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On the *action* parameter and one-period loops of oscillatory memristive circuits

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Abstract We consider memelements (memristors and memductors) with special periodic responses (mixed-mode oscillations) and 2D one-period loops yielding constant parameters describing the memelements as single units or components of oscillating circuits. One of the parameters is the *action* parameter having the dimensions of *energy × time* and the SI unit *Joule × second*. The remaining loops and parameters correspond to energy of magnetic and electric fields, power and *rms* current and voltage values. Special *mixed* one-period loops are also analyzed with pairs of signals associated with two different components of the circuits. The areas enclosed by various loops result in special equations which can be derived from the underlying ODE model of the circuits. The *action* of a memelement is equivalent to the time integral of the Lagrangian $L(w, w')$, where w is the internal state variable of a memelement. The analysis of memristive circuits and their parameters is considered in the framework of mixed-mode oscillations. Also, the unit of *action* for memelements is proposed to be called *Chua* (=*Joule × second*) to honor L.O. Chua for his work on memristors and memristive circuits.

1 Introduction

There have been quite a few results reported on oscillatory memristor circuits in the literature recently, see [1–11] and reference therein. The pinched hysteresis loop for a single memelement and the area enclosed by a one-period loop have been studied in [5,6,10,11], with the x-controlled memelement described by the following equations

$$y(t) = g(w(t))x(t), \quad w'(t) = x(t), \tag{1}$$

where y and x are the terminal memelement's variables and w is the internal state variable. For example, for a memductor the y, x and w are the current, voltage and flux, respectively. For a memristor the y, x and w are the voltage, current and charge, respectively. Extensions to memcapacitors and meminductors are also possible. For example, for a charge-controlled memcapacitor the y, x and w stand for voltage, charge and time-domain integral of charge, respectively. Interpreting the area of a one-period pinched hysteresis loop has lead various authors to move from the typical power and energy quantities to the memory-content quantities [6,11]. When a memelement is a component of a complex nonlinear circuit, one may expect a much richer spectrum of relations between variables than in a simple situation when a single memelement is considered

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only. Equation (1) involve only three variables (y, w and x) and one pinched hysteresis loop of y versus x. In the case of a memductor the charge variable $q = \int i(t)dt$ does not appear explicitly in (1). Similarly, for a memristor we have (1) in the form $v(t) = g(q(t))i(t)$ with $g(q(t))$ having the meaning of charge-dependent resistance. The flux variable $\phi = \int v(t)dt$ does not appear explicitly in (1) for a memristor. When all four variables v, ϕ, i and q for a memelement are taken into consideration and when the memelement is a component of an oscillating circuit, a number of closed one-period 2D loops can be analyzed, as shown through the numerical examples with figures in Sect. 5 of this paper. Most of such loops will not be pinched. Some of the one-period loops considered in this paper are such that a moving (in time) trajectory point has the enclosed area on one side only (left or right) for the entire time interval $0 \leq t < T$, with T denoting period, while for other loops a moving trajectory point traverses the area between the left and right sides in one period. Moreover, the interpretation of one-period areas enclosed by the loops depend on a particular pair of variables in a loop. Because of an interaction of a memelement with other elements in a circuit, it is also possible to consider closed loops of a *mixed* nature, in which one memelement's variable (e.g., charge) pairs with some other element's variable (e.g., inductor's flux). Examples of closed loops of *mixed* nature are also given in Sect. 5.

The purpose of writing this paper is to examine the phenomena occurring with various one-period loops for a memelement in an oscillatory circuit, including the loops of *mixed* nature. We provide an interpretation of the areas enclosed by such loops. In the process, we propose a new quantity called *action* to assign to memelements. The *action* quantity has the dimension of *energy* × *time*, with the SI unit *Joule* × *second*.

The paper is organized as follows. In Sect. 2 we show examples of various time-series mixed-mode oscillations (MMOs) and one-period loops obtained for the memristive circuits proposed in [7]. The circuits' responses in the form of MMOs are used only for illustrative purposes, as we introduce in Sect. 3 the definitions of six quantities in the general form

$$\int_{\Gamma} f \, dh = \int_0^T f(t)h'(t)dt \qquad (2)$$

for various pairs $(f(t), h(t))$ of periodic responses. Next, in Sect. 4, we provide an interpretation of the

six quantities as energy, power and root-mean-square (*rms*) values of the periodic responses. Special attention is paid to one of the six quantities, the *action* quantity, and its relation to the Lagrangian $L(w, w')$ of the memelement. Section 5 includes further analysis of the one-loop areas, including those of *mixed* nature. In Sect. 6 an extension of the analysis from Sects. 3 and 4 is carried out for memcapacitors and meminductors. Conclusions are given in Sect. 7.

The term memelements is used in Sects. 2–5 for memristors and memductors.

2 Memristive circuits with MMOs

MMOs are periodic responses in many electrical, mechanical, chemical, astronomical or biological systems [12–21]. The MMOs comprise both large- and small-amplitude oscillations, or LAOs and SAOs, respectively, in various periodic sequences denoted by $L_1^{s_1} L_2^{s_2}, \ldots, L_n^{s_n}$, where L_i and s_i are positive integers for $i = 1, 2, \ldots, n$. It is possible that MMOs occur for one set of system's parameters, while chaotic responses appear in the same system for another set of parameters. Also, bifurcations of systems, canards, Farey sequences, Arnold's tongues, fractals and devil's staircases are typical phenomena and properties associated with such systems [7,14,15,18,20,21].

