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Energy components in nonlinear dynamic response of SDOF systems

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Abstract The equation of motion for the response of SDOF systems is presented in terms of energy for a general class of hysteretic relationships that include stiffness degrading inelastic and bilinear elastic behavior, for systems subjected to earthquake ground motions in the presence or absence of P-delta effects. The evaluation of expended energy is presented in a continuous form, along with the evaluation of input energy associated with second-order geometric effects. Examples illustrate the dissipation of energy through damping and hysteretic response and the increase in input energy associated with P-delta effects.

Keywords Dissipated energy · Damping energy · Strain energy

1 Introduction

Energy associated with the dynamic response of bilinear and stiffness degrading single-degree-of-freedom (SDOF) systems in earthquake engineering has been described by [7]. Energy spectra and the potential use of earthquake input energy as a basis for seismic design is addressed by [11], who considered differences between absolute and relative energy and recognized value in using a scalar parameter in design. Essential terminology is summarized in textbooks on structural dynamics for earthquake engineering, such as [5]. The present article summarizes and extends this work to consider nonlinear elastic systems and second-order (P-delta) effects.

The equation of motion of a viscous damped single-degree-of-freedom (SDOF) system subjected to ground acceleration $\ddot{u}_g(t)$, can be expressed as:

$$m\ddot{u}(t) + c\dot{u}(t) + f_S = -m\ddot{u}_g(t) \quad (1)$$

where m is the mass, c is the viscous damping coefficient, f_S is the spring force, u_g is the earthquake ground displacement, and u is the relative displacement of the mass with respect to the ground (Fig. 1). [11] describe f_S as the “restoring force”, but it can also be called the spring force. The only potential sources of energy dissipation within Eq. (1) are the viscous damping and the spring force. The main point that we clarify in this review is the concept of energy expended during hysteretic behavior and energy associated with second-order (P-delta) effects. From an external point of view, during the loading process one does not know whether the work done in straining the system is recoverable

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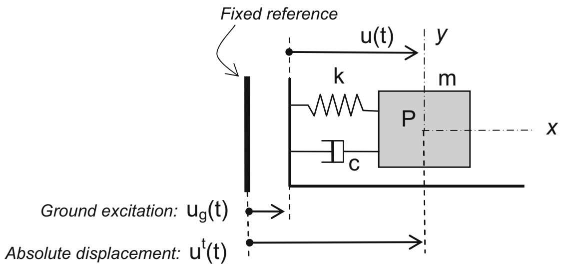


Fig. 1 Single-degree-of-freedom (SDOF) system used in earthquake engineering

or not, and to what degree. The strain energy consumed through inelastic deformation (e.g., movement of dislocations within the microstructure) can be known only from an internal point of view, or based on macroscopic hysteretic rules defined *a priori* for a particular hysteretic model, [9] and [1].

Structural engineers may contemplate the two systems illustrated in Fig. 2—both are bilinear: one is fully recoverable (i.e., elastic) and the other only partially recoverable (i.e., inelastic). The elastic one may represent a rocking wall, while the inelastic one may represent a common reinforced concrete frame member. Engineers usually consider the hysteretic behavior of such systems in terms of generic categories rather than considering in detail the internal behavior at a local level, and modeling is considered from the point of view of the response to externally applied loads. This means that Point 3 can be reached in the two systems (Fig. 2a, b) without considering whether all strain energy is recoverable (Fig. 2a) or if part of the strain energy is expended and not recoverable (Fig. 2b).

Under some circumstances, stiffness degradation may be modeled using the unloading stiffness degradation index (USD index: a), see Eq. 2 [8].

$$k_u = k\left|\frac{u_y}{u_{\max}}\right|^a \tag{2}$$

where k is the stiffness, αk is the post-yield stiffness and k_u is the degraded unloading stiffness. The degraded stiffness is tracked separately in the positive and negative directions of loading. In reinforced concrete structures a typically varies from 0.0 to 0.5, $u_{\max}$ is the peak displacement amplitude reached in the past in the direction of loading (and increases as the peak displacement increases).

A "snap-back" simulation can be represented by the free release from Point 3 (Fig. 2) in the absence of ground excitation. Under these circumstances, the oscillator would reach the abscissa (Point 1 for bilinear elastic and Point 4 for bilinear degrading) with a velocity that depends on the amount of damping and on the hysteretic losses associated with inelastic behavior. In the case of the bilinear elastic system (Fig. 2), Point 1 coincides with the origin, and thus the oscillator has a re-centering capability, returning to the origin after the external excitation ceases and the kinetic energy has damped out. In the case of the bilinear degrading system, stiffness degradation is seen to move Point 4 closer to the origin, thus providing a degree of re-centering capability. This is helpful for reducing peak and residual displacements under earthquake loading.

2 Damping energy and total energy

In physics the traditional definition of damping is *an influence that restricts oscillations*. Damping occurs because there are processes that dissipate energy during the oscillation. Damping typically has been modeled as proportional to velocity of the mass relative to the base. With the simple SDOF model of Eq. (1), only the damping term provides for energy dissipation in the case of elastic systems. Energy may also be dissipated through the f_S term if inelastic behavior is represented in the hysteretic model.

The terms of Eq. (1) represent the forces involved in the dynamic equilibrium of the mass as represented in Fig. 1. We multiply the terms of Eq. (1) by a differential of relative displacement du and integrate to obtain the equation of motion in terms of relative energy [11]:

$$\int m\ddot{u}(t)\mathrm{d}u + \int c\dot{u}(t)\mathrm{d}u + \int f_S \mathrm{d}u = -\int m\ddot{u}_g(t)\mathrm{d}u$$
$$E_K + E_D + E_S = E_i \tag{3}$$

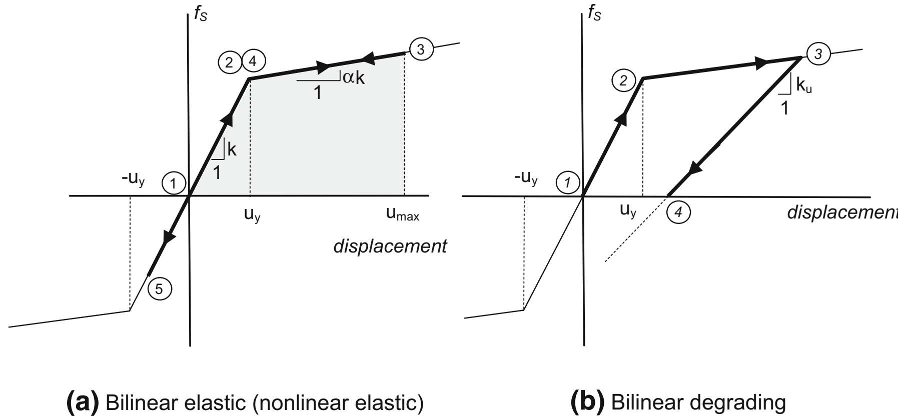
The system is excited by the ground excitation $u_g(t)$; the integral on the right-hand side of Eq. (3) defines the input energy, E_i.

