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An adaptive fast terminal sliding mode control combined with global sliding mode scheme for tracking control of uncertain nonlinear third-order systems

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Abstract In this paper, an adaptive fast terminal sliding mode control technique combined with a global sliding mode control scheme is investigated for the tracking problem of uncertain nonlinear third-order systems. The proposed robust tracking controller is formulated based on the Lyapunov stability theory and guarantees the existence of the sliding mode around the sliding surface in a finite time. Under the uncertainty and nonlinearity effects, the reaching phase is removed and the chattering phenomenon is eliminated. This scheme guarantees robustness against nonlinear functions, parameter uncertainties and external disturbances. The derivative of the state variable is replaced by a delay term in the form of an Euler approximation of the derivative function. Furthermore, the knowledge of upper bounds of the system uncertainties is not required, which is more flexible in the real implementations. Simulation results are presented to show the effectiveness of the suggested method.

Keywords Fast terminal sliding mode · Adaptive tuning · Global sliding surface · Third-order nonlinear system · Tracking · Robustness

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1 Introduction

1.1 Background and motivations

A rigorous foundation for the theory of finite time stabilization was primarily presented by Bhat and Bernstein [1]. The stabilization and tracking control of nonlinear and time-varying systems have important applications in electronics, mechanics and robotic systems [2–6]. Conventional feedback control methods do not obtain robustness and high performance when facing with the nonlinearities, uncertainties and external disturbances [7–10]. Sliding mode control (SMC) as an effective robust control technique that has been successfully applied to control or track certain linear and nonlinear systems such as robotic manipulators [11], nonholonomic systems [12], aircraft [13], underwater vehicles [14], spacecraft [15], flexible space structures [16], chaotic systems [17], electrical motors [18] and power systems [19]. The significant features of SMC are the fast response, robustness against uncertainties, insensitivity to the bounded disturbances, good transient performance and computational easiness with respect to other robust control methods [20–22]. The procedure of SMC design can be divided into two phases, namely, the sliding phase and the reaching phase. In the sliding phase, a switching surface is defined such that the closed-loop system exhibits desired dynamic behavior during sliding mode [23]. In the reaching phase, the sliding mode controller is employed to derive the system states to the sliding surface [24]. Because of the influence of sliding surface on the stability and transient performance of the system, the design of switching surface is the major subject in the SMC method [25]. In the conventional SMC, during the reaching phase, the controlled system is not robust and even matched disturbances can destabilize the system [26]. Actually, the robust tracking performance is guaranteed only after the system states reach the sliding surface, and hence robustness is not satisfied during the reaching phase [24–26]. Recently, there is more interest in the use of SMC and new researches have been proposed.

1.2 Literature review

In [27], an adaptive self-tuning fuzzy SMC compensator is proposed for velocity control of electro-hydraulic displacement-controlled system for which the proposed scheme can design the SMC with no requirement on the system dynamical model. In [28], an adaptive fuzzy controller is applied to active vibration control of a smart flexible beam with mass uncertainties. In [29], a stable decentralized adaptive fuzzy SMC structure is offered for reconfigurable modular manipulators to fulfill the concept of modular software and a first-order Takagi–Sugeno fuzzy-logic system is presented to approximate the unknown dynamics of subsystem by using adaptive SMC. In [30], an adaptive fuzzy SMC law is used for the stabilization of unstable periodic orbits in a chaotic pendulum where the adaptive fuzzy inference system is embedded in a smooth SMC to cope with both structured and unstructured uncertainties. In [31], an adaptive fuzzy SMC is applied for a benchmark problem on a seismically excited highway bridge and is also integrated with clipped-optimal strategy to demonstrate its efficiency for semi-active dampers. In [32], an adaptive SMC scheme is proposed for the robust tracking and model following of uncertain time-delay systems with input nonlinearity. In [33], a robust adaptive SMC strategy is presented for a new introduced class of uncertain chaotic systems in the absence of any knowledge on the uncertainty bounds. In [34], an adaptive integral sliding mode controller is proposed for multi-input multi-output (MIMO) systems affected by unknown matched or mismatched uncertainties. All of the mentioned works are performed in infinite time. Actually, using the linear sliding surfaces, the switching control law may have unsatisfactory performance in finite time.

Compared with the linear SMC, the terminal sliding mode (TSM) control technique offers some superior properties such as fast response and finite time convergence [35]. This method is particularly suitable for high-precision control as it speeds up the rate of convergence near the origin. Unlike conventional SMC, TSM method is based on a set of recursive nonlinear non-smooth differential equations enabling finite time convergence, and in the latest years there is more interest in the use of it [36]. However, TSM may not offer the same convergence performance as SMC when the system states are far away from the equilibrium [37]. The fast terminal sliding mode (FTSM) concept has been adopted by Yu and Man [38], which guarantees fast transient convergence and strong robustness. In the recent years, there is more interest in the use of this method [39]. In [40], a neural adaptive SMC algorithm is planned to achieve the position tracking performance of the field-oriented control for permanent magnet synchronous motors by combining FTSM and the radial basis function (RBF). In [41], a novel FTSM control approach is investigated for the robust tracker design of a class of nonlinear second-order systems with time-varying uncertainties. In [42], an adaptive FTSM method is designed for the finite time motion/force control of robotic manipulators in the presence of environmental constraint and modeling uncertainties. In [43], a systematic control method using non-singular FTSM for a linear motor positioner is studied which guarantees fast convergence of the tracking errors in the attendance of perturbations containing payload variations, friction, disturbances, and measurement noises. Nevertheless, it should be pointed out that the FTSM technique still needs to be further considered on robustness performance and reaching phase elimination. The global sliding mode control (GSMC) method has been presented to offer a general framework to remove the reaching interval so that the sliding mode exists right from the beginning, and then the system response is completely invariant to system perturbations [44,45]. GSMC causes superior robustness and performance when an additional term is inserted to sliding surface [46,47].

1.3 Contributions

To the best of the author’s knowledge, a very little attention [48,49] has been paid to the problem of combination of FTSM and GSMC methods for robust tracking control of uncertain nonlinear systems, which is still open in the literature. This motivates the current research. In this paper, we consider the robust tracking problem for third-order systems with time-varying uncertainties and nonlinearities. We employ a novel adaptive FTSM controller combined with GSMC surface structure that guarantees the existence of the sliding mode around the sliding surface in a finite time and eliminates the reaching phase to improve the robustness and performance of the system. The robustness of the controlled system can be guaranteed right from the beginning of the entire response. The proposed method ensures the robustness against nonlinearities, parametric uncertainties and external disturbances. Moreover, the knowledge of upper bounds of system uncertainties is not required.

