Nonlinear Dyn (2015) 82:333–347
DOI 10.1007/s11071-015-2161-7

ORIGINAL PAPER

Soliton and Riemann theta function quasi-periodic wave solutions for a $(2+1)$-dimensional generalized shallow water wave equation

Yiren Chen · Ming Song · Zhengrong Liu

Received: 20 February 2015 / Accepted: 12 May 2015 / Published online: 4 June 2015

Abstract In this paper, a $(2+1)$-dimensional generalized shallow water wave equation is investigated through bilinear Hirota method. Interestingly, the breather-type and lump-type soliton solutions are obtained. Furthermore, dynamic properties of the soliton waves are revealed by means of the asymptotic analysis. Based on Hirota bilinear method and Riemann theta function, we succeed in constructing quasi-periodic wave solutions with a generalized form. We also display the asymptotic properties of these quasi-periodic wave solutions and point out the relation between the quasi-periodic wave solutions and the soliton solutions.

Keywords $(2+1)$-dimensional GSWW equation · Hirota bilinear method · Riemann theta function · Quasi-periodic wave solution · Asymptotic analysis · Breather-type soliton · Lump-type soliton

Y. Chen · Z. Liu (✉)
School of Mathematics, South China University of Technology, Guangzhou 510640, People's Republic of China
e-mail: liuzhr@scut.edu.cn

M. Song
Department of Mathematics, Shaoxing University, Shaoxing 312000, People's Republic of China

M. Song
Department of Mathematics, Yuxi Normal University, Yuxi 653100, People's Republic of China
e-mail: songming12_15@163.com

Mathematics Subject Classification 35C07 · 34C25 · 76B25

1 Introduction

A lot of phenomena in physics and engineering can be described by nonlinear partial differential equations. When we try to study the physical mechanism of phenomena in nature which is described by nonlinear partial differential equations, exact solutions often are investigated. Many mathematicians and physicists are interested in looking for solitary wave solutions to these equations. In physics and other fields, these solutions may well describe miscellaneous phenomena, such as solitons and propagation with a finite speed. Thus, they may give a good insight into the physical aspects of the problems. The shallow water equations have been widely applied in hydraulic engineering, ocean and atmospheric modeling. A generalized shallow water wave (GSWW) equation is given by

$$u_t - u_{xxt} - \alpha u u_t - \beta u_x \int^x u_t \mathrm{d}x + u_x = 0. \qquad (1)$$

This equation can be derived from the classical shallow water theory in the so-called Boussinesq approximation [1]. The Hirota bilinear form [2] and a series of exact solutions for Eq. (1) were investigated in [3–9].

The study of soliton for nonlinear equations is of great interest not only in $(1+1)$-dimensional systems, but also in higher-dimensional systems. Other exact solutions also have been investigated by many researchers, and some of powerful methods have been presented, such as the extended Jacobi elliptic function expansion method [10–13], inverse scattering transformation method [14,15], multiple exp-function method [16,17], extended F-expansion method [18], the Hirota method [19–25], $\frac{G'}{G}$-expansion method [26,27], the Weierstrass elliptic function method [28–30] and so on. Nakamura [31,32] proposed a convenient way to construct a kind of quasi-periodic solutions of nonlinear equations in his two serial papers. Recently, Fan and his collaborators [33] have extended this method to investigate the discrete Toda lattice. This approach possesses powerful features that make it practical for the determination of quasi-periodic solutions [34–38].

Many authors had studied the $(2+1)$-dimensional generalized shallow water wave equation

$$v_t + v_{xxx} - 3(vr)_x = 0, \qquad v_x = r_y, \tag{2}$$

which can be reduced to the famous KdV equation if $y = x$. In [39], an inverse scattering scheme was developed to solve the Cauchy problem for Eq. (2). A set of solitary-like solutions for Eq. (2) were acquired by means of a symbolic-computation-based method [40,41]. In [42], the generalized dromion solutions for Eq. (2) were obtained. In [43], the author pointed out that the symmetries of integrable model for Eq. (2) can be obtained from the conformal invariance of its Schwartz form. Lou et al. [44–46] exposed that Eq. (2) is an asymmetric part of the NNV equation and revealed its abundant dromion structures. A series of soliton-like solutions and double-like periodic solutions for Eq. (2) were constructed by the generalized algebraic method in [47]. A series of exact solutions for Eq. (2) were obtained by using a linear variable separation approach and a projective equation in [48]. In [49,50], the authors acquired multi-periodic (quasi-periodic) wave solutions for Eq. (2) by employing Hirota bilinear method and Riemann theta function. In [51], based on the binary Bell polynomials and the bilinear form for Eq. (2), some exact solutions were presented with an arbitrary function in y. In [52], the multi-soliton solutions for Eq. (2) were obtained by means of the multiple exp-function method.

