1 s_1(t) - \hat{\kappa}_1 \mathrm{sign}(s_1(t)) - \hat{d}_1(t) \\ u_2(t) &= -e_3(t) - \eta_2 s_2(t) - \hat{\kappa}_2 \mathrm{sign}(s_2(t)) - \hat{d}_2(t) \\ u_3(t) &= \varepsilon_1 e_1(t) + e_2(t) + \varepsilon_2 e_3(t) + f_3(y(t)) \\ &\quad - f_3(x(t)) - \eta_3 s_3(t) - \hat{\kappa}_3 \mathrm{sign}(s_3(t)) - \hat{d}_3(t) \end{aligned} \tag{52}$$

Substituting (52) into (51), we have

$$\begin{aligned} D^\alpha e_1(t) &= -\eta_1 s_1(t) - \hat{\kappa}_1 \mathrm{sign}(s_1(t)) + \tilde{d}_1(t) \\ D^\alpha e_2(t) &= -\eta_2 s_2(t) - \hat{\kappa}_2 \mathrm{sign}(s_2(t)) + \tilde{d}_2(t) \\ D^\alpha e_3(t) &= -\eta_3 s_3(t) - \hat{\kappa}_3 \mathrm{sign}(s_3(t)) + \tilde{d}_3(t) \end{aligned} \tag{53}$$

where $D^\alpha \hat{\kappa}_i = \gamma_i(|s_i(t)| - \hat{\kappa}_i)$ with $\gamma_i > 0$ and $i = 1, 2, 3$.

To demonstrate the effectiveness of the proposed nonlinear FODO-based adaptive sliding mode synchronization control scheme, the numerical simulation results are presented for the modified fractional-order Jerk system under the following conditions: the initial conditions $x_0(t) = (1, 1, 1)^T$, $y_0(t) = (1.2, 0.6, 0.5)^T$, $\hat{\kappa}_0 = (0.1, 0.1, 0.1)^T$ and $\hat{\phi}_0(t) = (0.1, 0.1, 0.1)^T$, and the designed parameters are chosen as $\alpha = 0.98$, $\sigma_1 = \sigma_2 = \sigma_3 = 50$, $\gamma_1 = \gamma_2 = \gamma_3 = 0.1$ and $\eta_1 = \eta_2 = \eta_3 = 50$.

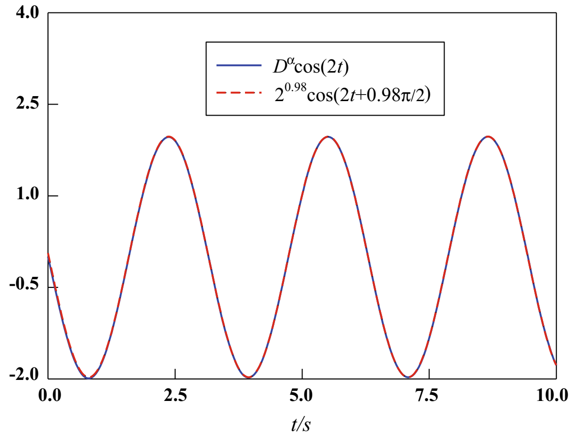

Fig. 2 Comparison result of $2^{0.98}\cos(2t + \frac{0.98\pi}{2})$ and $D^{\alpha}\cos(2t)$

The disturbance is assumed as $d_1(t) = \cos(2t)$, $d_2(t) = \cos(2t)$ and $d_3(t) = \cos(2t)$. On the basis of the result in [52], we have $\rho_1 D^{\alpha}\cos(\rho_2 t) = \rho_1\frac{1}{2}(j\rho_2)^m t^{m-\alpha}(E_{1,m-\alpha+1}(j\rho_2 t) + (-1)^n E_{1,m-\alpha+1}(-j\rho_2 t))$ with j denotes the unit of imaginary part with ρ_1 and ρ_2 which are arbitrary numbers. In this paper, the parameter $m = 1$ and the fractional-order $\alpha = 0.98$. Thus, $\rho_1\rho_2^{\alpha}\cos(\rho_2 t + \frac{\pi\alpha}{2})$ can be used to approximate $\rho_1 D^{\alpha}\cos(\rho_2 t)$. The comparison result is shown in Fig. 2 for the case of $\rho_1 = 1$ and $\rho_2 = 2$. According to Fig. 2, Assumption 1 is satisfied.