1.4 Paper organization

The rest of the paper is organized as follows: The problem formulation and preliminaries are presented in Sect. 2. The new FTSM controller design and stability analysis are discussed in Sect. 3. Simulation results are provided in Sect. 4, and the conclusions are drawn in Sect. 5.

2 Problem formulation

Consider the following time-varying and nonlinear third-order system [50]:

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= x_3, \\ \dot{x}_3 &= f(x,t) + \Delta f(x,t) + (b(x,t) + \Delta b(x,t))\, u \\ &\quad + d_0(x,t), \end{aligned} \tag{1}$$

where x_1, x_2 and x_3 are the state variables of the system, $x = [x_1, x_2, x_3]^T$ is the state vector, and $u \in R$ is the control input. The terms $b(x,t) \in R$ and $f(x,t) \in R$ are deemed to represent the known bounded nonlinear functions, and they belong to the smooth vector fields in a neighborhood of the origin $x = 0$ with $f(0,t) = 0$ and $b(x,t) \neq 0$. Moreover, the nonlinear functions $\Delta b(x,t)$, $\Delta f(x,t)$ and $d_0(x,t)$ are assumed to be the system uncertainties and external disturbances, and all of them are continuous functions depending on the state x. Now, defining $d(x,t) = \Delta f(x,t) + \Delta b(x,t)\, u + d_0(x,t)$, one can rewrite the nonlinear third-order system (1) as:

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= x_3, \\ \dot{x}_3 &= f(x,t) + b(x,t)\, u + d(x,t). \end{aligned} \tag{2}$$

The uncertain nonlinear system (2) is supposed to track the desired trajectory $x_d = [x_{1d}, x_{2d}, x_{3d}]^T$, where $x_{2d} = \dot{x}_{1d}, x_{3d} = \dot{x}_{2d}$ and x_{3d} is differentiable function of time. The tracking error is defined as:

$$E(t) = x - x_d = [e, \dot{e}, \ddot{e}]^T, \tag{3}$$

where $e = x_1 - x_{1d}$.

Assumption 1 The nonlinear functions $f(x,t)$, $b(x,t)$, $\Delta f(x,t)$, $\Delta b(x,t)$ and $d_0(x,t)$ are assumed to be differentiable.

Assumption 2 [50]: There exists a strictly positive constant $\bar{\delta}$ which is the lower bound of $b(x,t)$, i.e., $0 < \bar{\delta} = \inf\{|b(x,t)|\}$.

Assumption 3 [50] There exists a constant $\bar{\mu}$ which for every pair (x,t) satisfies the condition $|d(x,t)| \leq \bar{\mu}$.

Lemma 1 [51] *Assume that a continuous positive-definite function $V(t)$ satisfies the following differential inequality:*

$$\dot{V}(t) \leq -\alpha V(t) - \beta V^{\eta}(t) \qquad \forall t \geq t_0,\ V(t_0) \geq 0, \tag{4}$$

where α and β are positive constants, and η is a ratio of two odd positive integers with $1 > \eta > 0$. Then for initial time t_0, $V(t)$ converges to zero at least in a finite time:

$$t_s = t_0 + \frac{1}{\alpha(1-\eta)} \ln \frac{\alpha V^{1-\eta}(t_0) + \beta}{\beta}. \tag{5}$$

Proof By dividing two sides of the inequality (4) to $V^{\eta}(t)$, one can obtain:

$$V^{-\eta}(t)\dot{V}(t) \leq -\alpha V^{1-\eta}(t) - \beta, \tag{6}$$

and consequently:

$$\mathrm{d}t \leq -\frac{V^{-\eta}(t)}{\alpha V^{1-\eta}(t) + \beta} dV(t). \tag{7}$$

Now, integrating two sides of (7) from t_0 to t_s yields:

$$\begin{aligned} t_s - t_0 &\leq -\int_{V(t_0)}^{0} \frac{V^{-\eta}(t)}{\alpha V^{1-\eta}(t) + \beta} \mathrm{d}V(t) \\ &= -\frac{1}{\alpha(1-\eta)} \left[\ln\beta - \ln\left(\alpha V^{1-\eta}(t_0) + \beta\right)\right] \\ &= \frac{1}{\alpha(1-\eta)} \ln \frac{\alpha V^{1-\eta}(t_0) + \beta}{\beta}. \end{aligned} \tag{8}$$

which completes the proof of the lemma. □

2.1 Controllability analysis

The uncertain nonlinear system (1) can be transformed into the controllability canonical form using the global diffeomorphism specified by Isidori [52].

Assumption 4 [53] The distributions in the form of $D_i = \text{span}\left\{b, ad_f b, \ldots, ad_f^i b\right\}, 0 \le i \le 2$ are involutive and have constant rank $i+1$.

Assumption 5 The uncertain functions $\Delta f(x,t)$ and $d_0(x,t)$ satisfy parametric-strict-triangle condition such that $[\Delta f(x,t), d_0(x,t), D_i] \in D_i, 0 \le i \le 1$.

According to Assumptions 4 and 5, one can find a function $h(x_i)$ and obtain a diffeomorphism $\phi(x) = \left[h, L_f h, L_f^2 h\right]^T$ such that the system (1) can be transformed into the following form:

$$\begin{aligned}\dot{x}_1 &= x_2,\\ \dot{x}_2 &= x_3,\\ \dot{x}_3 &= L_f^3 h(\phi^{-1}(x)) + L_b L_f^2 h(\phi^{-1}(x))u + \Delta_3,\end{aligned} \tag{9}$$

where $\Delta_3 = L_{\Delta f} L_f^2 h(\phi^{-1}(x)) + L_{d_0} L_f^2 h(\phi^{-1}(x)) + L_{\Delta b} L_f^2 h(\phi^{-1}(x))$. The term Δ_3 is the function relating to the uncertainties and disturbances of the system. In order to investigate the stabilization of the uncertain nonlinear systems, the system (9) can be rewritten in the following state-space model:

$$\begin{aligned}\dot{x} = Ax + B\{&f(x,t) + \Delta f(x,t) + d_0(x,t)\\ &+ (b(x,t) + \Delta b(x,t))u\},\end{aligned} \tag{10}$$

where $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

Using the pole-placement technique, one can consider a term kx where $k = [k_0, k_1, k_2]$, and $k_i's$ are chosen such that $s^3 + k_2 s^2 + k_1 s + k_0 = 0$ is a stable polynomial. Then, the exponentially stable dynamics is obtained as:

$$\dddot{x} + k_2 \ddot{x} + k_1 \dot{x} + k_0 x = 0, \tag{11}$$

which indicates that $x \to 0$. Thus, system (10) can be rewritten as:

$$\begin{aligned}\dot{x} = \{A - Bk\}x + B\{&kx + f(x,t) + \Delta f(x,t)\\ &+ d_0(x,t) + (b(x,t) + \Delta b(x,t))u\},\end{aligned} \tag{12}$$

where the following transformation is considered for the control input:

$$u = b(x,t)^{-1}\{v - f(x,t)\}, \tag{13}$$

where v is a new control signal. The control input (13) involves two parts: One part is the term $-b(x,t)^{-1} f(x,t)$ which is used to cancel the nonlinearities of the system, and the other term is $b(x,t)^{-1}v$ which is employed to attenuate the effects of the uncertainties and disturbances of the system. Substituting (13) into (12) obtains:

$$\dot{x} = \{A - Bk\}x + B\{(1+G)v + H\}, \tag{14}$$

where $G = \Delta b(x,t) b(x,t)^{-1}$ and $H = \Delta f(x,t) + kx + d_0(x,t) - \Delta b(x,t) b(x,t)^{-1} f(x,t)$. The functions $\Delta b(x,t)$, $\Delta f(x,t)$ and $d_0(x,t)$ are continuous functions, and then G and H are also continuous functions. The system (14) is completely controllable and can be controlled using various robust control techniques.

3 Main results

3.1 Sliding surface design

The global switching function for system (2) can be defined as:

$$s(e) = C\left(E(t) - E(0)e^{-\varphi t}\right), \tag{15}$$

where $C = [c_1, c_2, c_3]$ are the gain constants and φ is an appropriate positive constant.

Remark 1 Compared with the sliding surface $s_c(e) = CE(t)$, the global sliding surface (15) makes the state trajectories to arrive at the sliding surface right from the beginning. As a result, the reaching phase is eliminated and the global robustness of the whole system can be guaranteed.

In light of (15), the global sliding mode $s(e) = 0$ indicates that:

$$CE(t) - CE(0)e^{-\varphi t} = 0, \tag{16}$$

where based on Remark 1, one can obtain:

$$s_c(e) = s_c(0)e^{-\varphi t}, \tag{17}$$

where this is the unique solution of the following first-order differential equation:

$$\dot{s}_c(e) + \varphi s_c(e) = 0. \tag{18}$$

In order to satisfy $s(e)$ converge to zero in finite time and remove the chattering, the following FTSM surface is proposed:

$$\sigma(e) = \dot{s}(e) + \lambda s(e) + \mu s^{\eta}(e), \tag{19}$$

where λ and μ are positive constant values, and η is a ratio of two odd positive integers with $1 > \eta > \frac{1}{2}$.

In order to guarantee the FTSM surface and tracking errors converge to zero in finite time, the following theorem is presented.

Theorem 1 *Consider the uncertain nonlinear system (2). Applying the control law:*

$$\dot{u} = -(c_3 b(x,t))^{-1}\Big(\Pi_1 + \mu\eta\Pi_2\Pi_3 + \Pi_4 u + \kappa\,\mathrm{sgn}(\sigma)\,|\sigma|^{\eta} + \gamma\sigma + \chi\,\mathrm{sgn}(\sigma)\Big), \tag{20}$$

with arbitrary positive coefficients γ and κ, and considering that χ is a scalar value which satisfies:

$$\chi \geq \left(c_3\dot{d} + (c_2 + \lambda c_3 + \mu\eta c_3\Pi_3)\,d\right)_{\max}, \tag{21}$$

where:

$$\Pi_1 = (c_1 + \lambda c_2)(x_3 - x_{3d}) + (c_2 + \lambda c_3)(f - \dot{x}_{3d}) + \lambda c_1 (x_2 - x_{2d}) + c_3\left(\dot{f} - \ddot{x}_{3d}\right) - \varphi(\varphi - \lambda)e^{-\varphi t}\sum_{i=1}^{3} c_i e(0)^{(i-1)}, \tag{22}$$

$$\Pi_2 = c_1(x_2 - x_{2d}) + c_2(x_3 - x_{3d}) + c_3(f - \dot{x}_{3d}) + \varphi e^{-\varphi t}\sum_{i=1}^{3} c_i e(0)^{(i-1)}, \tag{23}$$

$$\Pi_3 = \left(c_1(x_1 - x_{1d}) + c_2(x_2 - x_{2d}) + c_3(x_3 - x_{3d}) - e^{-\varphi t}\sum_{i=1}^{3} c_i e(0)^{(i-1)}\right)^{\eta-1}, \tag{24}$$

$$\Pi_4 = (\mu\eta\Pi_3 c_3 + c_2 + \lambda c_3)\,b(x,t) + c_3\dot{b}(x,t), \tag{25}$$

then the trajectory of the system (2) is forced to move from any initial condition to the sliding surface (15) in finite time and to remain on it.