In this work, we investigate the soliton solutions and quasi-periodic wave solutions with an arbitrary function in y for Eq. (2), which have not been reported before. Furthermore, their dynamic properties, interaction mechanisms and limit behavior are analyzed. This paper is organized as the following. In Sect. 2, we present one-soliton solutions via the simplified bilinear method and acquire the rational function solution in virtue of the limit method. By means of the asymptotic analysis and graphical simulations, we reveal the dynamic properties of the solitons and investigate the breather-type and lump-type solitons. In Sect. 3, besides the multiple-soliton solutions are acquired by means of the simplified bilinear method, we also investigate their dynamic properties and interaction mechanisms. Furthermore, the breather-type and lump-type multiple solitons are analyzed. In Sect. 4, we construct Riemann theta function one-periodic wave solutions with a generalized form and establish the relation between the one-periodic solutions and one-soliton solutions. In Sect. 5, we investigate the two-periodic wave solutions similar to one-periodic wave solutions. A short conclusion is given in Sect. 6.

2 One-soliton solution

To obtain the soliton solution directly, we use the simplified version of Hirota bilinear method [23] to study Eq. (2). Letting

$$v = u_y, \qquad r = u_x, \tag{3}$$

then Eq. (2) becomes

$$u_{yt} + u_{xxxy} - 3u_{xx}u_y - 3u_x u_{xy} = 0. \tag{4}$$

Based on the special structure of Eq. (4), we look for the solutions of Eq. (4) with the form

$$u(x, y, t) = \varphi(\xi), \qquad \xi = kx + q\phi(y) - \gamma t + p, \tag{5}$$

where k, q, γ and p are arbitrary constants and $\phi(y)$ is an arbitrary function of y. Since $\varphi(\xi)$ contains an arbitrary function $\phi(y)$, it is different from the previous form. Substituting

$$u(x, y, t) = \mathrm{e}^{\omega_1}, \qquad \omega_1 = k_1 x + q_1\phi(y) - \gamma_1 t + p_1, \tag{6}$$

into the linear terms of Eq. (4), and solving the equation for γ_1, we get $\gamma_1 = k_1^3$. Consequently, the dispersion variable ω_1 becomes

$$\omega_1 = k_1 x + q_1 \phi(y) - k_1^3 t + p_1. \tag{7}$$

Now, we use the transformation

$$u(x, y, t) = R\frac{f_x(x, y, t)}{f(x, y, t)}, \tag{8}$$

to determine R, where

$$f(x, y, t) = 1 + \mathrm{e}^{\omega_1}. \tag{9}$$

Substituting (8) into (4), it follows that $R = -2$. Therefore, a solution for Eq. (4) is given by

$$u(x, y, t) = -k_1 - k_1 \tanh\left(\frac{\omega_1}{2}\right). \tag{10}$$

From (3) and (10), we obtain one-soliton solution for Eq. (2) in the form

$$v(x, y, t) = -\frac{k_1 q_1}{2}\phi'(y)\mathrm{sech}^2\left(\frac{\omega_1}{2}\right), \tag{11}$$

$$r(x, y, t) = -\frac{k_1^2}{2}\mathrm{sech}^2\left(\frac{\omega_1}{2}\right). \tag{12}$$

The dynamic properties for the solitary waves are revealed by mean of the asymptotic analysis and graphical simulations as follows. From (11), we see that the characteristic plane of the wave is defined by (7).

Thus, the following two interesting solitons are acquired by selecting the special $\phi(y)$.

Case 1 Breather-type soliton
Through choosing $\phi(y)$ as periodic function, the breather-type soliton is shown in Fig. 1, where $\phi(y) = \mathrm{sn}(y, 0.3)$ in Fig. 1a and $\phi(y) = \mathrm{cn}(y, 0.9)$ in Fig. 1b.