Memristive circuits with interesting dynamical properties and MMOs have been proposed in [7] (see also [17]). A two-dimensional bifurcation diagram for MMOs of type L^s (single values of L and s) and pinched hysteresis loops for both LAOs and SAOs have also been shown in [7]. Several typical MMOs are shown in Figs. 1 (time-series responses) and 2 (pinched hysteresis loops). The pinched hysteresis loops in Fig. 2 are for the three types of MMOs shown on the left side in Fig. 1. Those three graphs denoted as 6^1, 2^3 and 1^1 illustrate the MMOs having periods consisting of 6 LAOs and 1 SAO, 2 LAOs with 3 SAOs, and 1 LAO with 1 SAO, respectively. Also, Fig. 3 shows examples of four one-period loops for MMOs 2^3. The time instants $a, \ldots, v$ in Fig. 3 are also shown explicitly in the time-series graphs of one-period MMO 2^3 responses $x(t)$ and $w(t)$ in Fig. 4. Each period begins at the time instant $t = a$, goes through the instants $b, c, \ldots$, and ends at the instant $t = v$. Placing those time instants in the two-dimensional plots in Fig. 3 clarifies the time-domain motion of the four trajectories.

Fig. 1 A few typical time-series MMOs responses with t and s (*horizontal axes*) denoting the time variable and its unit (s), respectively
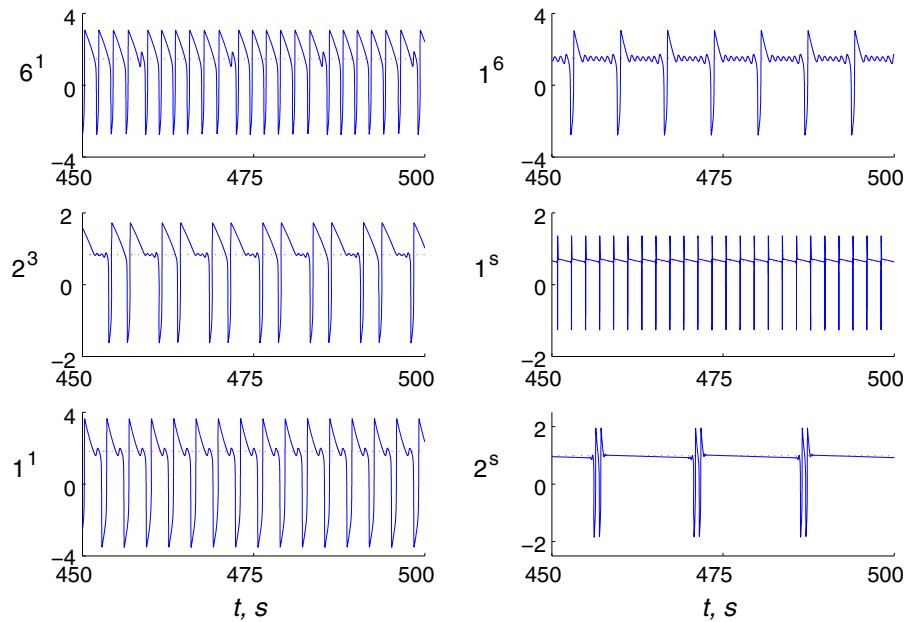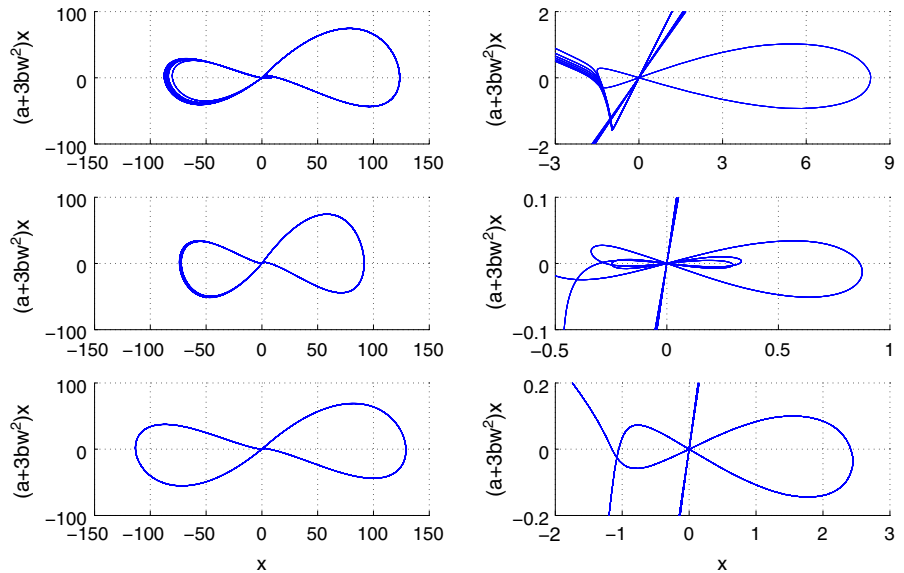

Fig. 2 Pinched hysteresis loops: LAOs (*left panels*), SAOs (*right panels*) for oscillations 6^1 (*top*), 2^3 (*middle*) and 1^1 (*bottom*). These are the y versus x plots as defined in (1) with $g(w) = a + 3bw^2$

All the above responses illustrate interesting dynamics of two dual memristive circuits shown in Fig. 7a, b in the Appendix. Each circuit is described by the following ODE model [7]

$$
\begin{aligned}
\epsilon \overline{x}' &= s_c[-y/\eta - g(w)\overline{x}] \\
y' &= s_c\alpha(\eta\overline{x} - Ky - z) \\
z' &= -s_c\beta y \\
w' &= s_c\eta\overline{x}
\end{aligned}
\tag{3}
$$

where $g(w) = a + 3bw^2$, the prime $'$ denotes the time derivative, $\epsilon = C_1 \ll 1$, $\alpha = 1/L$, $K = R$, $\beta = \gamma/C_2$ for the circuit in Fig. 7a and $\epsilon = L_1 \ll 1$, $\alpha = 1/C$, $K = G$, $\beta = \gamma/L_2$ for the circuit in Fig. 7b. The current-controlled current source and voltage-controlled voltage source in the circuits are described through the expression $(1+\gamma)y$ with $\gamma > 0$. The $g(w) = a + 3bw^2$ with various values of a and b were used in the above simulations. Those a and b