Proof Consider the candidate Lyapunov function as:

$$V(\sigma) = \frac{1}{2}\sigma^2. \tag{26}$$

From (3), the derivatives of the tracking error e are obtained as:

$$\dot{e} = x_2 - x_{2d}, \tag{27}$$

$$\ddot{e} = x_3 - x_{3d}, \tag{28}$$

$$\dddot{e} = f(x,t) + b(x,t)u + d(x,t) - \dot{x}_{3d}, \tag{29}$$

$$e^{(4)} = \dot{f}(x,t) + \dot{b}(x,t)u + b(x,t)\dot{u} + \dot{d}(x,t) - \ddot{x}_{3d}. \tag{30}$$

From (19), (27)–(30), the derivative of $\sigma(e)$ can be obtained as:

$$\begin{aligned}\dot{\sigma}(e) &= \ddot{s}(e) + \lambda\dot{s}(e) + \mu\eta\dot{s}(e)s^{\eta-1}(e)\\ &= \sum_{i=1}^{3} c_i e^{(i+1)} - \varphi(\varphi-\lambda)e^{-\varphi t}\sum_{i=1}^{3} c_i e(0)^{(i-1)} + \lambda\sum_{i=1}^{3} c_i e^{(i)}\\ &\quad + \mu\eta\left(\sum_{i=1}^{3} c_i e^{(i)} + \varphi e^{-\varphi t}\sum_{i=1}^{3} c_i e(0)^{(i-1)}\right)\\ &\quad \times\left(\sum_{i=1}^{3} c_i e^{(i-1)} - e^{-\varphi t}\sum_{i=1}^{3} c_i e(0)^{(i-1)}\right)^{\eta-1},\end{aligned} \tag{31}$$

where using the equalities (22)–(25), then (31) can be rewritten as:

$$\dot{\sigma}(e) = \Pi_1 + \mu\eta\Pi_2\Pi_3 + \Pi_4 u + c_3 b(x,t)\dot{u} + \big(c_2 + \lambda c_3 + \mu\eta c_3\Pi_3\big)d(x,t) + c_3\dot{d}(x,t). \tag{32}$$

Differentiating $V(\sigma)$ and using (32) yields:

$$\begin{aligned}\dot{V}(\sigma) &= \sigma\dot{\sigma}\\ &= \sigma\Big(\Pi_1 + \mu\eta\Pi_2\Pi_3 + \Pi_4 u + c_3 b(x,t)\dot{u}\\ &\quad + \big(c_2 + \lambda c_3 + \mu\eta c_3\Pi_3\big)d(x,t)\\ &\quad + c_3\dot{d}(x,t)\Big),\end{aligned} \tag{33}$$

where substituting (20) in (33), one can obtain:

$$\begin{aligned}\dot{V}(\sigma) &= -\sigma\kappa\,\mathrm{sgn}(\sigma)\,|\sigma|^{\eta} - \gamma\sigma^2 - \sigma\chi\,\mathrm{sgn}(\sigma)\\ &\quad + \sigma\big((c_2 + \lambda c_3 + \mu\eta c_3\Pi_3)\,d(x,t)\\ &\quad + c_3\dot{d}(x,t)\big),\end{aligned} \tag{34}$$

where based on the condition (21) follows that:

$$\begin{aligned}\dot{V}(\sigma) &\leq -\gamma\,|\sigma|^2 - \kappa\,|\sigma|^{\eta+1}\\ &= -\alpha V(\sigma) - \beta V^{\bar{\eta}}(\sigma)\end{aligned} \tag{35}$$

where $\bar{\eta} = (\eta+1)/2 < 1, \alpha = 2\gamma > 0$ and $\beta = 2^{\bar{\eta}}\kappa > 0$. This means that the Lyapunov function (26) decreases gradually and the sliding surface converges to zero in finite time. Then, the tracking errors are convergent to zero in the finite time. This completes the proof. □

3.2 Adaptive sliding mode controller design

After design of the sliding surface and control input, the next phase is to design an adaptation law for the parameter χ in (21). Actually, it is not easy to achieve the upper bound of the term $(c_3\dot{d} + (c_2 + \lambda c_3 + \mu\eta c_3\Pi_3)d)$ due to the complexity of the system uncertainties and external disturbances. Then, an adaptation law is presented to overcome this problem. The control law (20) can be modified as:

$$\dot{u} = -(c_3 b(x,t))^{-1} \left(\Pi_1 + \mu\eta\Pi_2\Pi_3 + \Pi_4 u + \kappa\,\mathrm{sgn}(\sigma)\,|\sigma|^{\eta} + \gamma\sigma + \hat{\chi}\,\mathrm{sgn}(\sigma)\right), \tag{36}$$

where $\hat{\chi}$ is the estimate of χ in (36). The adaptation parameter $\hat{\chi}$ can be found by the adaptation law as follows:

$$\dot{\hat{\chi}} = \rho^{-1}\,|\sigma(e)|\,, \tag{37}$$

where $\rho > 0$ is an adaptation gain. By choosing appropriate values of ρ, the rate of the parameter adaptation can be adjusted. Defining $\tilde{\chi} = \hat{\chi} - \chi$, the following expression is obtained:

$$\dot{\tilde{\chi}} = \dot{\hat{\chi}} = \rho^{-1}\,|\sigma(e)|\,. \tag{38}$$

Theorem 2 *Consider the uncertain nonlinear system (2). If the FTSM surface (19), the control law (36) and the adaptation law (38) are applied, then the trajectories of the system (2) are forced toward the sliding surface and the reaching condition is guaranteed.*

Proof Consider the candidate Lyapunov function as:

$$V(\sigma) = \frac{1}{2}\left(\sigma^2 + \rho\tilde{\chi}^2\right). \tag{39}$$

Taking the time derivative of $V(\sigma)$ and using (32) and (38) yields:

$$\begin{aligned}\dot{V}(\sigma) &= \sigma\dot{\sigma} + \rho\tilde{\chi}\dot{\tilde{\chi}} \\ &= \sigma\left\{\Pi_1 + \mu\eta\Pi_2\Pi_3 + \Pi_4 u + c_3 b(x,t)\dot{u} \right.\\ &\quad \left. + (c_2+\lambda c_3+\mu\eta c_3\Pi_3)\,d(x,t)+c_3\dot{d}(x,t)\right\} \\ &\quad + \rho\tilde{\chi}\dot{\hat{\chi}},\end{aligned} \tag{40}$$

where substituting (36) in (40), one can find:

$$\begin{aligned}\dot{V}(\sigma) &= \sigma\left\{-\kappa\,\mathrm{sgn}(\sigma)\,|\sigma|^{\eta} - \gamma\sigma - \hat{\chi}\,\mathrm{sgn}(\sigma) \right.\\ &\quad \left. + (c_2+\lambda c_3+\mu\eta c_3\Pi_3)\,d(x,t)+c_3\dot{d}(x,t)\right\} \\ &\quad + \rho\tilde{\chi}\dot{\hat{\chi}} \\ &\le -\left\{\kappa\,|\sigma|^{\eta+1} + \gamma\,|\sigma|^2\right\} - \hat{\chi}\,|\sigma| + \chi\,|\sigma| \\ &\quad - \chi\,|\sigma| + |\sigma|\left\{(c_2+\lambda c_3+\mu\eta c_3\Pi_3)\,d(x,t) \right.\\ &\quad \left. + c_3\dot{d}(x,t)\right\} + \rho\tilde{\chi}\dot{\hat{\chi}},\end{aligned} \tag{41}$$