The relative kinetic energy is:

$$E_K = \int m\ddot{u}(t)\mathrm{d}u = \int m\frac{\mathrm{d}\dot{u}}{\mathrm{d}t}\mathrm{d}u = \int m\dot{u}\mathrm{d}\dot{u} = m\frac{\dot{u}^2}{2} \tag{4}$$

The damping energy, considering linear viscous damping, is:

Fig. 2 Two bilinear SDOF systems **a** fully recoverable, **b** partially recoverable



$$E_D = \int c \frac{du}{dt} du = \int c \left(\frac{du}{dt}\right)^2 dt \tag{5}$$

In order to calculate E_D, it is necessary to know $u(t)$, which can be determined by solving Eq. (1) for the damped system.

In structural dynamics, damping is often specified by setting the fraction of critical damping, ξ, equal to a fixed value (e.g., 5 %) and computing the damping coefficient c from the definition

$$\xi = \frac{c}{2m\omega} \rightarrow c = 2m\omega\xi \tag{6}$$

Different approaches are in use for determining the damping coefficient for use in nonlinear analysis. If c is calculated to obtain a desired ξ based on initial elastic properties, relatively large damping forces can result in systems undergoing nonlinear response [4]. Alternatively, c can be recomputed based on the instantaneous stiffness and associated natural circular frequency, ω. This approach (e.g., Example 2) avoids the relatively high damping forces but is not generally applicable should the hysteretic response develop a zero or negative post-yield stiffness.

3 Expended energy

Traditionally the expended strain energy (E_{SE}) has been presented as energy dissipated through inelastic hysteretic response, and has been calculated as the difference between the strain energy used in the loading process (E_S) and the strain energy recovered during the unloading process (E_{SR}), as illustrated in Fig. 3.

During the loading process (going from Point 1 to Points 2 and 3), the external work used to deform the system causes strain and possible damage within the material. The energy (E_S) absorbed in going from Point 1 to Point 3 is composed of recoverable strain energy (E_{SR}) and energy dissipated by inelastic deformation (E_{SE}). The recoverable strain energy is associated with elastic deformation of the microstructure, while the hysteretic losses, associated with damage and inelastic deformation, are due to physical processes such as movement of dislocations within the steel microstructure, cracking and fracture, and friction across cracked surfaces. The work associated with these changes to the internal microstructure, taken together, is termed strain energy, E_S, and is always positive (i.e., $f_S > 0$ for $du > 0$ or $f_S < 0$ for $du < 0$); this is a precept of the energy-based pushover method of [6]:

$$E_S = \int f_S(u) du > 0 \tag{7}$$

The strain energy associated with the recovery process (3–4), shown in Fig. 3, involves elastic unloading and is termed recoverable strain energy (E_{SR}), shown in Fig. 3. It is negative (i.e., $f_S > 0$ for $du < 0$ or $f_S < 0$ for $du > 0$):

$$E_{SR} = \int f_S(u) du < 0 \tag{8}$$

Using the traditional approach, the expended strain energy (E_{SE}) is calculated at the end of each half-cycle as the difference between the work done during loading and the strain energy recovered during unloading ($E_{SE} = E_S - E_{SR}$); in this expression and in what

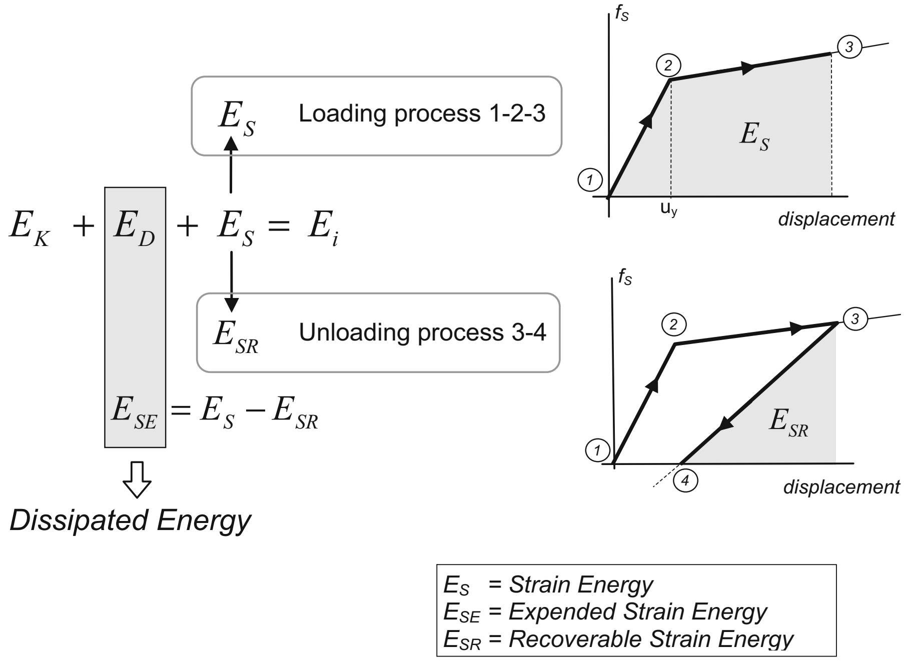


Fig. 3 Decomposition of strain energy in a half-cycle of response of a nonlinear system

follows the negative sign of E_{SR} is explicited, meaning that the negative sign is due to the fact that E_{SR} is < 0 (Eq. 8).

The dissipated energy represents losses under dynamic loading and is given by the sum of the damping energy, E_D, and the expended strain energy, E_{SE}.

Nevertheless, in case of the model of Fig. 2b, when Point 3 is reached, the damage associated with the expended strain energy has already occurred; the recovery segment from Point 3 to Point 4 is just a linear elastic process with no hysteretic losses. Traditionally, the loss of strain energy is evaluated based on internal mechanisms or based on a previously defined hysteretic model. As an alternative, we introduce a continuous approach, in which the expended strain energy is evaluated incrementally.