Fig. 3 Four one-period loops of MMOs 2^3 for $g(w) = a + 3bw^2$ and $G(w) = \int g(w)\mathrm{d}w = aw + bw^3$, $G(0) = 0$
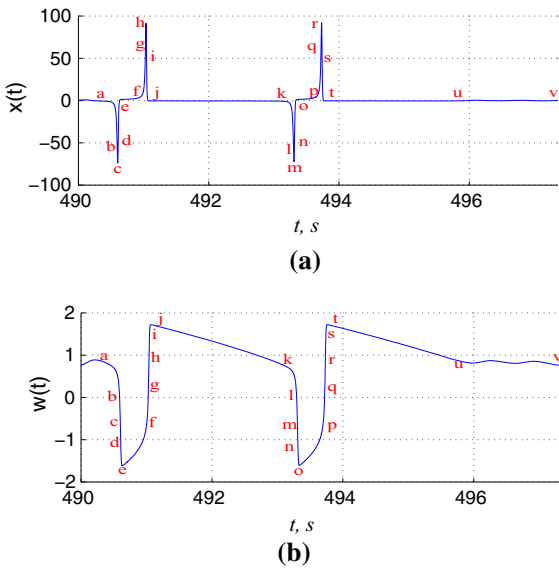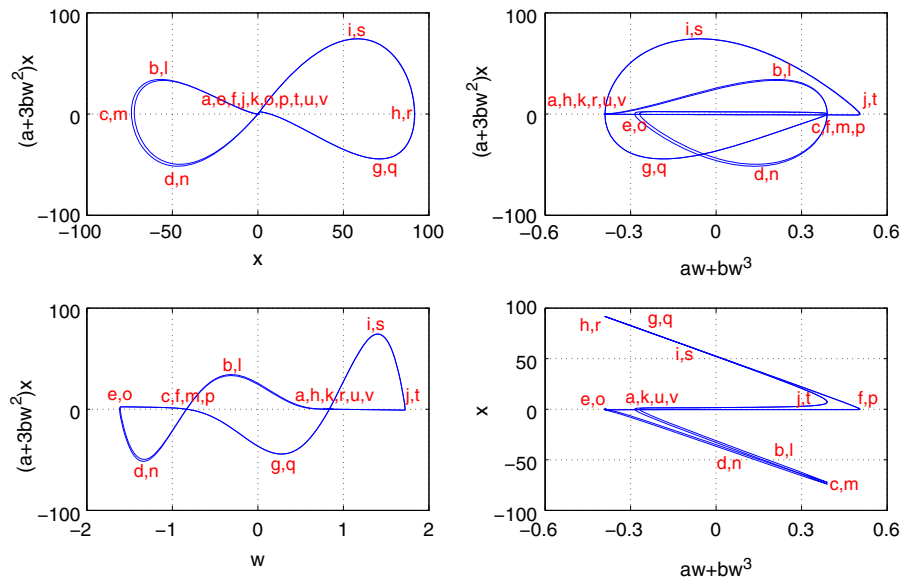

Fig. 4 One-period MMOs 2^3 with consecutive time instants $a, b, \ldots, u, v$. The SAOs occur between u and v. **a** One period of $x(t)$ for MMOs 2^3. **b** One period of $w(t)$ for MMOs 2^3

values in $g(w)$ should not be confused with the time instants a and b in the series of time instants $a, b, \ldots, v$ used in Figs. 3 and 4. The scaling factor $\eta > 1$ was chosen to reduce the variable x and its derivative (important in circuit simulation in SPICE [22]), since $\bar{x} = x/\eta$, with x being the memductor's voltage in the circuit in Fig. 7a and memristor's current in the circuit in Fig. 7b. The $s_c > 0$ is a time-scaling coefficient. The *small* val-

ues of C_1 and L_1 in Fig. 7a, b make both circuits singularly perturbed ones. In some examples in this paper we also consider (3) with the second equation replaced by $y' = s_c \alpha(\eta \bar{x} - Ky - z \pm a_s)$, that is, a small biasing constant source a_s of order ϵ is used. When $a_s \neq 0$ we used zero initial conditions in our analysis. Otherwise, with $a_s = 0$, the following nonzero initial conditions were used $[x(0), y(0), z(0), w(0)] = [2.22, 1.105, -0.00628, 0.375]$. The current through the memelement in Fig. 7a equals $g(w)x$, where w and x are the flux and voltage, respectively. For the dual circuit in Fig. 7b the voltage equals $g(w)x$ with w and x being the memristor's charge and current. The particular choice of $g(w(t)) = a + 3bw^2$ in (1) was for the purpose of illustration only (creation of all the above figures). A general integrable function $g(w)$ is considered in the next sections of the paper.