The numerical results are shown in Figs. 3 and 4 under the proposed nonlinear FODO-based adaptive sliding mode control scheme. The state synchronization results of drive system (45) and response system

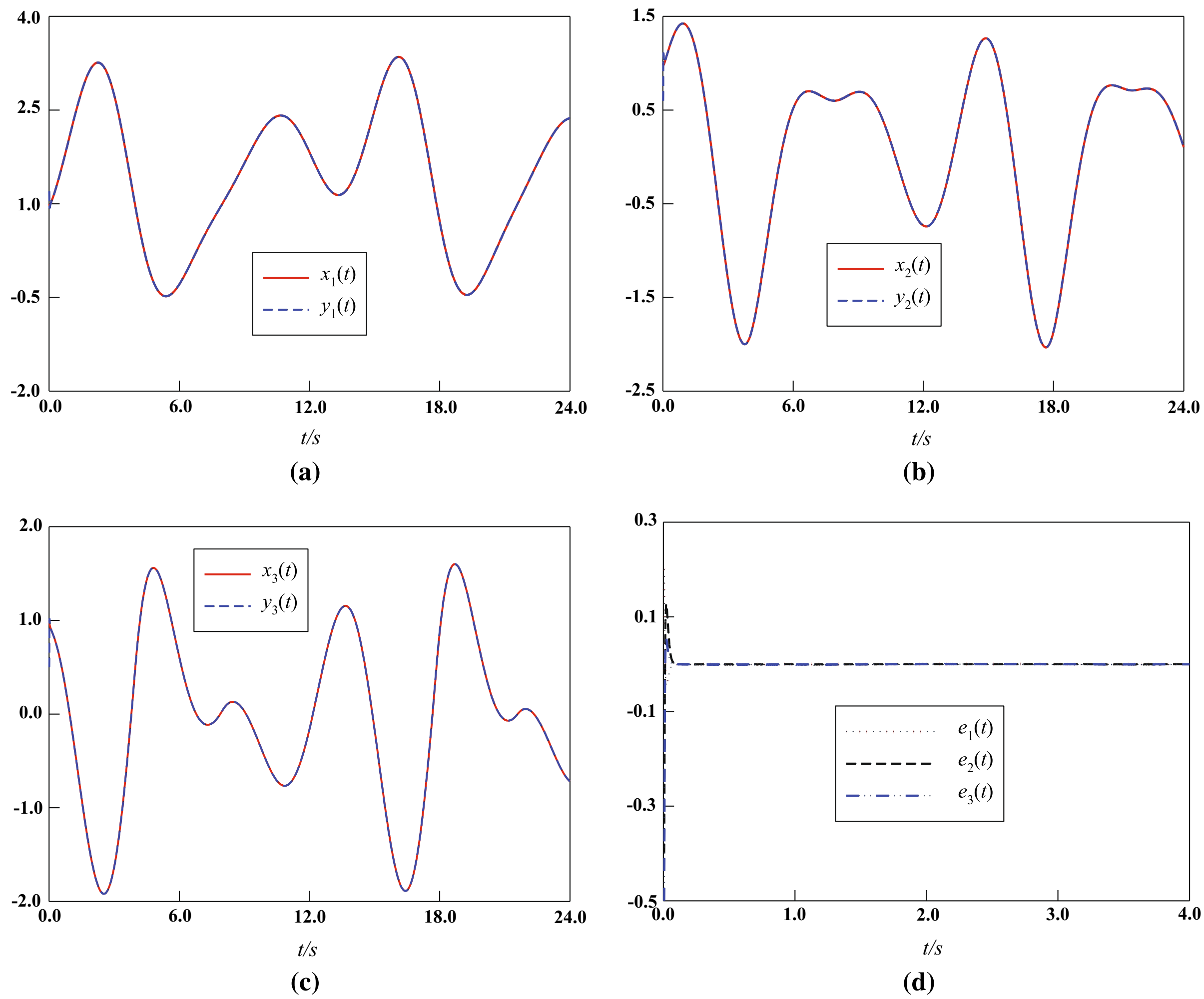

Fig. 3 Synchronization control results of modified fractional-order Jerk system. **a** Synchronization state of $x_1(t)$ and $y_1(t)$, **b** synchronization state of $x_2(t)$ and $y_2(t)$, **c** synchronization state of $x_3(t)$ and $y_3(t)$, **d** synchronization error $e_1(t)$, $e_2(t)$ and $e_3(t)$