With reference to Fig. 4, at displacement u_i, the recoverable strain energy is E_{SRi}; and at u_{i+1} the recoverable strain energy is E_{SRi+1}. The recoverable strain energy during the process from u_i to u_{i+1} is $\Delta E_{SR} = E_{SRi+1} - E_{SRi}$. In the case of the bilinear degrading hysteretic model:

$$E_{SRi} = \frac{1}{2} f_{Si}\tilde{u}_i = \frac{f_{Si}^2}{2k_i} \tag{9a}$$

$$E_{SRi+1} = \frac{f_{Si+1}^2}{2k_{i+1}} \tag{9b}$$

$$\Delta E_{SR} = \frac{1}{2}\left(\frac{f_{Si+1}^2}{k_{i+1}} - \frac{f_{Si}^2}{k_i}\right) \tag{9c}$$

During this loading process (from u_i to u_{i+1}), the incremental strain energy is ΔE_S and the incremental expended strain energy is $\Delta E_{SE} = \Delta E_S - \Delta E_{SR}$. In case of the bilinear degrading model of Fig. 4:

$$\Delta E_S = \frac{f_{Si} + f_{Si+1}}{2}(u_{i+1} - u_i) \tag{10a}$$

$$\Delta E_{SE} = \frac{1}{2}\left((f_{Si} + f_{Si+1})(u_{i+1} - u_i - \frac{f_{Si+1}^2}{k_{i+1}} + \frac{f_{Si}^2}{k_i}\right) \tag{10b}$$

At any time, the principle of conservation of energy can be expressed as:

$$\begin{aligned}\Delta E_i &= \Delta E_K + \Delta E_D + \Delta E_S \\ &= \Delta E_K + \Delta E_D + \Delta E_{SE} + \Delta E_{SR} \\ &= \Delta E_K + \Delta E_{SR} + (\Delta E_D + \Delta E_{SE})\end{aligned} \tag{11}$$

where E_i is the input energy for the damped nonlinear oscillator. The term in parentheses is the dissipated energy during the process, due to damping and inelasticity.

In the following, several examples are used to illustrate the partitioning of the energy during response. In most cases, the oscillator is loaded quasi-statically to an initial displacement of $u_0 = 0.03$ m and released.

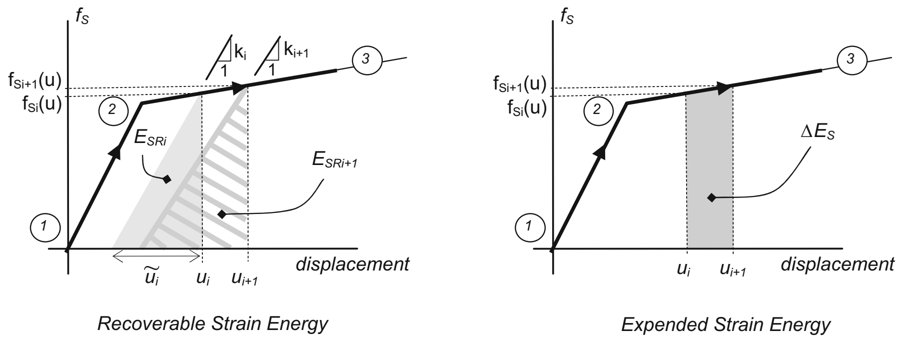


Fig. 4 Continuous decomposition of the strain energy in a nonlinear system

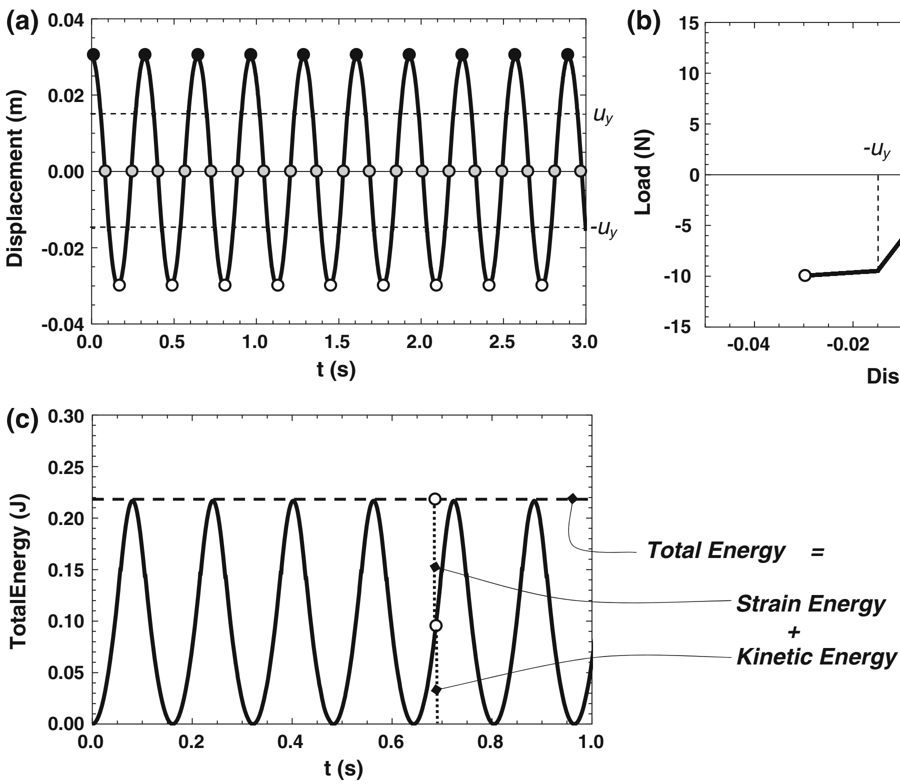


Fig. 5 Example 1: free vibration response of a bilinear elastic system having no damping

In this "snap-back" type of analysis, the input energy is fixed and the dissipated energy is free of external influence.

Example 1 A bilinear elastic system is considered in this example, having: $m = 1$ kg, $\xi = 0$ (no damping), $T = 0.25$ s, $u_y = 0.015$ m, $u_0 = 0.03$ m, and $\alpha = 0.05$. The solution was obtained using the linear acceleration method [5] with a time step of 0.001 s. The numerical error is negligible for the time interval studied. The oscillator was loaded quasi-statically up to Point 3 (see Fig. 2a), being this point the starting point of the dynamic problem. In accordance with the principle of conservation of energy, the total energy $(E_K + E_D + E_S = E_i)$ is constant along the duration of free vibration response and is equal to 0.217 J; during the loading process this energy was stored as strain energy, to be released at the start of the free (unforced) vibration response. Figure 5 shows the first 3 s of dynamic response and the partitioning of the total energy into strain energy and kinetic energy during this time interval.