Case 2 Lump-type soliton
Through choosing some appropriate function $\phi(y)$, the lump-type soliton is shown in Fig. 2, where (a) $\phi(y) = \frac{\mathrm{sn}(y,0.6)}{1+y^2}$ in Fig. 2 and $\phi(y) = \frac{\mathrm{sn}(y,0.6)}{1+y^2[\sin(\ln(y^2+0.001))]}$ in Fig. 2b.

Similarly, applying the Hirota bilinear method to Eq. (2), we get its single singular-soliton solution

$$v(x, y, t) = \frac{k_1 q_1}{2}\phi'(y)\mathrm{csch}^2\left(\frac{\omega_1}{2}\right), \tag{13}$$

$$r(x, y, t) = \frac{k_1^2}{2}\mathrm{csch}^2\left(\frac{\omega_1}{2}\right). \tag{14}$$

Let $q_1 = k_1$, $p_1 = 0$ and $k_1 \to 0$, and we obtain the rational function solution for Eq. (2) in the form

$$v(x, y, t) = \frac{2\phi'(y)}{[x + \phi(y)]^2}, \tag{15}$$

$$r(x, y, t) = \frac{2}{[x + \phi(y)]^2}. \tag{16}$$

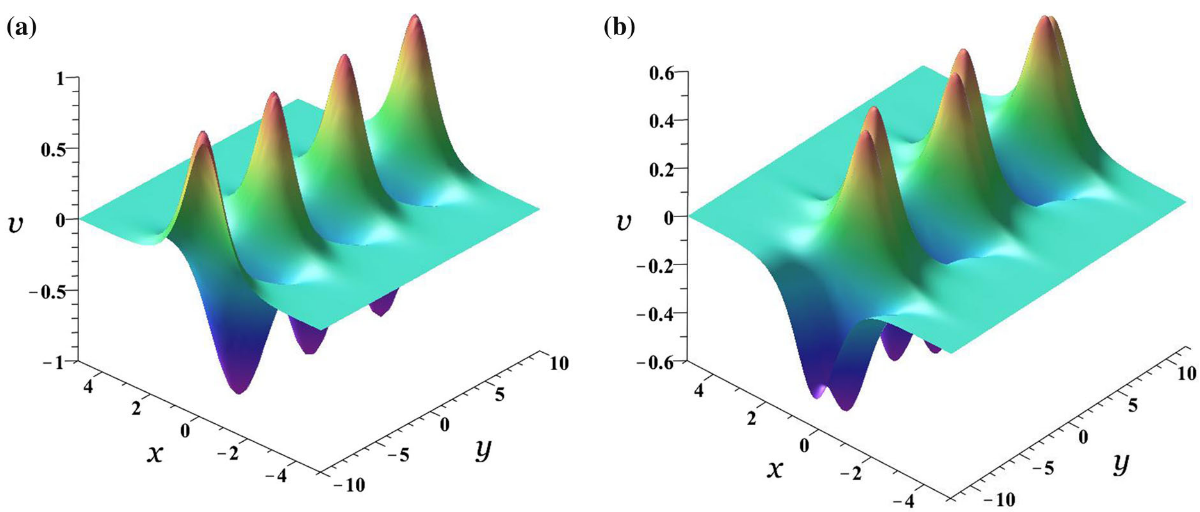

Fig. 1 Steady propagation of $v(x, y, t)$ given by (11) at $t = 0$, where $k_1 = 2, q_1 = 1, p_1 = 0$ and **a** $\phi(y) = \mathrm{sn}(y, 0.3)$; **b** $\phi(y) = \mathrm{cn}(y, 0.9)$