The MATLAB's procedure *ode15s* with *abserr* = *relerr* = 10^{-10} and various parameters in (3) were used to obtain the solutions shown in Figs. 1, 2, 3 and 4 for MMOs L^s:

6^1: $\alpha = 1$, $\beta = 0.1$, $a = -0.5$, $b = 0.078$,
  $K = 0$, $a_s = 0$
2^3: $\alpha = 1$, $\beta = 0.1$, $a = -0.7$, $b = 0.335$,
  $K = 0$, $a_s = 0$
1^1: $\alpha = 1.55$, $\beta = 0.15$, $a = -0.5$, $b = 0.05$,
  $K = 2$, $a_s = 0.04$

1^6: $\alpha = 1.55$, $\beta = 0.15$, $a = -0.34$, $b = 0.05$,
   $K = 2$, $a_s = 0.04$
1^s: $\alpha = 1$, $\beta = 0.15$, $a = -0.64$, $b = 0.5$,
   $K = 0$, $a_s = 0.04$
2^s: $\alpha = 1$, $\beta = 0.15$, $a = -0.44$, $b = 0.15$,
   $K = 2.3$, $a_s = 0.04$

Parameter $\eta = 1$ in all six cases considered above, while $s_c = 1$ in the first four cases, $s_c = 26$ in the 1^s case and $s_c = 2$ for the 2^s case above. The s in 1^s and 2^s indicates very large numbers of SAOs, all with very small amplitudes.

3 One-period loops

The pinched hysteresis loops shown in Fig. 2 (and in the left upper corner in Fig. 3), i.e., the graphs of the memelement's current *versus* voltage, are not the only loops one may consider for MMOs. The current versus charge loop for the circuit in Fig. 7a (or voltage vs. flux for the circuit in Fig. 7b) shown in the upper right corner in Fig. 3 is another example. Also, the current versus flux loop (circuit in Fig. 7a) or voltage versus charge (circuit in Fig. 7b) shown in the lower left corner in Fig. 3 can also be considered. A fourth example of a one-period loop is shown in the right lower corner in Fig. 3 (x vs. $\int g(w)\mathrm{d}w$ with $g(w) = a + 3bw^2$). The four loops in Fig. 3 are examples of one-period loops with pairs of variables associated with one memelement only. However, when a memristive circuit is analyzed, we can also construct one-period loops linking variables of two different elements, for example, the memelement's current versus the current of inductor L in Fig. 7a or the memelement's current versus the voltage of capacitor C_2. Equivalently, one may consider a loop of the memelement's voltage versus the voltage of capacitor C in Fig. 7b, or a loop of the memelement's voltage versus the current of inductor L_2. Several other one-period loops are also possible.

Two natural questions arise now: What do the surface areas enclosed by various one-period loops represent and how are the enclosed areas related to each other?

The literature on memelements provides an analysis of the pinched hysteresis loop and an interpretation of the enclosed area only in the case of a single memelement driven by periodic sinusoidal signal $x(t)$ in (1), either voltage or current [6,10,11]. The analysis

of various types of closed loops for complex circuits with memelements is different not only because of the nonsinusoidal shapes of the involved signals, but also because other components (parameters) of the circuits must be taken into account.

Let $g(w)$ in (1) be an integrable function with the integral $G(w) = \int g(w)\mathrm{d}w$. Consider the following quantities (with their units) representing the above-mentioned areas of one-period loops of (the SI units for the memristor circuit in Fig. 7b) are given in parentheses if they differ from the units for the memductor circuit in Fig. 7a:

$$\mathcal{P} \equiv \int_{\Gamma_1} g(w)x\mathrm{d}x, \quad [\mathcal{P}] = V \times A$$

$$\mathcal{E}_{C(L)} \equiv \int_{\Gamma_2} x\mathrm{d}(G(w)), \quad [\mathcal{E}_{C(L)}] = V \times A \times s$$

$$\mathcal{E}_{L(C)} \equiv \int_{\Gamma_3} g(w)x\mathrm{d}w, \quad [\mathcal{E}_{L(C)}] = V \times A \times s$$

$$\mathcal{TE}_M \equiv \int_{\Gamma_4} G(w)\mathrm{d}w, \quad [\mathcal{TE}_M] = V \times A \times s^2$$

$$\mathcal{D}_{I(V)} \equiv \int_{\Gamma_5} g(w)x\mathrm{d}(G(w)), \quad [\mathcal{D}_I] = A^2 \times s,$$
$$[\mathcal{D}_V] = V^2 \times s$$

$$\mathcal{D}_{V(I)} \equiv \int_{\Gamma_6} x\mathrm{d}w, \quad [\mathcal{D}_V] = V^2 \times s,$$
$$[\mathcal{D}_I] = A^2 \times s$$

and examples of closed loops for pairs of *mixed* variables

$$\mathcal{W} \equiv \int_{\Gamma_7} z\mathrm{d}(G(w)), \quad [\mathcal{W}] = V \times A \times s$$

$$\mathcal{Y} \equiv \int_{\Gamma_8} g(w)x\mathrm{d}y, \, [\mathcal{Y}]$$
$$= A^2 \text{ or } V^2 \text{ (depends on memelement)}$$

$$\mathcal{Z} \equiv \int_{\Gamma_9} g(w)x\mathrm{d}z, \quad [\mathcal{Z}] = V \times A.$$

The notation $\mathcal{E}_{C(L)}$ means that we use $\mathcal{E}_C$ for the circuit in Fig. 7a and $\mathcal{E}_L$ for the circuit in Fig. 7b. On the other hand, the notation $\mathcal{E}_{L(C)}$ means that we use $\mathcal{E}_L$ for the circuit in Fig. 7a and $\mathcal{E}_C$ for the circuit in Fig. 7b. The quantities $\mathcal{E}_C$ and $\mathcal{E}_L$ have the meaning of one-period energy with $\mathcal{E}_C$ being the energy due to an electric field and $\mathcal{E}_L$ is the energy due to a magnetic field. Also, the $\mathcal{D}_{I(V)}$ indicates that this quantity is linked to current of the memductor in Fig. 7a and to voltage of the memristor in Fig. 7b. The reverse holds
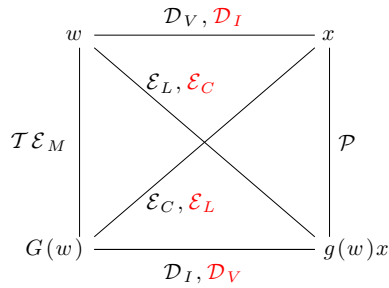
Fig. 5 Six quantities $\mathcal{P}, \mathcal{TE}_M, \mathcal{E}_C, \mathcal{E}_L, \mathcal{D}_I$ and $\mathcal{D}_V$ for the circuits in Fig. 7a, b

for $\mathcal{D}_{V(I)}$. More details about the physical interpretation of the above quantities are given is Sect. 4.