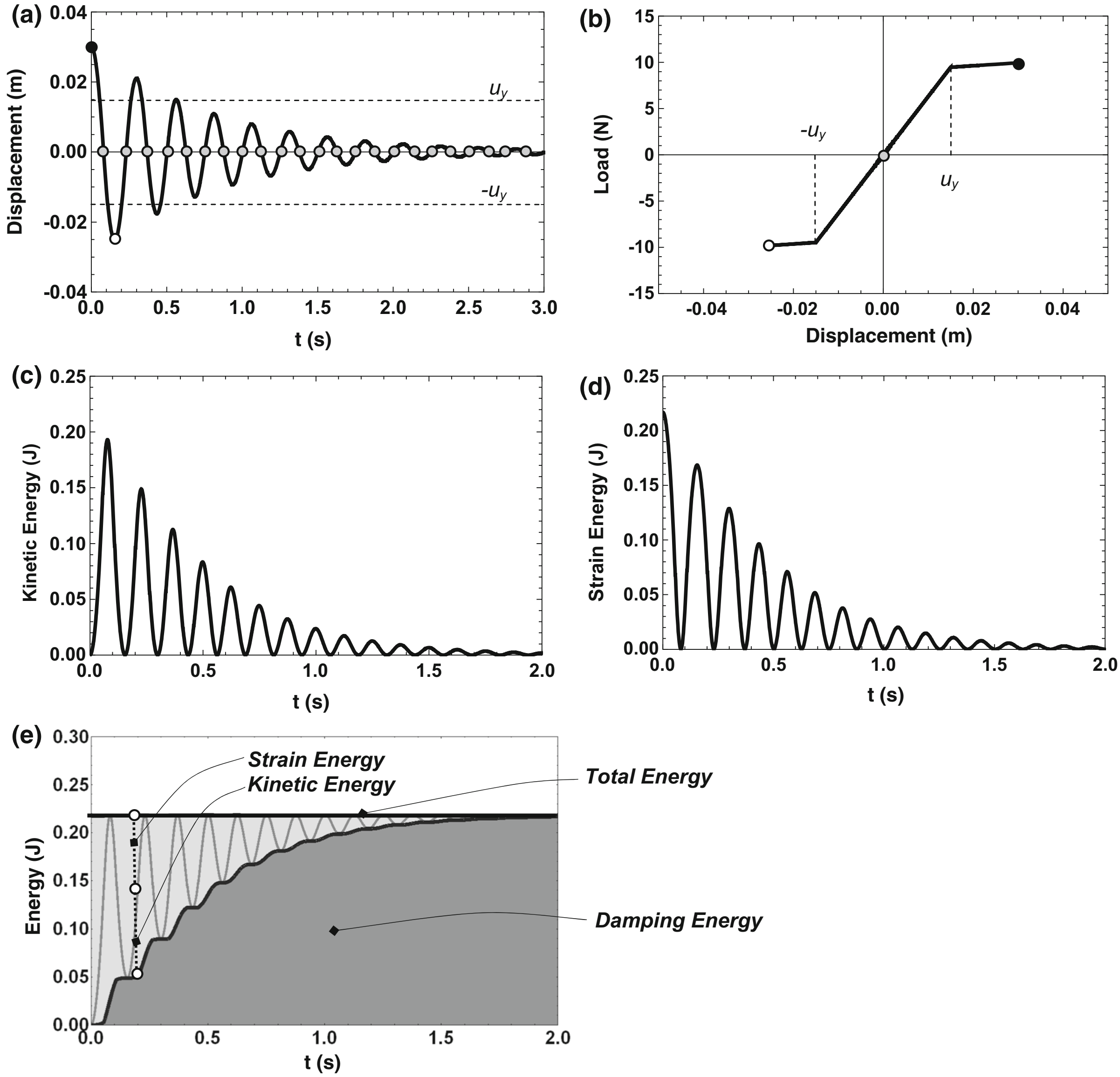


Fig. 6 Example 2: free vibration of a damped bilinear elastic system ($\xi = 0.05$)

Example 2 The only change in this example relative to Example 1 is the introduction of instantaneous damping equal to 5 % of critical damping ($\xi = 0.05$). To evaluate the damping energy, Eqs. 5 and 7 are considered in their incremental form:

$$E_D = 2\xi m \sum_i \omega v^2 \Delta t_i$$

$$E_S = \sum_i \frac{f_{Si+1} + f_{Si}}{2}(u_{i+1} - u_i)$$

where Δt_i is the time interval associated with the ith step.

At any instant of time the sum of the accumulated damping energy and instantaneous kinetic and strain energies is the total energy (0.217 J); this sum ($E_K + E_D + E_S$) is constant during the free vibration (snap-back) response. In this system the only energy sink is associated with viscous damping. In Fig. 6e, four plateaus can be observed in the damping energy plot before 0.5 s. These plateaus occur in the displace-

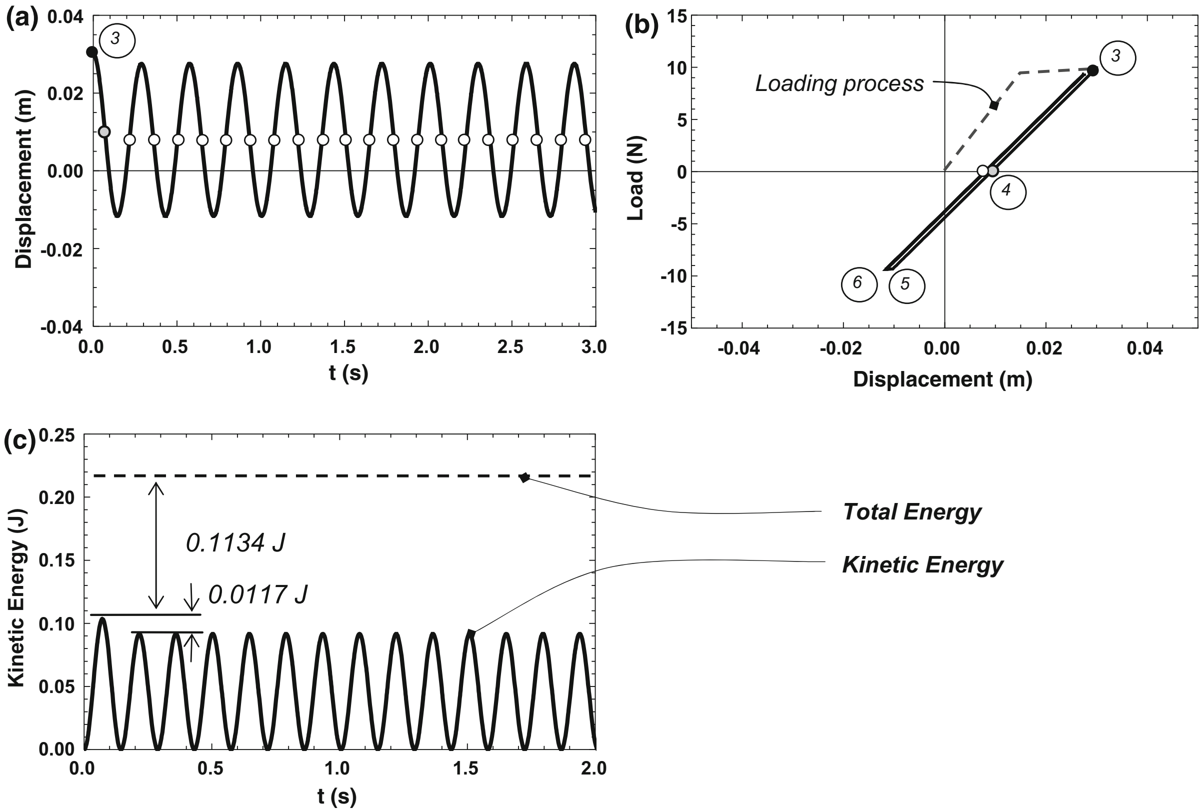


Fig. 7 Example 3: free vibration of a bilinear degrading model having no damping

ment versus time plot (Fig. 6a), where the displacement exceeds $\pm u_y$. These plateaus are due to the fact that the damping force associated with maintaining $\xi = 0.05$ is very small ($c = 2\xi m\omega$) where the stiffness has reduced from k to αk. More specifically:

$$c = \begin{cases} 2\xi m\sqrt{\frac{k}{m}} & \text{in the elastic zone} \\ 2\xi m\sqrt{\frac{\alpha k}{m}} & \text{in the plastic zone} \end{cases}$$