where using (21) and (38), it yields:

$$\begin{aligned}\dot{V}(\sigma) &\le -\kappa\,|\sigma|^{\eta+1} - \gamma\,|\sigma|^2 - \hat{\chi}\,|\sigma| + \chi\,|\sigma| \\ &\quad + \rho\left(\hat{\chi} - \chi\right)\rho^{-1}\,|\sigma| = -\kappa\,|\sigma|^{\eta+1} - \gamma\,|\sigma|^2.\end{aligned} \tag{42}$$

Then, using the adaptive tuning control law, it is concluded that $\dot{V}(\sigma) \le 0$ for $\kappa > 0$ and $\gamma > 0$. This completes the proof. □

Remark 2 Using the adaptive tuning controller, the finite time convergence of the tracking errors is not guaranteed. Then, by the adaptive control law (36) in Theorem 2, the Lyapunov function decreases gradually and only the reaching condition is satisfied.

Remark 3 From (42), the term $\dot{V}(\sigma)$ is negative semi-definite and ensures that $V(\sigma), \sigma(e)$ and $\tilde{\chi}$ are all bounded. It is concluded from (19) that $s(e)$ and $\dot{s}(e)$ are also bounded. Since $V(0)$ is bounded and $V(\sigma)$ is non-increasing and bounded, it can be concluded that $\lim_{t\to\infty}\int_0^t \|\sigma\|\,\mathrm{d}t$ and $\lim_{t\to\infty}\int_0^t \|s\|\,\mathrm{d}t$ are bounded. Since $\lim_{t\to\infty}\int_0^t \|s\|\,\mathrm{d}t$ and $\dot{s}(e)$ are bounded, according to Barbalat's lemma, $s(e)$ will asymptotically converge to zero, i.e., $\lim_{t\to\infty}\int_0^t s(e)\mathrm{d}t = 0$. Consequently, from (15), $E(t)$ is bounded and will also asymptotically converge to zero. Then, because of the boundedness of the functions Π_i, $(i = 1, \ldots, 4)$, the control signal of (36) is bounded.

To reduce the chattering behavior, the control law (36) can be modified to be:

$$\dot{u} = -(c_3 b(x,t))^{-1}\left(\Pi_1 + \mu\eta\Pi_2\Pi_3 + \Pi_4 u + \kappa\,|\sigma|^{\eta}\,\mathrm{sat}\left(\frac{\sigma}{a}\right) + \gamma\sigma + \hat{\chi}\,\mathrm{sat}\left(\frac{\sigma}{a}\right)\right), \tag{43}$$

where sat(.) is a saturation function and a is the boundary layer thickness.

Remark 4 Since the saturation function is used in (43), the FTSM surface $\sigma(e)$is not equal to zero for all of the time and the adaptive parameter increases slowly and boundlessly. To overcome this problem, the adaptation law (37) can be modified to the following formula:

$$\dot{\hat{\chi}} = \begin{cases} 0 & \text{if } |\sigma(e)| \le a \\ \rho^{-1}\,|\sigma(e)| & \text{if } |\sigma(e)| > a \end{cases} \tag{44}$$

Remark 5 In general it is required to feedback the entire state variables to synthesize a reasonable controller. In the cases in which some of the variables are not measurable, the delayed feedback control technique can be employed. In particular, the derivative of the

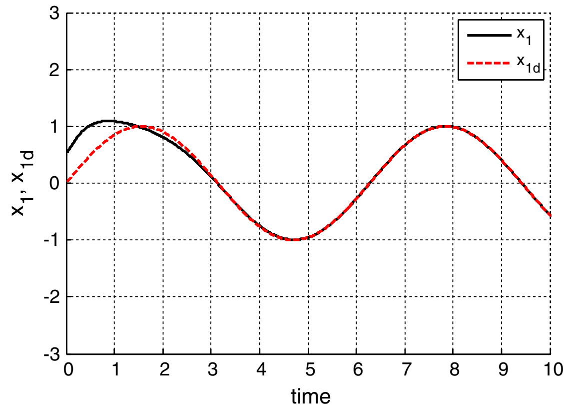

Fig. 1 State trajectory x_1

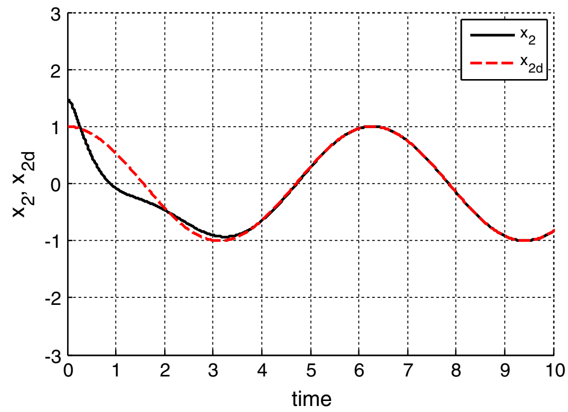

Fig. 2 State trajectory x_2

state variable can be replaced by a delay term in the form of an Euler approximation of the derivative function [54]. The state-derivative $\dot{x}_3$ is replaced by a state delay function as $\dot{x}_3 = \frac{1}{h}\left[x_3(t) - x_3(t-h)\right]$ provided the delay $h > 0$ is sufficiently small.

4 Simulation results

Example 1 In this section, in order to verify the performance of the proposed FTSM control method, the following uncertain nonlinear third-order system is considered [50]:

$$\begin{aligned}
\dot{x}_1 &= x_2, \\
\dot{x}_2 &= x_3, \\
\dot{x}_3 &= -3x_3 - 4x_2 - 2x_1 + x_1 x_2 \\
&\quad + \Delta f(x,t) + u + d(x,t),
\end{aligned} \tag{45}$$

where $\Delta f(x,t) = 0.39\sin\left(x_1 x_2 + x_3\sqrt{t}\right)$ and $d(x,t) = 0.6\sin(10t)$. The initial conditions of the system are chosen as: $x(0) = \left[0.5\ 1.5\ -1\right]^T$. The desired trajectory is determined as $x_{1d} = \sin(t)$. The control parameters are selected by trial and error as: $a = 0.2, c_1 = 2, c_2 = 1, c_3 = 0.5, h = 0.1, \varphi = 10, \lambda = 5, \chi = 6, \mu = 10, \gamma = 3, \kappa = 15$, and $\eta = 3/5$.