Notice that the first six quantities defined above involve the four variables (signals) x, w, $g(w)x$ and $G(w)$ whose interpretation depends on the memelement under consideration. For example, for a memductance we have the following: $x =$ voltage, $w =$ flux, $g(w)x =$ current and $G(w) =$ charge and for a memristance: $x =$ current, $w =$ charge, $g(w)x =$ voltage and $G(w) =$ flux. Similar assignments can be done for memcapacitors and meminductors, see Sect. 6.

A taxonomy of the six quantities for memelements is shown in Fig. 5. Similar diagrams can be created for charge- and voltage-controlled memcapacitors and flux- and current-controlled meminductors (see Sect. 6). As shown below, the $\mathcal{D}_I$ and $\mathcal{D}_V$ quantities in Fig. 5 are equivalent to the period of MMOs multiplied by the squared values of memductor's and memristor's *rms* current and voltage, respectively.

Each of the remaining three quantities ($\mathcal{Y}, \mathcal{Z}$ and $\mathcal{W}$) involves pairs of variables of two different elements. For example, the $\mathcal{Y}$ defines an area of a loop with the memelement's variable $g(w)x$ (current in Fig. 7a and voltage in Fig. 7b) and the variable y, which is the inductor's current in Fig. 7a and capacitor's voltage in Fig. 7b. Other cases of one-period loops with *mixed* variables are defined in a similar way.

Interestingly, some of the loops are such that a moving (in time) trajectory point always has the enclosed area on the right side (e.g., trajectories for quantities $\mathcal{D}_I$ and $\mathcal{D}_V$), left side (e.g., trajectory for quantity $\mathcal{P}$), while for other loops the orientation is changed a number of times in one period (e.g., the trajectory for quantities $\mathcal{E}_C$ and $\mathcal{E}_L$). This statement is true when we ignore SAOs and consider LAOs only as shown in Fig. 3 with the consecutive time instants marked by lowercase letters $a, \ldots, v$ covering one period of MMOs 2^3. The

big hysteresis loop ($g(w)x$ vs. x) occurs in very short time intervals: a–e and k–o for the loop with negative $x(t)$ and the intervals f–j and p–t for the loop with positive $x(t)$—compare the graph $g(w)x$ versus x in Fig. 3 with the graphs in Fig. 4. Ignoring SAOs is justified by a simple fact that the areas enclosed by SAOs are a tiny fraction (negligible) compared with the areas enclosed by LAOs and therefore can be neglected. An obvious consequence of ignoring SAOs in the pinched hysteresis graph is that the crossing of the point $(0, 0)$ is of type II, as described in [10], rather than type I as it would be if SAOs were not ignored. In Sects. 4 and 5 we provide an interpretation of various one-period loops defined above and show relationships between the loops. Without losing generality we assume that $s_c = 1$ in (3).

4 Interpretation of the quantities $\mathcal{P}, \mathcal{E}_{C(L)}, \mathcal{E}_{C(L)}, \mathcal{D}_{I(V)}, \mathcal{D}_{V(I)}$ and $\mathcal{TE}_M$

Notice that

$$\mathcal{D}_{I(V)} \equiv \int_{\Gamma_5} g(w)x\, d(G(w)) = \int_0^T g(w)x \frac{d(G(w))}{dw} w'\, dt. \tag{4}$$

Since $w' = x$, and $d(G(w))/dw = g(w)$, therefore

$$\mathcal{D}_{I(V)} = \int_0^T (g(w)x)^2\, dt = T \left(\{g(w)x\}_{\text{rms}} \right)^2. \tag{5}$$

Thus, the area enclosed by $\mathcal{D}_{I(V)}$ is equal to the period T multiplied by the square of the *rms* value of $g(w)x$, the current for a memductor, voltage for a memristor and, as discussed in Sect. 6, the respective *rms* values of charge and voltage for memcapacitors, as well as the *rms* values of flux and current for meminductors. Also, it is obvious, that since $w' = x$, we have $\mathcal{D}_{V(I)} = \int_0^T xw'\, dt = T(\{x\}_{\text{rms}})^2$.

Next, we have $\mathcal{E}_{C(L)} \equiv \int_{\Gamma_2} x\, d(G(w)) = \int_0^T x \frac{d(G(w))}{dw} w'\, dt = \int_0^T x^2 g(w)\, dt$, which has the meaning of one-period energy. Similarly, $\mathcal{E}_{L(C)} \equiv \int_{\Gamma_3} g(w)x\, dw = \int_0^T g(w)x^2\, dt$ with the same meaning.

For the pinched hysteresis loop and quantity $\mathcal{P} \equiv \int_{\Gamma_1} g(w)x\, dx$ the interpretation is reported in [11] in terms of the *memory effect* and the concept of *content*. However, in the framework of our analysis, the quantity $\mathcal{P}$ has the meaning of power with its unit $V \times A$.