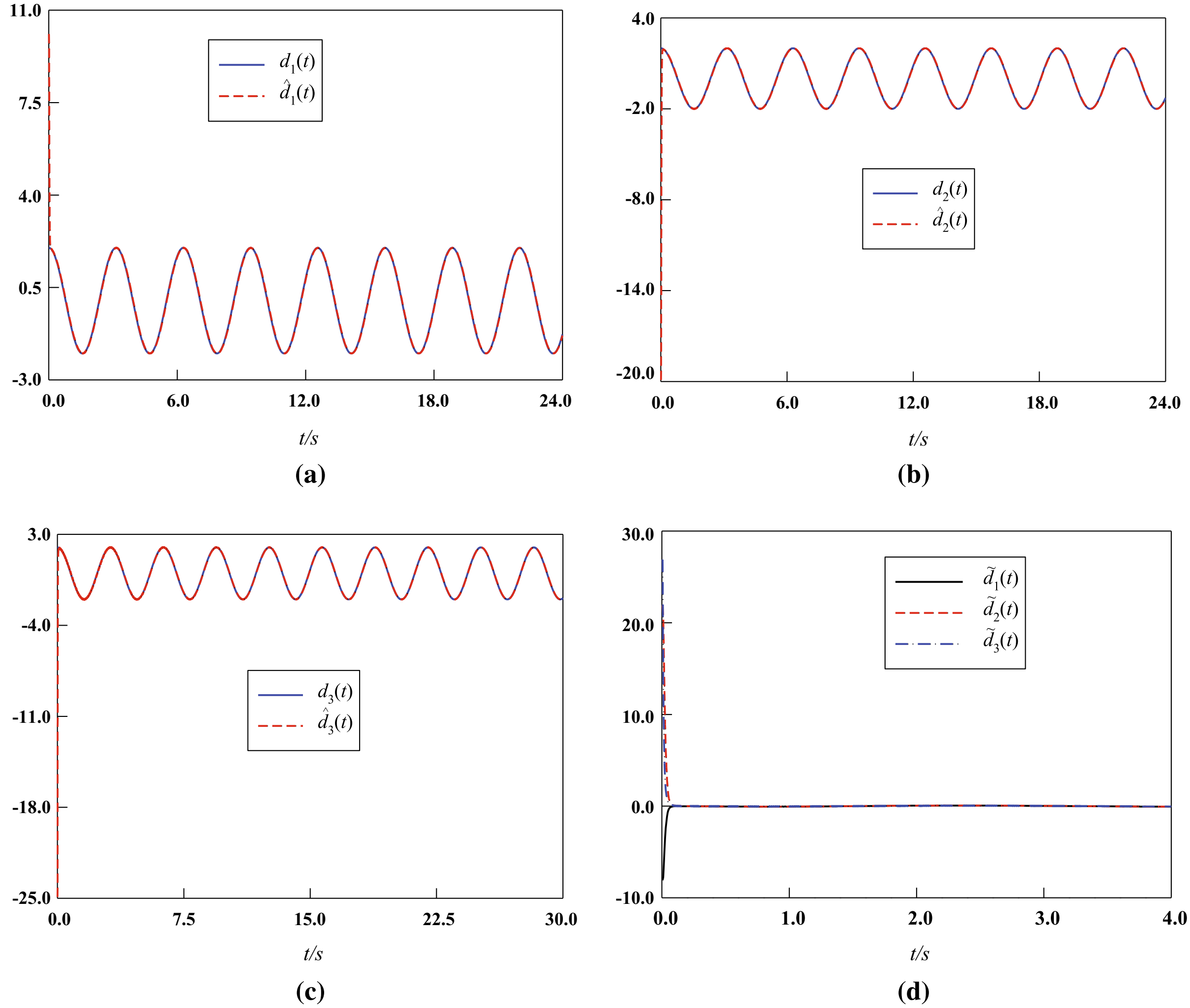

Fig. 4 Disturbance observer results. **a** $d_1(t)$ and $\hat{d}_1(t)$, **b** $d_2(t)$ and $\hat{d}_2(t)$, **c** $d_3(t)$ and $\hat{d}_3(t)$, **d** observation errors $\tilde{d}_1(t)$, $\tilde{d}_2(t)$ and $\tilde{d}_3(t)$

(49) are given in Fig. 3a–c. It is shown that good synchronization performance is achieved. Figure 3d shows the synchronization errors $e_1(t)$, $e_2(t)$ and $e_3(t)$ are convergent. Furthermore, the observation performance of the proposed FODO (13) and (14) is presented in Fig. 4. It is evident in Fig. 4 that the observer is effective and feasible. According to the simulation results, the drive system (45) and the response system (49) are bounded synchronization under the designed sliding mode controller (26) and the adaptive update law (27). Therefore, the proposed nonlinear FODO-based adaptive sliding mode synchronization control scheme is valid for fractional-order chaotic systems with external disturbance.

6 Conclusion

In this paper, the nonlinear FODO-based adaptive sliding mode synchronization control scheme has been studied for fractional-order chaotic systems in the presence of external disturbance. A nonlinear FODO has been developed to approximate the unknown disturbances. A sliding mode synchronization controller has been designed based on the nonlinear FODO for synchronization of fractional-order chaotic systems. Furthermore, an example is given in the present paper, i.e., the synchronization between two modified fractional-order Jerk systems. The numerical simulations show the effectiveness of the proposed nonlinear FODO-based adaptive sliding mode synchronization control scheme.

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