The position tracking trajectories of states x_1, x_2 and x_3 are demonstrates in Figs. 1, 2 and 3, respectively. The trajectory of the control input is shown in Fig. 4. It is shown that the proposed control method can obtain the superior position tracking performance and high robustness and is able to overcome the uncertainties and nonlinearities. The time responses of the GSMC surface s and the FTSM surface σ are plotted in Figs. 5 and 6. Obviously, it can be seen that the switching surface and FTSM surface converge to the origin quickly. Noticeably, this numerical simulation confirms the theoretical analysis.

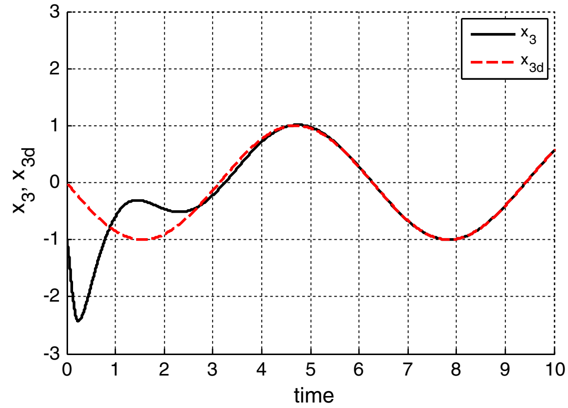

Fig. 3 State trajectory x_3

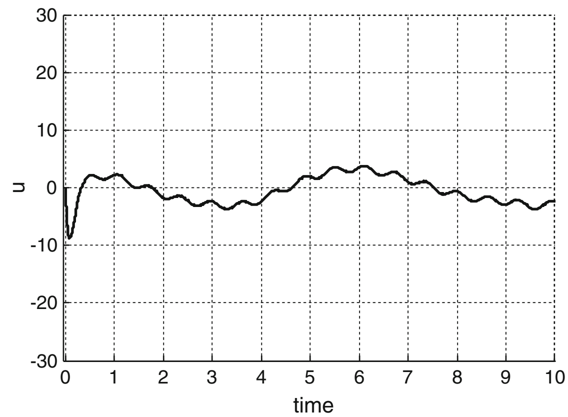

Fig. 4 Control input u

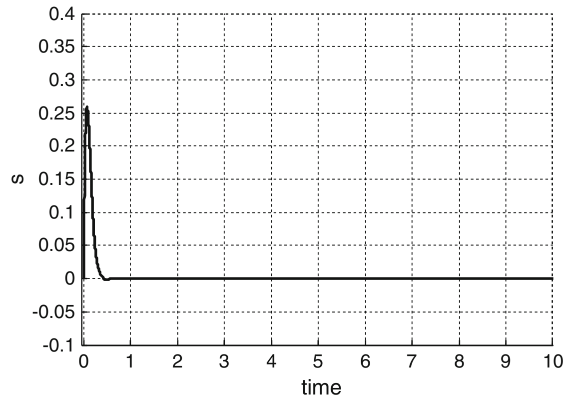

Fig. 5 Sliding surface s

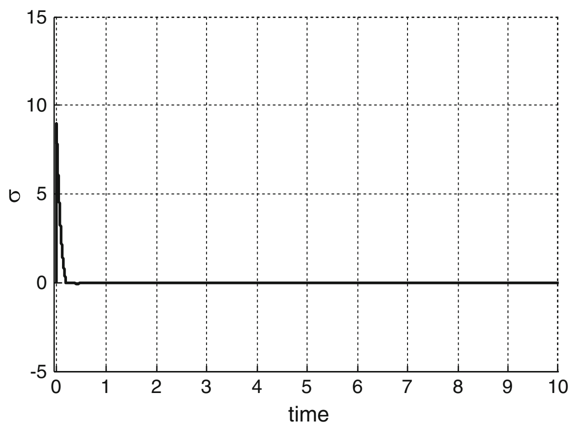

Fig. 6 FTSM surface σ

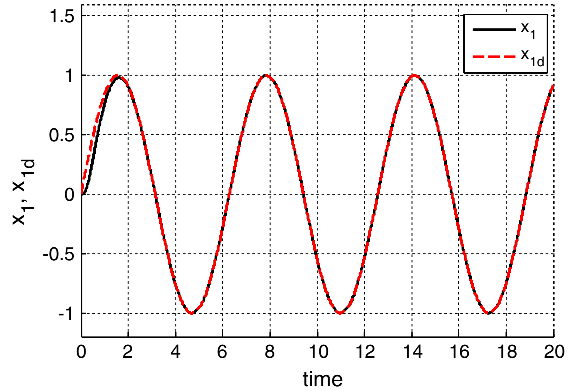

Fig. 7 State trajectory x_1

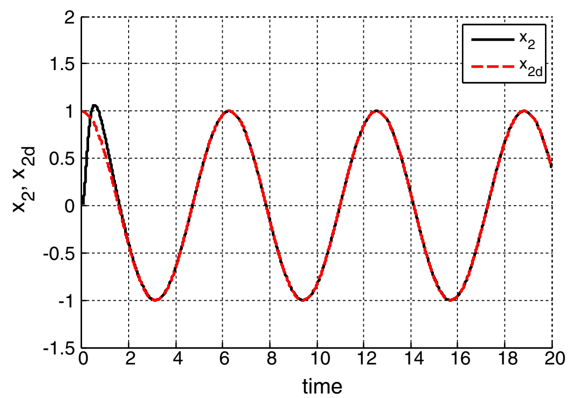

Fig. 8 State trajectory x_2

Example 2 Consider the following nonlinear system [55]:

$$
\begin{aligned}
\dot{x}_1 &= x_2, \\
\dot{x}_2 &= x_3, \\
\dot{x}_3 &= -(1 + 0.3 \sin t)x_1^2 - (1.5 + 0.2 \cos t)x_2 \\
&\quad - (1 + 0.4 \sin t)x_3 + (3 + \cos x_1)u + w
\end{aligned} \tag{46}
$$