Example 3 A bilinear degrading model is used in Example 3, as illustrated in Fig. 2b, with stiffness degradation described by Eq. 2. In this case no damping is considered: $m = 1$ kg, $\xi = 0$, $T = 0.25$ s, $u_y = 0.015$ m, $u_0 = 0.03$ m, $a = 0.4$, and $\alpha = 0.05$.

It can be observed in the load-displacement graph of Fig. 7 that the starting point (Point 3) is in the plastic range. During the loading process, the expended strain energy (associated with inelastic behavior) is 0.1134 J. During segment 3–4–5, the system has linear elastic behavior with no loss of energy. Due to strain hardening ($\alpha = 0.05$), the recoverable strain energy at Point 3 (or potential energy) is sufficient to drive the oscillator past Point 5 where it begins yielding to Point 6. During segment 5–6, the expended energy associated with inelastic response is 0.0117 J. After Point 6, the system oscillates in perpetuity with no further energy dissipation.

Table 1 Expended strain energy per half-cycle

Half-cycle	Expended strain energy
1st	0.1134 J
2nd	0.0117 J
3rd and subsequent	0 J

Following the traditional approach, the expended strain energy is computed for each half-cycle, i.e., each time that the spring force reaches is null. Only the first two half-cycles present expended strain energy (Table 1); the system remains elastic for all subsequent cycles (Fig. 8).

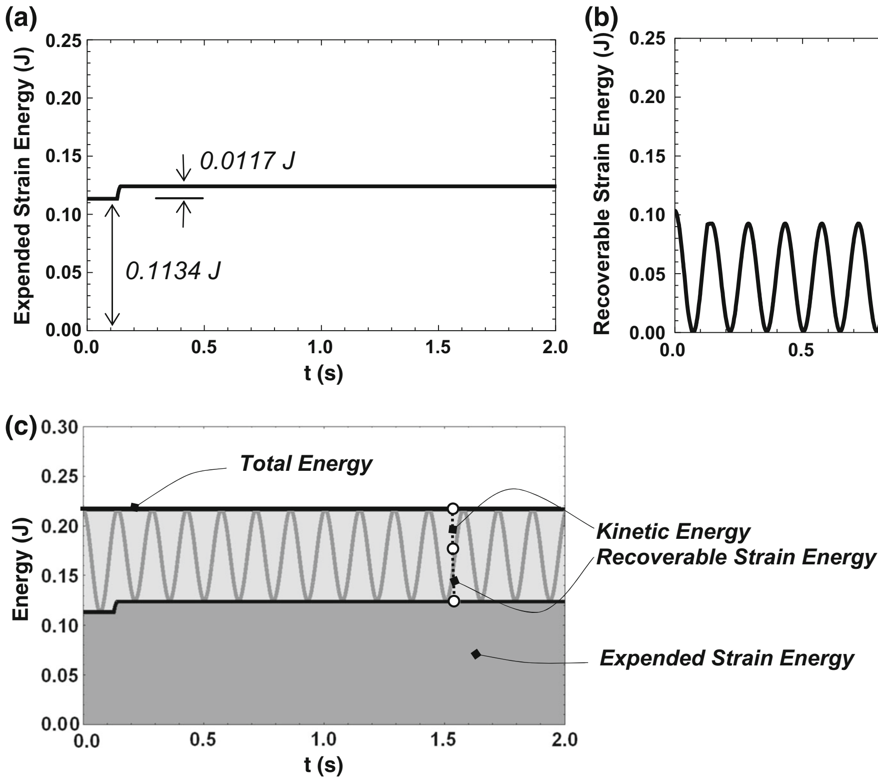


Fig. 8 Example 3: recoverable and expended strain energy

The use of the proposed formulation allows a continuous accounting of the expended strain energy, plotted in Fig. 8c. The initial expended energy of 0.217 J was required to reach Point 3 in quasi-static loading from the origin; this energy consists of components ($E_K + E_D + E_S = E_i$) whose sum remains constant throughout the free vibration snap-back response.

Example 4 Several changes in relation with previous examples are considered, to better observe the energy losses: $m = 1$ kg, $\xi = 0.04$, $T = 0.25$ s, $u_y = 0.015$ m, $u_0 = 0.05$ m, $a = 0.4$, and $\alpha = 0.10$. The total dissipated energy (i.e., damping energy plus expended strain energy) is shown in Fig. 9c. The quasi-static loading to $u_0 = 0.05$ m requires 0.441 J.

4 Consideration of P-delta effects

The P-delta effect is a second-order effect associated with the evaluation of equilibrium in the deformed configuration. In order to account for P-delta effects, additional variables are considered for the SDOF system: the height of the system, h, and applied vertical load, P (e.g., [10]). As shown in Fig. 10, the deformation of the SDOF system is concentrated at an elasto-plastic spring located at the base of the rigid column. The rotational stiffness at the base is k_r. Equilibrium of moments in the undeformed configuration (first-order equilibrium) provides:

$$fh = k_r \varphi \Rightarrow f = \frac{k_r}{h}\varphi = \frac{k_r}{h^2}u = f_S(u) \quad (12)$$

Equilibrium in the deformed configuration (second-order equilibrium), not accounting for large displacements, is given by:

$$fh + Pu = k_r \varphi \Rightarrow f = \left(\frac{k_r}{h^2} - \frac{P}{h}\right)u = f_S(u) - \frac{P}{h}u \quad (13)$$

Thus, one way to view the two previous equations is to consider that for a fixed relative displacement u, a smaller external force f is needed to equilibrate the system when P-delta effects are considered. This is obvious from the equilibrium of moments (left side of the arrow in Eqs. 12 and 13): in order to obtain the same