We now focus on the remaining quantity, namely $\mathcal{TE}_M \equiv \int_{\Gamma_4} G(w)\mathrm{d}w$. Notice that if we consider one full period $0 \le t < T$, then $\mathcal{TE}_M = 0$. This follows from the fact that for $G(w) = \int g(w)\mathrm{d}w$ and periodic $w(t)$ we have $w(0) = w(T) = w^*$ for some value of w^*. This yields $TE_M \equiv \int_{w^*}^{w^*} G(w)\mathrm{d}w = 0$. The quantity $\mathcal{TE}_M \equiv \int_{\Gamma_4} G(w)\mathrm{d}w = \int_{w_0}^{w_1} G(w)\mathrm{d}w$, $w_0 = w(t_0)$, $w_1(t_1)$, defined over time interval $t_0 \le t \le t_1$ does not represent any area enclosed by a loop, but $\mathcal{TE}_M$ is a very interesting quantity of memelements as shown below.

Action $= \mathcal{TE}_M$: Let $\mathcal{TE}_M = \int_{t_0}^{t_1} G(w)w'\mathrm{d}t$ and denote $L(w, w') = G(w)w'$. It is easy to check that with $G(w) = \int g(w)\mathrm{d}w$, the $L(w, w')$ satisfies the Euler–Lagrange equation

$$\frac{\partial L}{\partial w} - \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial w'}\right) = 0, \qquad (6)$$

indicating that $G(w)w'$ is a Lagrangian and since the time integral of Lagrangian is called *action*[1] with dimensions *energy* × *time*; therefore, we propose to assign the quantity *action* ($= \int G(w)\mathrm{d}w$) to any memductor or memristor. The unit of *action* is *Joule* × *second*. Notice, that $\mathcal{TE}_M = \int_\Gamma (charge) \cdot \mathrm{d}(flux)$ for a memductor, while $\mathcal{TE}_M = \int_\Gamma (flux) \cdot \mathrm{d}(charge)$ for a memristor. The term *action* fits well with the dynamical properties of a memelement, being a circuit component whose present state depends on the history of trajectory motion (memory). The number $\mathcal{TE}_M = \int_{\Gamma_4} G(w)\mathrm{d}w$ takes into account the amount of charge $G(w)$ passed in the interval $t_0 \le t < t_1$ for a memductor or the amount of flux $G(w)$ through a memristor for $t_0 \le t < t_1$. There is also a link of $\mathcal{TE}_M$ with quantum mechanics since the Planck's constant has also the same unit (*Joule* × *second*) as $\mathcal{TE}_M$ does [24]. The Planck's constant is used in the relationship between energy and frequency of an electromagnetic wave, known as the Planck–Einstein equation $E = h\nu$ (or $h = E/\nu = TE$), where E is the energy of the charged atomic oscillator, ν is the frequency of an associated electromagnetic wave and h is the Planck's constant.

The unit of action The flux and charge quantities are the two variables defining memductors and memristors, whose existence was predicted in 1971 by professor Chua [1,2], leading to a successful construction of a memelement by the HP team [4]. The unit of the quantity $\mathcal{TE}_M$ is of course the same as the unit of the product flux×charge, that is $V \times A \times s^2 = Wb \times C = J \times s$ (with V, A, Wb, C and J chosen to honor Volta, Ampère, Weber, Coulomb and Joule, respectively). Since the unit $V \times s \times A \times s$ links the magnetic flux with electric charge, which is the essence of the memelements' existence, we propose that the unit of flux × charge is called *Chua*, to honor L.O. Chua, the pioneer of memelements. That is, let $1\ Chua = 1\ V \times A \times s^2 = 1\ Wb \times C = 1\ J \times s = 1\ \mathcal{T} \times A \times m^2 \times s^2 = 10^8\ Mx \times A \times s$ with $\mathcal{T}$ and Mx standing for Tesla and Maxwell, respectively.

We discuss further the concept of *action* for memcapacitors and meminductors in Sect. 6.

5 Further relations between areas of various one-period loops

Based on the ODE model (3) with $s_c = 1$ and various one-period loops (including those of *mixed* nature) we have the following.

Theorem 1 $\epsilon\mathcal{P} = \mathcal{Z}/\beta - \mathcal{D}_{I(V)}$.

Proof Substituting $x' = -[y + g(w)x]/\epsilon$ from (3) into $\mathcal{P} = \int_0^T g(w)x'\mathrm{d}t$ we obtain

$$\mathcal{P} = -\frac{1}{\epsilon}\int_0^T g(w)x[y + g(w)x]\mathrm{d}t, \qquad (7)$$

which can be, with the help of $y = -z'/\beta$ (see (3)), further changed to

$$\mathcal{P} = \frac{1}{\epsilon}\left[\frac{1}{\beta}\int_0^T g(w)xz'\mathrm{d}t - \int_0^T (g(w))^2 x^2\mathrm{d}t\right]. \qquad (8)$$

From (8) we obtain
$$\epsilon\mathcal{P} = \frac{1}{\beta}\int_0^T g(w)x\mathrm{d}z - \int_0^T g(w)x\mathrm{d}(G(w))$$
$$= \mathcal{Z}/\beta - \mathcal{D}_{I(V)}. \qquad (9)$$

This completes the proof. $\qquad\square$

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[1] "In physics, action is an attribute of the dynamics of a physical system. It is a mathematical functional which takes the trajectory, also called path or history, of the system as its argument and has a real number as its result. Generally, the action takes different values for different paths. Action has the dimensions of (energy) × (time), and its SI unit is joule-second. This is the same unit as that of angular momentum" [23].

A note of caution: Theorem 1 should be understood with all the quantities representing the enclosed areas bearing the correct signs, either positive or negative. In this paper the following rule is applied: If a trajectory point moves clockwise around the enclosed area, then the integral is assumed to be positive. On the other hand, if a trajectory point is moving counterclockwise around the encircled area, then the integral is negative [10]. This is important in the case of numerical computation of the enclosed areas by using, for example, the procedure *polyarea* in MATLAB, which uses the opposite rule.