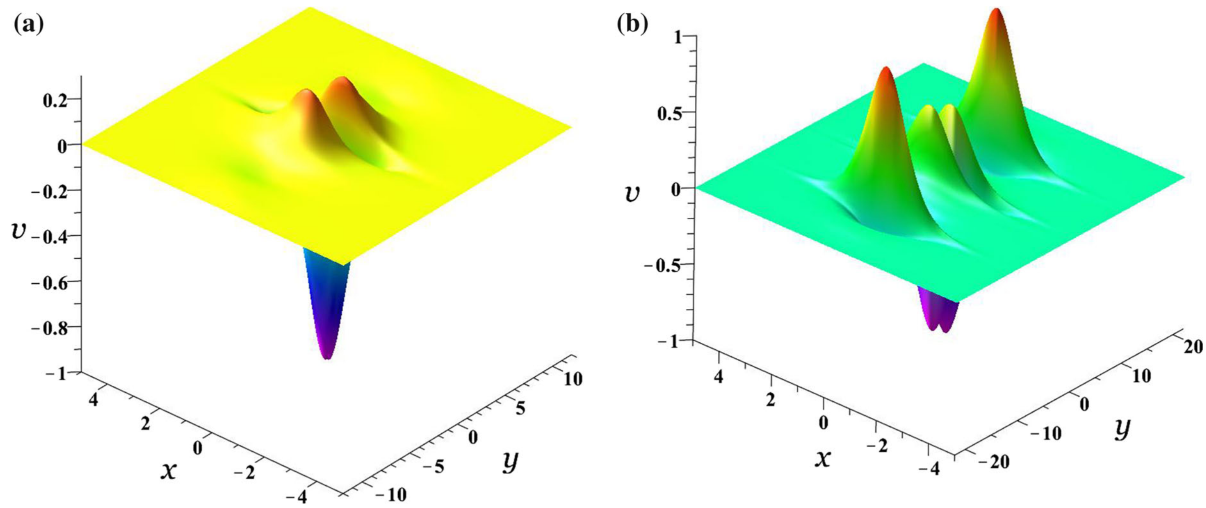

Fig. 2 Steady propagation of $v(x, y, t)$ given by (11) at $t = 0$, where $k_1 = 2, q_1 = 1, p_1 = 0$ and **a** $\phi(y) = \frac{\mathrm{sn}(y,0.6)}{1+y^2}$; **b** $\phi(y) = \frac{\mathrm{sn}(y,0.6)}{1+y^2[\sin(\ln(y^2+0.001))]}$

3 Multiple-soliton solution

To get two-soliton solutions, let

$$u(x, y, t) = -2[\ln f(x, y, t)]_x, \tag{17}$$

where ω_1 is given in (7) and

$$\omega_2 = k_2x + q_2\phi(y) - k_2^3 t + p_2, \tag{18}$$

$$f(x, y, t) = 1 + \mathrm{e}^{\omega_1} + \mathrm{e}^{\omega_2} + a_{12}\mathrm{e}^{\omega_1+\omega_2}. \tag{19}$$

Substituting (17) into (4), the phase shift a_{12} is obtained as

$$a_{12} = \frac{(k_1 - k_2)(q_1 - q_2)}{(k_1 + k_2)(q_1 + q_2)}. \tag{20}$$

These imply that Eq. (2) has two-soliton solution

$$v(x, y, t) = \chi_y, \tag{21}$$

$$r(x, y, t) = \chi_x, \tag{22}$$

where

$$\chi = \frac{-2\left[k_1\mathrm{e}^{\omega_1} + k_2\mathrm{e}^{\omega_2} + a_{12}(k_1 + k_2)\mathrm{e}^{\omega_1+\omega_2}\right]}{1 + \mathrm{e}^{\omega_1} + \mathrm{e}^{\omega_2} + a_{12}\mathrm{e}^{\omega_1+\omega_2}}. \tag{23}$$

Now, we consider the asymptotic property of $v_4(x, y, t)$. First, assume that $a_{12} > 0, k_1 > k_2 > 0$. From (7) and (18), we get the relation between ω_1 and ω_2 as

$$\omega_2 = k_2\left(k_1^2 - k_2^2\right)t + \frac{k_2}{k_1}\omega_1 + \left(q_2 - \frac{k_2}{k_1}q_1\right)\phi(y) + \left(p_2 - \frac{k_2}{k_1}p_1\right). \tag{24}$$

If let $\omega_1 =$ constant, then we get

$$v \to \begin{cases} -\dfrac{k_1q_1}{2}\phi'(y)\mathrm{sech}^2\left(\dfrac{m_1}{2}\right), & t \to +\infty, \\ -\dfrac{k_1q_1}{2}\phi'(y)\mathrm{sech}^2\left(\dfrac{\omega_1}{2}\right), & t \to -\infty, \end{cases} \tag{25}$$

where

$$m_1 = \omega_1 + \ln(a_{12}). \tag{26}$$

Similarly, if we let $\omega_2 =$ constant, then

$$v \to \begin{cases} -\dfrac{k_2q_2}{2}\phi'(y)\mathrm{sech}^2\left(\dfrac{\omega_2}{2}\right), & t \to +\infty, \\ -\dfrac{k_2q_2}{2}\phi'(y)\mathrm{sech}^2\left(\dfrac{m_2}{2}\right), & t \to -\infty, \end{cases} \tag{27}$$

where

$$m_2 = \omega_2 + \ln(a_{12}). \tag{28}$$

In order to understand above asymptotic property intuitively, we give four examples in Fig. 3. Also, we

