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COMMENT

Comments on "Solving nonlinear stochastic differential equations with fractional Brownian motion using reducibility approach" [Nonlinear Dyn. 67, 2719–2726 (2012)]

Khosro Khandani · Vahid Johari Majd · Mahdieh Tahmasebi

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Abstract In this note, some comments are pointed out to the paper (Zeng et al. in Nonlinear Dyn 67:2719–2726, 2012). It is shown that the authors in the mentioned paper have wrongly utilized the fractional Ito formula to derive the reducibility conditions of nonlinear fractional stochastic differential equations (SDEs) to linear fractional SDEs. This incorrect use of the fractional Ito formula has led to fundamental flaws in the proposed theorems.

Keywords Fractional Ito formula · Nonlinear stochastic differential equations · Fractional Brownian motion

1 Introduction

In [1], the authors have claimed that they have derived some necessary and sufficient conditions for reducing the nonlinear stochastic differential equations (SDEs) driven by fractional Brownian motion (fBm) to linear SDEs. The nonlinear SDE with fBm considered in [1] is:

$$\mathrm{d}x(t) = F(t, x(t))\mathrm{d}t + G(t, x(t))\mathrm{d}B^H(t) \quad (1)$$

where $F(t, x(t))$ and $G(t, x(t))$ are real-valued nonlinear functions, and $B^H(t)$ is a standard fBm with Hurst parameter $1/2 < H < 1$. It has been claimed in [1] that by applying the fractional Ito formula in [2] to process $y(t) = h(t, x(t))$, the following can be obtained:

$$\begin{aligned}\mathrm{d}y(t) = &\left[\frac{\partial h(t, x(t))}{\partial t} + \frac{\partial h(t, x(t))}{\partial x}F(t, x(t))\right.\\ &+\frac{\partial^2 h(t, x(t))}{\partial x^2}G(t, x(t))\\ &\left.\int_0^t G(s, x(s))\phi(s, t)\mathrm{d}s\right]\mathrm{d}t\\ &+\frac{\partial h(t, x(t))}{\partial x}G(t, x(t))\mathrm{d}B^H(t)\end{aligned} \quad (2)$$

where the function $\phi(s, t)$ is defined as

$$\phi(s, t) = H(2H-1)\,|s-t|^{2H-2} \quad (3)$$

Based on this claim, some necessary and sufficient conditions for the reducibility of nonlinear SDEs with fBm to linear homogenous and nonhomogenous SDEs have been presented in two theorems. Furthermore, following the reducibility conditions, an explicit solution of (1) has been obtained. The authors of [1] have also pro-

K. Khandani · V. J. Majd (✉)
Intelligent Control Systems Laboratory, School of Electrical and Computer Engineering, Tarbiat Modares University, P.O. Box 14115-194, Tehran, Iran
e-mail: majd@modares.ac.ir
URL: http://www.modares.ac.ir/~majd

M. Tahmasebi
School of Mathematics, Department of Applied Mathematics, Tarbiat Modares University, Tehran, Iran
e-mail: tahmasebi@modares.ac.ir

vided some simulation examples to illustrate the effectiveness of their approach. However, in the next section, we show that the authors have inconsiderately used the fractional Ito formula for their problem. Due to this negligence, the solutions obtained for nonlinear SDEs with fBm (1) by reducing them to linear SDEs are not valid.

2 Main discussion

To prove that the fundamental relation (2) is wrong, we review the results in [2] on fractional Ito formula.

Theorem 1 (Theorem 4.5 in [2]) *Let* $G_u, u \in [0, T]$ *satisfy the conditions of theorem 4.3 in* [2], *and Let* $sup_{0\le s<T} |F_s| < \infty$. *Denote* $x_t = \xi + \int_0^t F_u \mathrm{d}u + \int_0^t G_u \mathrm{d}B_u^H$, $\xi \in \mathbb{R}$ *for* $t \in [0, T]$. *Let* $\left(\frac{\partial f}{\partial x}(s, x_s) F_s, s \in [0, T]\right) \in \mathcal{L}(0, T)$. *Then, for* $t \in [0, T]$, *the following holds:*

$$\begin{aligned} f(t, x_t) = f(0, \xi) &+ \int_0^t \frac{\partial f}{\partial s}(s, x_s)\,\mathrm{d}s \\ &+ \int_0^t \frac{\partial f}{\partial x}(s, x_s)\, F_s \mathrm{d}s \\ &+ \int_0^t \frac{\partial f}{\partial x}(s, x_s)\, G_s \mathrm{d}B_s^H \\ &+ \int_0^t \frac{\partial^2 f}{\partial x^2}(s, x_s)\, G_s D_s^{\phi} x_s \mathrm{d}s \end{aligned} \tag{4}$$

where $D_t^{\phi} x_t$ is the Malliavin derivative of x_t defined in Definition 3.1 in [2].