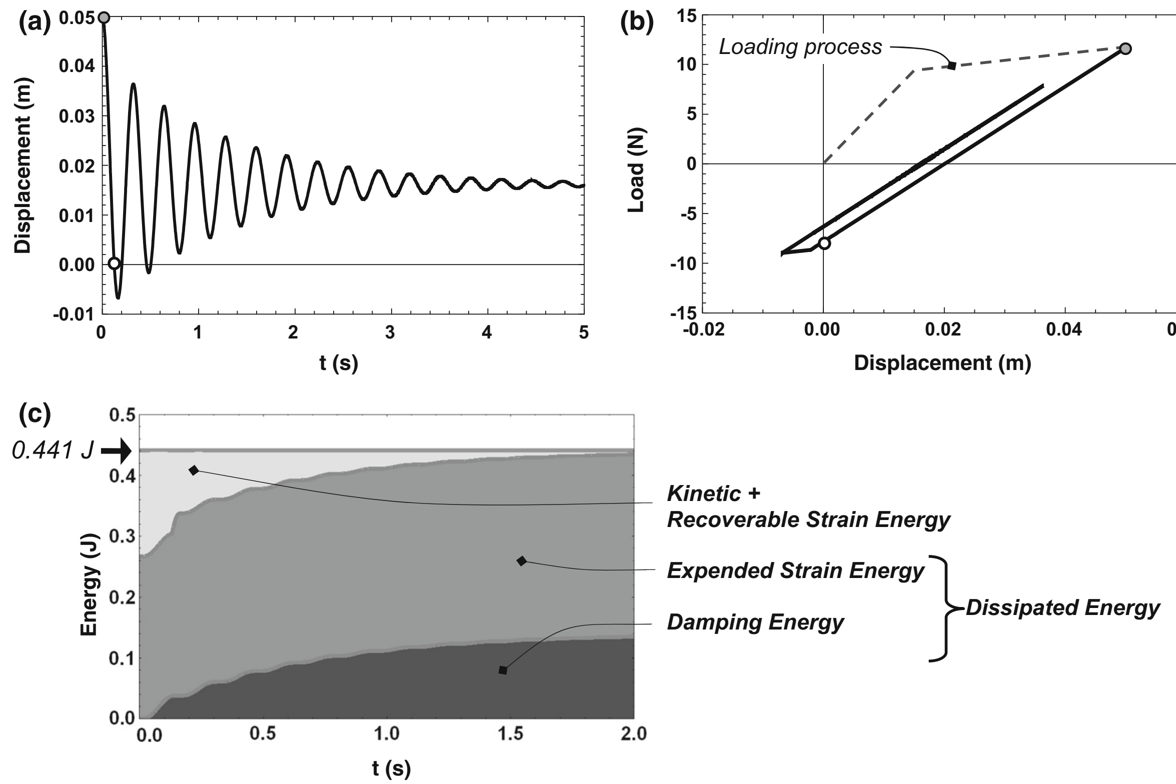


Fig. 9 Example 4: bilinear degrading model with damping

rotation φ, only f contributes in the first case, while P also contributes in the second case. Thus, it has been convenient in the case of linear analysis to consider the compressive load P to cause a reduction in stiffness (from k to $k - P/h$) and therefore a reduction in the period of vibration of the system.

In Eqs. 12 and 13 $f_S(u)$ is the internal force associated with deformation of the spring component. In the first-order approximation (Eq. 12), the spring force $f_S(u)$ coincides with the applied force f. In the second-order approximation, the external force f induces the spring force $f_S(u)$ given by Eq. 13. This is illustrated in Fig. 10, where the external applied force f_i causes a displacement u_i, with spring force f_{Si} greater than f_i.

In many cases, the applied force P represents the self-weight or dead load of the oscillator mass. In a more general formulation, the vertical load P is considered to be composed of dead (D) and live (L) load components. An effective height, h_{eff}, associated with self-weight (dead load) alone can be used to simplify the representation of seismic demands in the presence of P-delta effects [2]. With reference to Fig. 11, h_{eff} can be established as

$$\frac{P}{h}u = \frac{D+L}{h}u = \frac{D}{h_{\text{eff}}}u \Rightarrow h_{\text{eff}} = \frac{D}{D+L}h \tag{14}$$

where W is the reactive weight of the system (equal to D) and V_y is the base shear at yield. Thus, the conventional representation of Fig. 10 can be replaced by the normalized representation of Fig. 11. (Note that the first-order stiffness, k, is based on the actual geometry, and is not influenced by h_{eff}.)

To account for P-delta effects on dynamic response, a new term is introduced into the equation of motion. The response of a viscous damped SDOF system having bilinear hysteretic behavior, considering small deformations, and in the presence of a vertical load P (represented by gravity acting on the mass of the oscillator having height h_{eff}) is given by

$$m\ddot{u}(t) + c\dot{u}(t) + f_S(u(t)) - \frac{mg}{h_{\text{eff}}}u(t) = -m\ddot{u}_g(t) \tag{15}$$

The forces expressed in Eq. (15) can be multiplied by a differential relative displacement du and integrated to obtain the equation of motion in terms of energy:

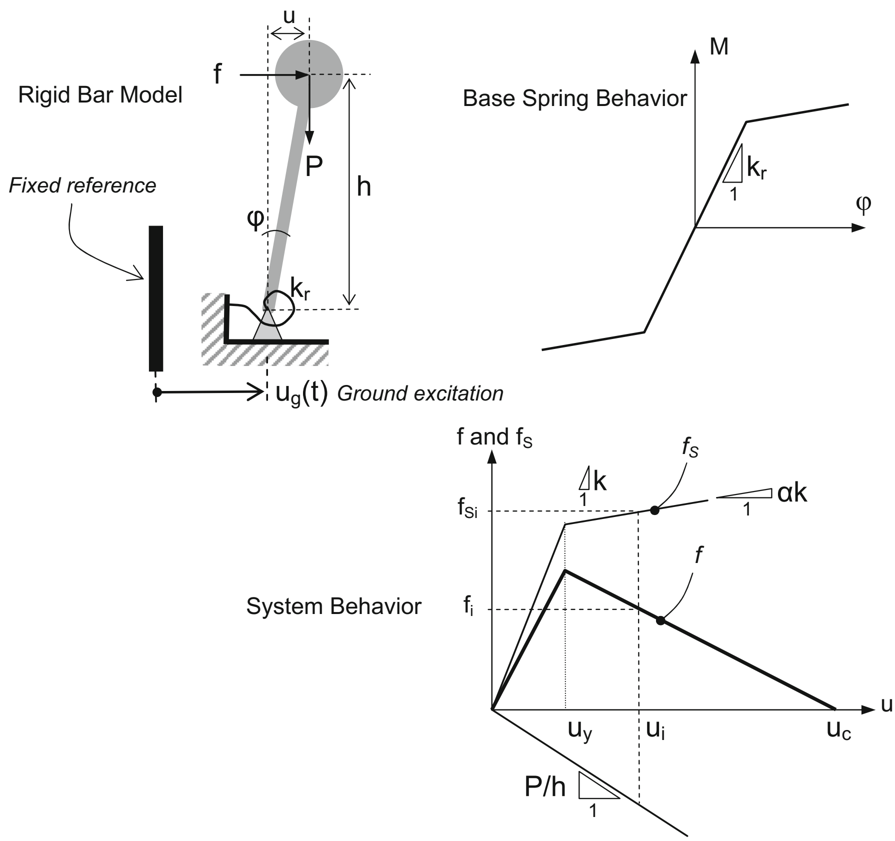


Fig. 10 Model for considering P-delta effects for a single-degree-of-freedom system