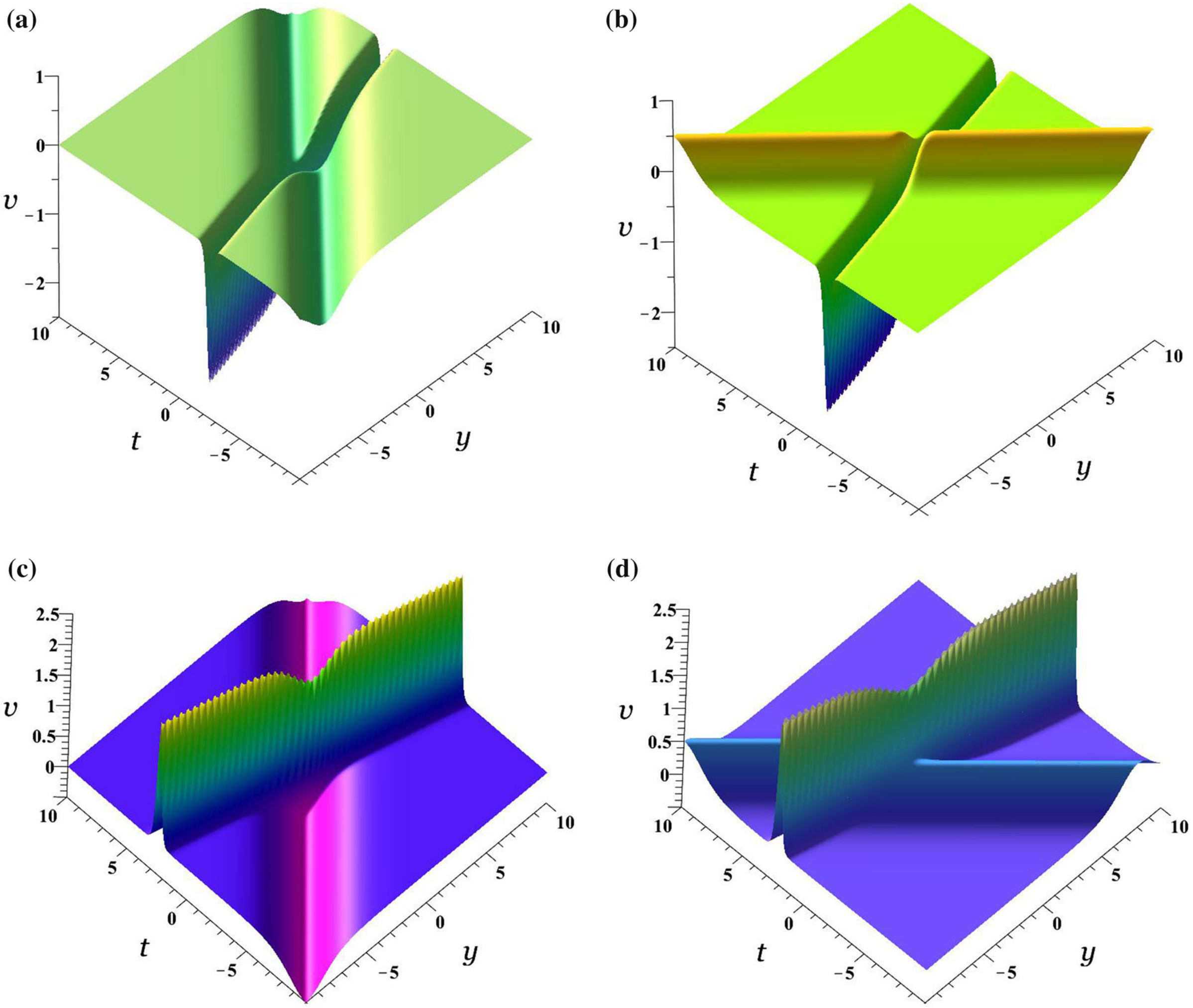

Fig. 3 Surface of the two-soliton via (21) on the $y - t$ plane, where $k_1 = 1, k_2 = 2, p_1 = p_2 = 0, x = 0, \phi(y) = y$ and **a** $q_1 = 1, q_2 = 2$; **b** $q_1 = -1, q_2 = 2$; **c** $q_1 = 1, q_2 = -2$; **d** $q_1 = -1, q_2 = -2$