For the nonlinear SDE with fBm given in (1), it has been claimed that:

$$D_r^{\phi} x_t = \int_0^t G(s, x(s))\phi(s, r)\mathrm{d}s \tag{5}$$

which is incorrect. As a matter of fact, (5) is true only in the case that in (1), $F(t, x(t)) = 0$ and $G(t, x(t))$ is not a function of $x(t)$ as given in relation 3.6 in [2]. For the nonlinear SDE (1), it can be shown that $D_t^{\phi} x_t$ satisfies the following equation:

$$\begin{aligned} D_r^{\phi} x_t = \int_0^t \frac{\partial F_s}{\partial x} D_r^{\phi} x_s \mathrm{d}s &+ \int_0^t \frac{\partial G_s}{\partial x} D_r^{\phi} x_s \mathrm{d}B_s^H \\ &+ \int_0^t G_s \phi(r, s)\,\mathrm{d}s \end{aligned} \tag{6}$$

By letting $Y_t := D_r^{\phi} x_t$, then:

$$\mathrm{d}Y_t = \left\{\frac{\partial F_t}{\partial x} Y_t + G_t \phi(r, t)\right\}\mathrm{d}t + \left\{\frac{\partial G_t}{\partial x} Y_t\right\}\mathrm{d}B_t^H \tag{7}$$

To obtain Y_t for the nonlinear SDEs with a nonzero drift term, one needs to solve the fractional SDE (7) which is a complicated task. However, Eq. (5) used by the authors of the paper reduces to:

$$\mathrm{d}Y_t = \{G_t \phi(r, t)\}\,\mathrm{d}t \tag{8}$$

It is clear that Eqs. (7) and (8) are not identical and will not yield the same solution. Therefore, the necessary and sufficient conditions for reducibility of nonlinear SDEs with fBm to linear SDEs should be modified by employing Theorem 1 with the correct Malliavin derivative of x_t as given in Eq. (6). Additionally, based on this discussion, the explicit solution of (1) that is obtained by the authors by using the reducibility approach in section 3 of [1] and also the illustrative examples in section 4 of [1] are not valid.

Remark 1 As a counterexample, consider a special case that SDE is linear and is given by:

$$\mathrm{d}x_t = A_t x_t \mathrm{d}t + C_t x_t \mathrm{d}B_t^H \tag{9}$$

where A_t and C_t are matrix-valued stochastic processes, and $\mathrm{d}B_t^H$ is the Ito-type differential of fractional Brownian motion with Hurst parameter H. For this SDE, $D_t^{\phi} x_t$ has been proved to be as follows (see Lemma 2.12 in [3]):

$$D_t^{\phi} x_t = x_t \int_0^t \phi(t, s)\, C_s \mathrm{d}s, \quad \forall t \in [0, T] \tag{10}$$

It is clear that the above equation is not identical to the linear form of (5). Moreover, by replacing (10) in fractional Ito formula (4), Eq. (2) will not be obtained.

Remark 2 In Remarks 1–3 in [1], it has been claimed that for $F(t, x(t)) = F(x)$ and $G(t, x(t)) = G(x)$, both independent of t, the reducibility conditions take some special forms. However, we should mention the fact that if any of F and G is independent of t, then $x(t)$ will become independent of t, which means that x is a constant. Therefore, the given results in Remarks 1–3 in [1] are not valid as well.

3 Conclusions

In this comment paper, we pointed out that the results in the paper [1] contain essential errors. It was shown that the necessary and sufficient conditions for reducibility of nonlinear SDEs to linear ones given in [1] cannot be trusted.

References

1. Zeng, C., Yang, Q., Chen, Y.Q.: Solving nonlinear stochastic differential equations with fractional Brownian motion using reducibility approach. Nonlinear Dyn. **67**, 2719–2726 (2012)
2. Duncan, T.E., Hu, Y., Pasik-Duncan, B.: Stochastic calculus for fractional Brownian motion. I. Theory. SIAM J. Control Optim. **38**, 582–612 (2000)
3. Hu, Y., Zhou, X.Y.: Stochastic control for linear systems driven by fractional noises. SIAM J. Control Optim. **43**(6), 2245–2277 (2005)