The desired state trajectory is determined as $x_d^T = [\sin t, \cos t, -\sin t]$. The control parameters and initial conditions are given as: $c_1 = 12, c_2 = 7, c_3 = 1, h = 0.1, \varphi = 8, \lambda = 3, \chi = 5, \mu = 5, \gamma = 2, \kappa = 10, \eta = 3/5$, and $x(0) = \begin{bmatrix} 0 & 0 & -0.35 \end{bmatrix}^T$. The boundary layer a is chosen as 0.15. The disturbance w is assumed to be a random noise with a mean value of 0.5 and $|w| \leq 0.1$. Figures 7, 8, 9, 10, 11 and 12 show the simulation results using the proposed control technique. As shown in Figs. 7, 8 and 9, the tracking performance of the states x_1, x_2 and x_3 is accurately achieved. The trajectory of the control signal is demonstrated in Fig.10. It is shown that the control input is smooth and is capable of overcoming the nonlinearities and disturbances. The trajectories of the surfaces s and σ are shown in Figs. 11 and 12, correspondingly. It can be concluded that these surfaces converge to zero quickly. Therefore, this simulation confirms the theoretical results.

Example 3 Consider the Genesio's chaotic system described by [17]:

$$
\begin{aligned}
\dot{x}_1 &= x_2, \\
\dot{x}_2 &= x_3, \\
\dot{x}_3 &= -6x_1 - 2.92x_2 - 1.2x_3 \\
&\quad + (1 - 0.1 \sin(t))\, u + x_1^2.
\end{aligned} \tag{47}
$$

The reference signal is determined as: $x_{1d} = \sin(t)$. The whole system is numerically simulated using the

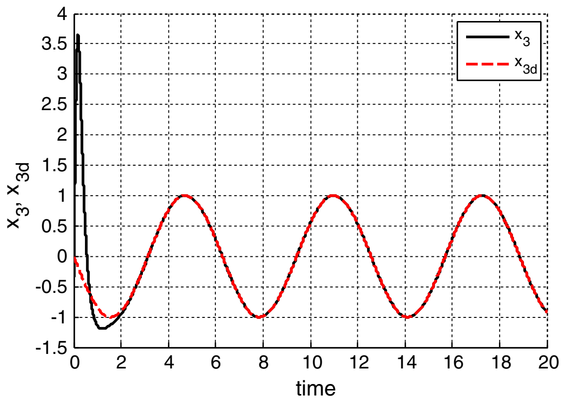

Fig. 9 State trajectory x_3

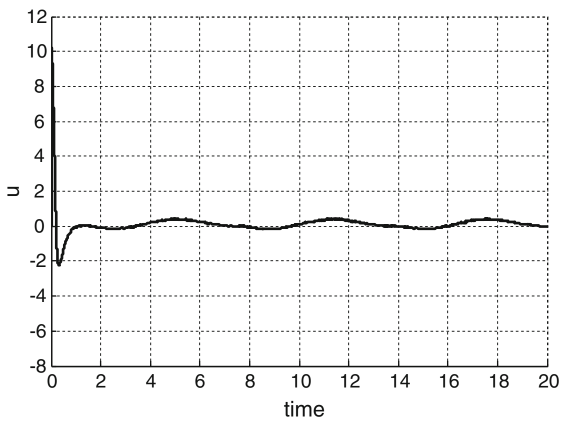

Fig. 10 Control input u

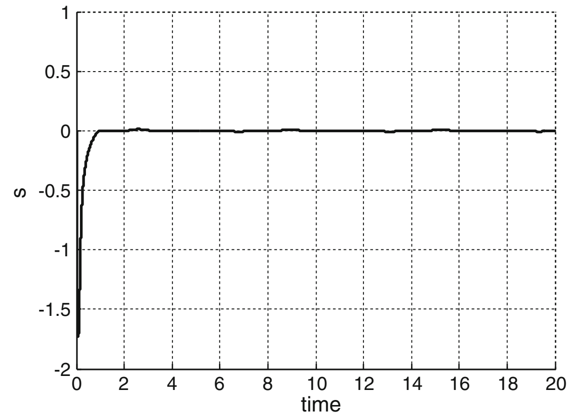

Fig. 11 Sliding surface s

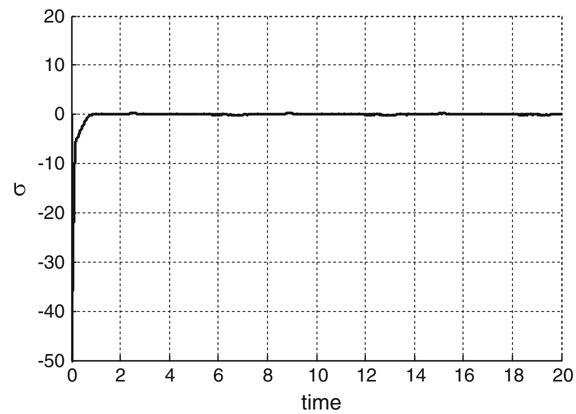

Fig. 12 FTSM surface σ

proposed controller with the following initial parameters and initial conditions: $a = 0.2, c_1 = 3, c_2 = 1, c_3 = 2, h = 0.1, \varphi = 8, \lambda = 3, \chi = 5, \mu = 5, \gamma = 2, \kappa = 10, \eta = 3/5$, and $x(0) = \left[-1\ 1\ 0\right]^T$.

The controller is applied at time $t = 15$. The tracking trajectories of states x_1, x_2 and x_3 are shown in Figs. 13, 14 and 15. It is demonstrated from these figures that as soon as the control is started, the synchronization of all states is realized with good performance. The trajectory of the control input is displayed in Fig.16. The time responses of the GSMC surface s and the FTSM surface σ are demonstrated in Figs. 17 and 18. Clearly, it is found that as soon as the controller is applied, the switching surface and FTSM surface converge to the origin quickly. Therefore, the uncertain Genesio's chaotic system is stabilized on an oscillating system by the action of the proposed controller.