In the MMOs 2^3 example with $\epsilon = 0.01$ and $\beta = 0.1$, we obtain by using the MATLAB *polyarea* function: $\mathcal{P} = -22281.1457$, $\mathcal{D}_{I(V)} = 222.7570$, $\mathcal{Z} = -0.0054$ and $|\epsilon\mathcal{P} - \mathcal{Z}/\beta + \mathcal{D}_{I(V)}| = 1.9192 \times 10^{-7}$ for the numerically identified period $T = 7.3518$. On the other hand, for the MMOs 6^1 we obtain: $\mathcal{P} = -78691.0617$, $\mathcal{D}_{I(V)} = 786.7352$ and $\mathcal{Z} = -0.0175$ with $T = 13.2690$ to yield $|\epsilon\mathcal{P} - \mathcal{Z}/\beta + \mathcal{D}_{I(V)}| = 4.14670 \times 10^{-7}$.

Theorem 2 $\mathcal{Y} = \alpha(\mathcal{E}_{L(C)} + K\mathcal{Z}/\beta - \mathcal{W})$.

Proof Substituting $y' = \alpha(x - Ky - z)$ (see (3)) into $\mathcal{Y} = \int_0^T g(w)xy'dt$, making use of $x = w'$ and $y = -z'/\beta$, yield $\mathcal{Y} = \alpha\int_0^T g(w)x(x - Ky - z)dt = \alpha\int_0^T g(w)xw'dt + (\alpha K/\beta)\int_0^T g(w)xz'dt - \alpha\int_0^T g(w)xzdt$. Since $g(w)x = d(G(w))/dt$, and therefore, the last expression is equivalent to $\mathcal{Y} = \alpha\mathcal{E}_{L(C)} +$

$\alpha K\mathcal{Z}/\beta - \alpha\int_0^T zd(G(w))$, which further yields $\mathcal{Y} = \alpha(\mathcal{E}_{L(C)} + K\mathcal{Z}/\beta - \mathcal{W})$. This ends the proof. □

Using the *polyarea* function from MATLAB for the MMOs 2^3 with $\alpha = 1$, $K = 0$ and $\beta = 0.1$, we obtain $\mathcal{Y} = 0.00181133$, $\mathcal{E}_{L(C)} = 0.00181305$, $\mathcal{Z} = -0.00544598$ and $\mathcal{W} = 8.6797 \times 10^{-8}$ to yield $|\mathcal{Y} - \alpha(\mathcal{E}_{L(C)} + K\mathcal{Z}/\beta - \mathcal{W})| = 1.6325 \times 10^{-6}$. In the case of MMOs 6^1 we obtain $\mathcal{Y} = 0.00808594$, $\mathcal{E}_{L(C)} = 0.00809422$, $\mathcal{Z} = -0.01753621$ and $\mathcal{W} = 7.6720 \times 10^{-7}$ to yield $|\mathcal{Y} - \alpha(\mathcal{E}_{L(C)} + K\mathcal{Z}/\beta - \mathcal{W})| = 7.5149 \times 10^{-6}$.

6 Memcapacitors and meminductors

It is possible to consider the *action* quantity for memcapacitors and meminductors, too. Variables x, y and w for these four memelements: voltage-controlled memcapacitor (VCMC), charge-controlled memcapacitor (QCMC), flux-controlled meminductor (FCML) and current-controlled meminductor (CCML) are given in Table 1 with the *TIQ* and *TIF* denoting the time-domain integrals of charge and flux, respectively. The $g(w)$ and $G(w)$ variables are also included for completeness. The i, v, q and ϕ denote, as usual, the current, voltage, charge and flux, respectively. Also, $MC =$ memcapacitance, $ML =$ meminductance.

The six quantities defined in Sect. 3 for memductors and memristors in terms of variables x, w, $g(w)x$ and $G(w)$, see Fig. 5, can also be computed for memcapacitors and meminductors as shown in Table 2. The meaning of the six quantities is now different than the meaning of the same quantities for memductors and memristors with the *action* parameters marked boldface in Table 2 and illustrated for a QCMC and a FCML in Fig. 6. The quantities in the last two columns in Table 2 are equivalent to the product of period T and square *rms* values of voltage and charge for memcapacitors and the product of period T and square *rms* values
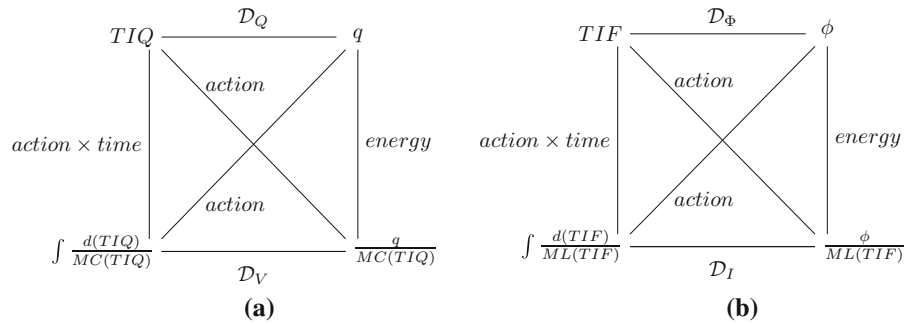
Table 1 Memcapacitors and meminductors

	y	x	w	$g(w)$	$G(w)$
QCMC	v	q	TIQ	$(MC(w))^{-1}$	$\int(MC(w))^{-1}dw$
VCMC	q	v	ϕ	$MC(w)$	$\int MC(w)dw$
FCML	i	ϕ	TIF	$(ML(w))^{-1}$	$\int(ML(w))^{-1}dw$
CCML	ϕ	i	q	$ML(w)$	$\int ML(w)dw$