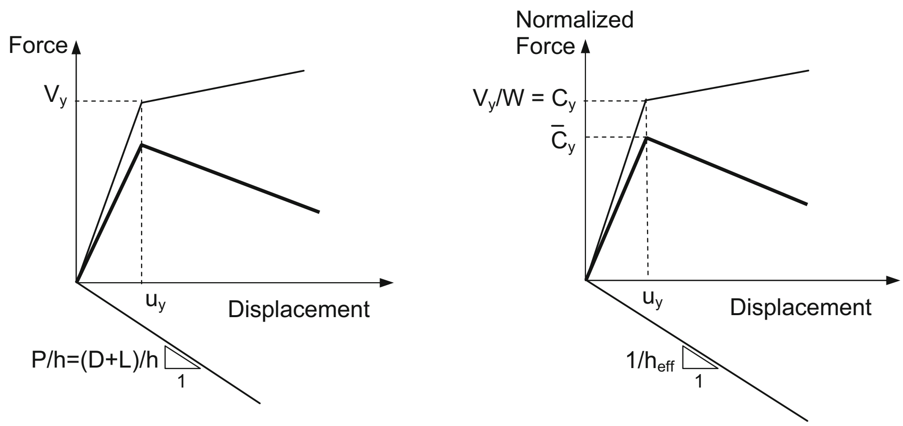


Fig. 11 Use of h_{eff} to define P-delta effect in normalized form

$$\int m\ddot{u}(t)\mathrm{d}u + \int c\dot{u}(t)\mathrm{d}u + \int f_S \mathrm{d}u - \int \frac{mg}{h_{\text{eff}}} u \mathrm{d}u = -\int m\ddot{u}_g(t)\mathrm{d}u$$

$$E_K + E_D + E_S - E_G = E_i \tag{16}$$

The geometric energy term, E_G, introduced above, is defined as:

$$E_G = \int \frac{mg}{h_{\text{eff}}} u \mathrm{d}u = \frac{mg}{2h_{\text{eff}}} u^2 \tag{17}$$

Of course, the term mg/h_{eff} can be replaced by P/h for a more traditional representation. The terms of Eq. (16) can be rearranged to show the geometric energy term (or P-delta effect) as an external loading, acting along with the input energy E_i:

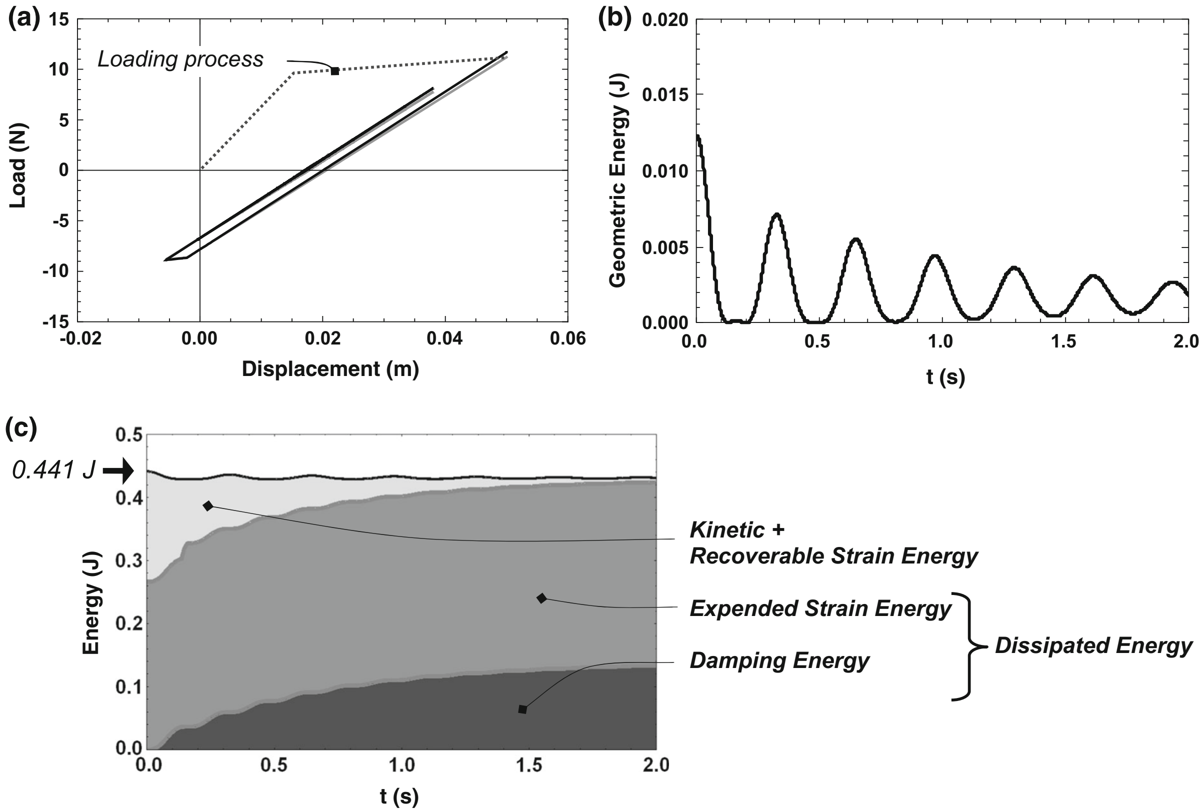


Fig. 12 Example 5: dynamic response of bilinear degrading model with P-delta effects present. Total energy during free vibration snap-back varies due to the vertical force, P, acting through the relative displacement

$$E_K + E_D + E_S = E_i + E_G \tag{18}$$

Example 5 Example 4 is reconsidered with P-delta effects present. Parameter values are unchanged ($m = 1\,\mathrm{kg}$, $\xi = 0.04$, $T = 0.25\,\mathrm{s}$, $u_y = 0.015$, $u_0 = 0.05\,\mathrm{m}$, $a = 0.4$ and $\alpha = 0.10$), with period T referenced to the first-order system, not considering P-delta. The assumed height of the system is 1 m, h = 1 m. The stability coefficient, defined in [3] as $\theta = \mathrm{k_G/k}$, is equal to 0.0155 for this example.