The effects of changing various design parameters on the tracking accuracy, control signal and convergence rate in all of the simulations are investigated with the following descriptions: (a) By increasing φ, we can meet worse tracking accuracy and overshoots in the control signal, but a fast convergence; (b) if the parameter λ is increased, the tracking accuracy and convergence rate are improved, but amplitude of the control signal is increased; (c) if the parameter μ is decreased, the overshoot in the tracking response and the amplitude of the control input are decreased, but the convergence rate is improved; (d) the increase in κ improves the convergence speed and increases the variations of the control signal, but no explicit influence on the tracking performance is obtained; (e) if the parameter χ is increased, the fast convergence is satisfied, but the amplitude of the control input is increased and the chattering phenomenon is observed in the control signal; (f)

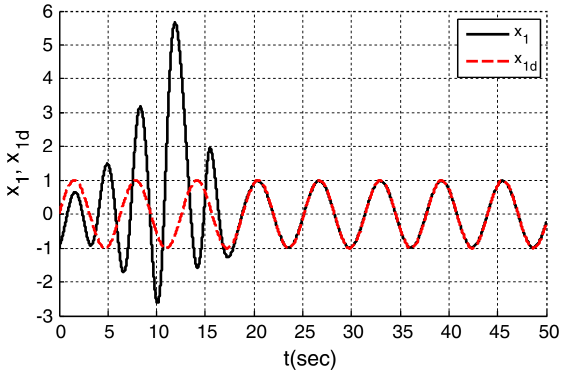

Fig. 13 State trajectory x_1

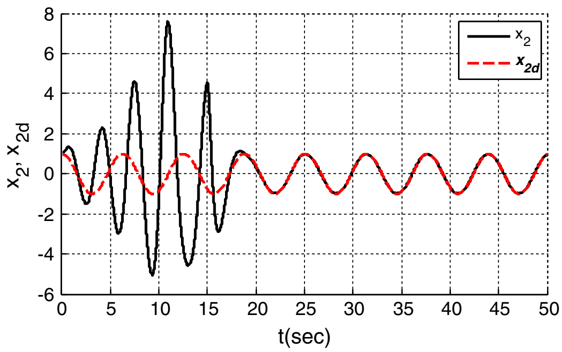

Fig. 14 State trajectory x_2

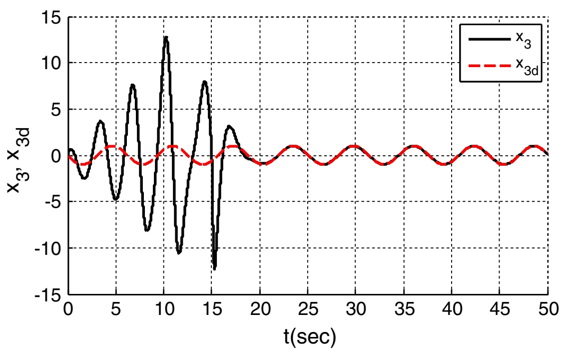

Fig. 15 State trajectory x_3

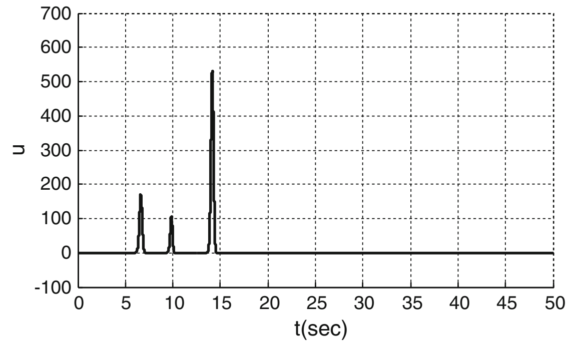

Fig. 16 Control input u

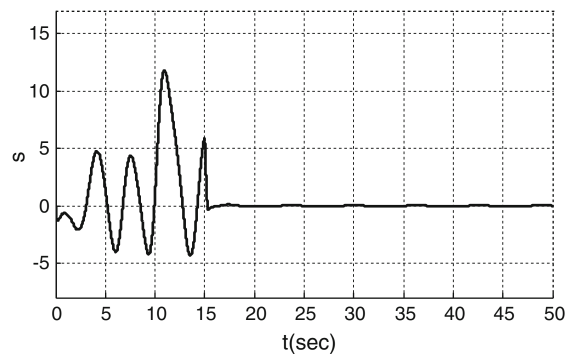

Fig. 17 Sliding surface s

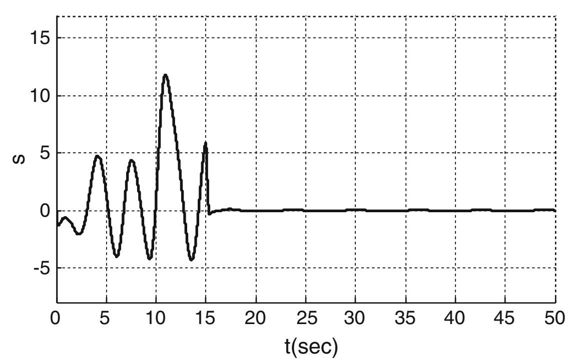

Fig. 18 FTSM surface σ

The increment in the parameters c_1, c_2 and c_3 causes the improvement in the convergence rate, decrease in the accuracy and increase in input amplitude; (g) if the parameter γ is increased, the tracking performance and convergence rate are improved, but the input signal is worsened.

5 Conclusions

Tracking control of uncertain nonlinear third-order systems is studied in this work. A novel scheme for the design of adaptive FTSM controller combined with a new GSMC surface is introduced. This technique provides the existence of the sliding mode around the switching surface in a finite time. Finally, the reaching phase is eliminated and the robustness of the system is improved. In general, compared with many existing researches on TSM and FTSM, the suggested method demonstrates four attractive features: (i) It eliminates chattering phenomenon in the control input and therefore is appropriate for practical applications; (ii) it is rather straightforward and guarantees the presence of the sliding mode around the surface in the finite time; (iii) by using on-line adaptation law, the information about upper bounds of the perturbations is not necessary; (iv) based on the elimination of the reaching phase, the robustness of the system is guaranteed right from the beginning of whole response. Intensive simulation results are showed to verify the efficiency of the offered technique, and satisfactory results are realized. The proposed control law can achieve favorable tracking performance for higher-order nonlinear systems. The further researches in this field can be extended to time-delayed neuron networks using the results reported in [56,57].

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