Table 2 The six parameters for memcapacitors and meminductors

	$\int_{\Gamma_1} g(w)xdx$	$\int_{\Gamma_2} xd(G(w))$	$\int_{\Gamma_3} g(w)xdw$	$\int_{\Gamma_4} G(w)dw$	$\int_{\Gamma_5} g(w)xd(G(w))$	$\int_{\Gamma_6} xdw$
QCMC	Energy	**Action**	**Action**	Action × time	$T\{v\}_{rms}^2$	$T\{q\}_{rms}^2$
VCMC	Energy	**Action**	**Action**	Action × time	$T\{q\}_{rms}^2$	$T\{v\}_{rms}^2$
FCML	Energy	**Action**	**Action**	Action × time	$T\{i\}_{rms}^2$	$T\{\phi\}_{rms}^2$
CCML	Energy	**Action**	**Action**	Action × time	$T\{\phi\}_{rms}^2$	$T\{i\}_{rms}^2$

Fig. 6 One-period quantities of QCMC and FCML. **a** One-period quantities of a QCMC, **b** one-period quantities of a FCML

of current and flux for meminductors. The previously discussed quantity *action* ($\int_{\Gamma_4} G(w)dw$) for memductors and memristors in Sect. 4 has now the meaning of *action × time* for memcapacitors and meminductors. Figure 6a, b show all the quantities for the QCMC and FCML only. Similar diagrams can be constructed for VCMC and CCML.

7 Conclusions

The well-known pinched hysteresis loops for oscillating memristive circuits have been considered in a broader spectrum of other one-period loops representing various quantities of memelements as well as the loops with pairs of *mixed* variables. The areas enclosed by various one-period loops for memductors and memristors are related to power, energy and *rms* values of periodic responses and other equations resulting from mathematical models of the circuits, as shown in Theorems 1 and 2. For a single memelement there exists six quantities relating the four variables x, w, $g(w)x$ and $G(w)$. One of the six quantities, $\mathcal{TE}_M$, has the meaning of *action*, the quantity that is well known in physics, but rather unknown (or forgotten) in the circuit theory. Finally, the author strongly believes that the proposal made in this paper of honoring professor Chua and assigning the unit *Chua* to the *action* quantity is justified and will be accepted.

8 Appendix

Equation (3) describes the memristive circuits with MMOs shown in Fig. 7. For bifurcation diagrams, more
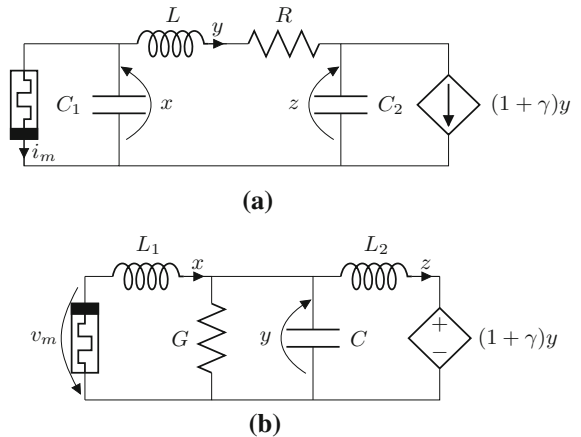

Fig. 7 Two dual memristive circuits described by (3) with $x = \eta\bar{x}$. **a** The M+CLRC circuit with memductance $g(w)$ for $w = \phi$, **b** The M+LGCL circuit with memristance $g(w)$ for $w = q$

time-domain responses and pinched hysteresis loops see [7].

Finally, suppose that we use $[x(0), y(0), z(0), w(0)]$ as the nonzero initial conditions of (3). It is possible to transform system (3) into a singularly perturbed scalar ODE in variable $w(t)$ and use it in a SPICE [22] realization of the circuits in Fig. 7a, b. The scalar equation is (the modified version of the second equation in (3) with a_s is used as discussed in Sect. 2):

$$\epsilon w'''' + \{s_c(a + 3bw^2) - \epsilon s_c \alpha K\}w'''$$
$$+ \{s_c^2\alpha(1 - K(a + 3bw^2)) - \epsilon s_c^2\alpha\beta$$
$$+ 18s_c bww'\}w'' - \Big\{s_c^3\alpha\beta(a + 3bw^2) \qquad (10)$$
$$- 6s_c bw'^2 + 6s_c^2\alpha bKww'\Big\}w' = 0.$$

In the process of deriving (10) we also obtain the following initial conditions in addition to $w(0)$

$$w'(0) = s_c\gamma x(0)$$
$$w''(0) = -s_c^2[y(0) + \gamma(a + 3bw^2(0))x(0)]/\epsilon$$

$$w'''(0) = \left\{ \left[-s_c^3 \alpha \gamma - s_c(a + 3bw^2(0)) \right. \right.$$
$$\left. - 6s_c^3 \gamma^2 bw(0)x(0) \right] x(0)$$
$$+ (s_c^3 \alpha K + s_c^3(a + 3bw^2(0)))y(0)/\epsilon$$
$$\left. + s_c^3 \alpha z(0) \mp s_c^3 \alpha a_s \right\} /\epsilon. \qquad (11)$$

A similar approach was successfully used in [17,21] for a third-order *jerk* equation yielding MMOs in a circuit with a nonlinear element of third-degree polynomial current–voltage characteristic. Moreover, it was shown that some of the singularly perturbed *jerk* equations were Newtonian, since they were obtained after differentiation of the Newton's second law of the type $w'' = F(t, w, w')/m$ with m being constant. In such a situation, the w'' has the meaning of *acceleration*, $F(t, w, w')$ is a nonlinear *force* (with a memory term) and w''', being derivative of w'' has the meaning of *jerk*. If the same analysis can be extended to (10), then w'''' will have the meaning of *jounce*, the second derivative of acceleration w''. Such an approach will link the memristive MMOs circuits with their equivalent mechanical realizations.

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