A slight difference in the hysteretic response can be appreciated. In Example 4 (without P-delta effects) a force of 11.7 N was applied. Now, with P-delta effects present, a force of 11.2 N is required to reach $u_0 = 0.05$ m. The hysteretic response in Fig. 12 compares both responses; the gray line is the applied horizontal force, while the black line corresponds to f_S.

It can be appreciated in Fig. 12c how the sum of the E_K, E_S and E_D is affected by the geometric energy. In the presence of P-delta effects, the total energy ($E_K + E_D + E_S$) increases in the presence of P-delta effects as displacements increase (or decrease) relative to the origin.

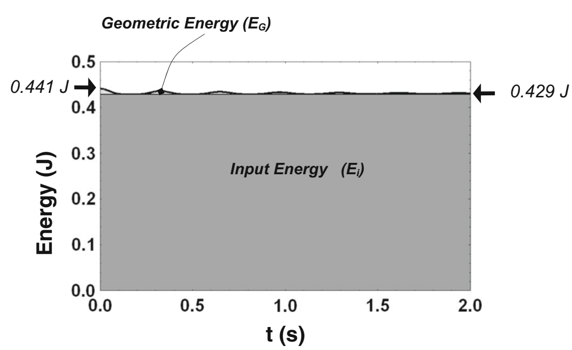


Fig. 13 Example 5: external loading of bilinear degrading model with P-delta effects: input energy and geometric energy

As previously mentioned, P-delta effects traditionally have been viewed as a reduction in resistance, but alternatively can be considered as an increment in the action. Figure 13 plots the right side of Eq. 18,

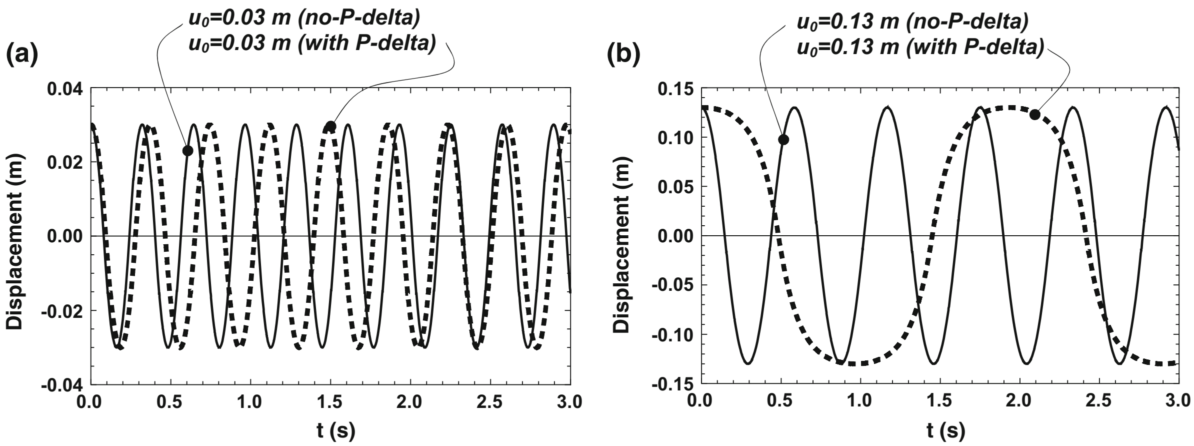


Fig. 14 Example 6: displacement versus time curves

(i.e., $E_i + E_G$). Because the horizontal force needed to reach u_0 is smaller (due to the presence of P-delta effects), the input energy E_i associated with the applied lateral force is smaller ($E_i = 0.429$ J); the additional energy to reach 0.441 J is provided by the P-delta effect. The total input energy is the sum of the input energy E_i and the geometric energy E_G.

Example 6 Example 1 is modified to show P-delta effects for a bilinear elastic system. Because P-delta effects depend on the height of the system, as shown in Fig. 10, a smaller value of h (=0.1 m) is considered in order to amplify this effect. Two different initial displacements are considered: 0.03 m (twice the displacement of 0.015 m associated with the change in stiffness) and 0.13 m (approaching the displacement of 0.136 m associated with collapse—where the applied lateral force is zero). Figure 14 shows the displacement versus time curves for these initial displacements with and without P-delta effects; it is apparent that P-delta effects (dashed lines) cause a large reduction in stiffness and increase the time required for each complete cycle of oscillation For values of u_0 greater than 0.136 m, the displacement increases without limit.

Figure 15 shows the hysteretic curves for the applied force $f(u)$ and for the strain force $f_S(u)$ and the distribution of potential and kinetic energy.

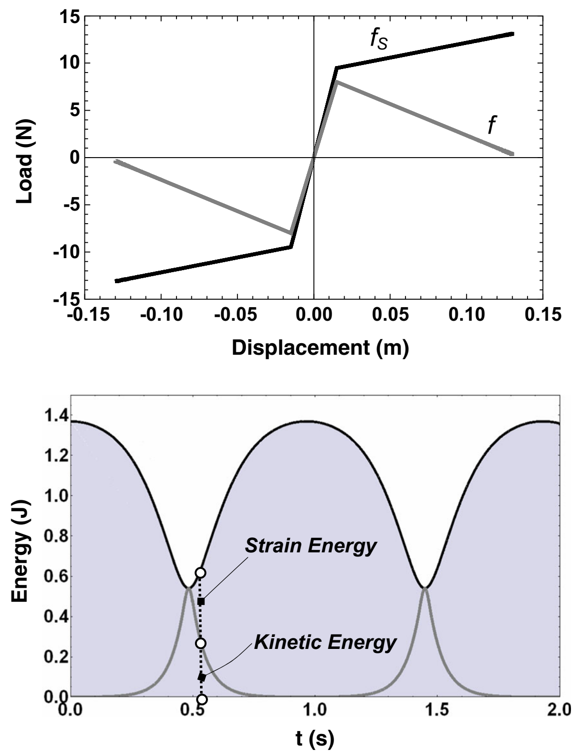


Fig. 15 Example 6: hysteretic curve and energy distribution for $u_0 = 0.13$ m. Total energy varies due to the variable contribution from geometric energy

5 Conclusions

The traditional treatment of dynamic response of SDOF oscillators in terms of energy was reviewed. Extensions were made to address bilinear elastic systems, the evaluation of expended energy in a continuous form, and to define the geometric energy associated with P-delta effects. An instantaneous form of viscous damping energy was used to illustrate interrelationships among these energy terms in several examples. The examples illustrated "snap-back" response, in which the oscillators are released from a quasi-statically imposed initial displacement. The examples illustrate that total energy is a constant in first-order systems, while total energy varies due to increases associated with P-delta effects as displacements deviate from zero.

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