draw the following conclusions: (1) If $q_1, q_2 > 0$, then the two-soliton consists of two dark solitons, as shown in Fig. 3a. (2) If $q_1 < 0, q_2 > 0$ or $q_1 > 0, q_2 < 0$, then the two-soliton consists of a bright soliton and a dark soliton, as shown in Fig. 3b, c. (3) If $q_1, q_2 < 0$, then the two-soliton consists of two bright solitons, as shown in Fig. 3d. From above figures, phase shifts are evidently shown.

Next, we show that the breather-type two-soliton is composed of two breather-type solitons in Fig. 4. Also, the lump-type two-soliton composed by two lump-type solitons is given in Fig. 5.

Similarly, in order to obtain the three-soliton solution, we substitute

$$
\begin{aligned}
v(x, y, t) &= -2\,[\ln f(x, y, t)]_{xy}\,,\\
r(x, y, t) &= -2\,[\ln f(x, y, t)]_{xx}\,,
\end{aligned}
\tag{29}
$$

$$
\begin{aligned}
f(x, y, t) = 1 &+ e^{\omega_1} + e^{\omega_2} + e^{\omega_3}\\
&+ a_{12}e^{\omega_1+\omega_2} + a_{23}e^{\omega_2+\omega_3}\\
&+ a_{13}e^{\omega_1+\omega_3} + b_{123}e^{\omega_1+\omega_2+\omega_3},
\end{aligned}
\tag{30}
$$

$$
a_{ij} = \frac{(k_i - k_j)(q_i - q_j)}{(k_i + k_j)(q_i + q_j)},
\tag{31}
$$

into (2), we get

$$
b_{123} = a_{12}a_{13}a_{23}.
\tag{32}
$$

To determine the three-soliton solution explicitly, we substitute the last result for $f(x, y, t)$ in the formula (29).

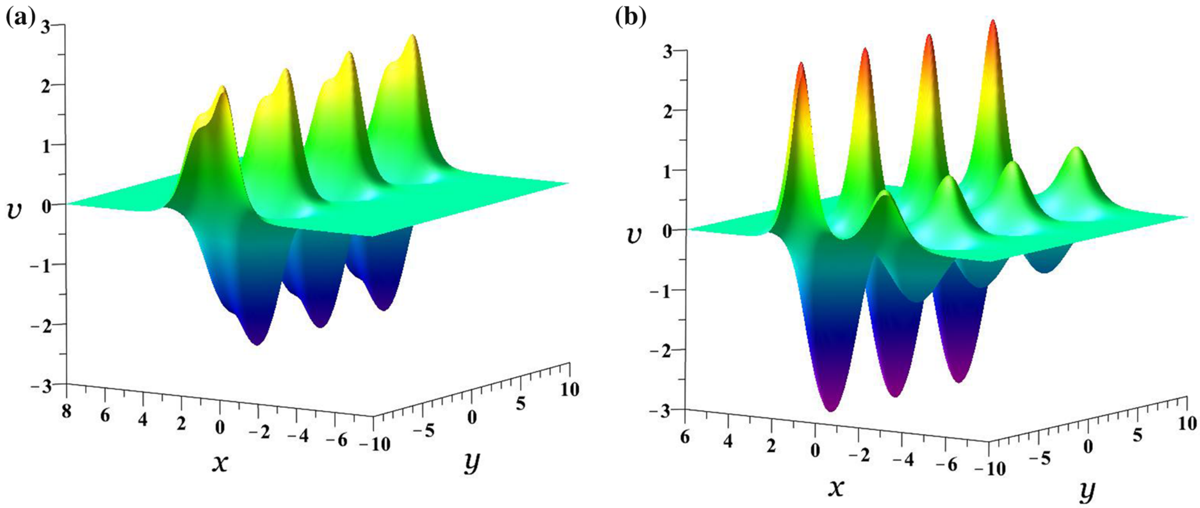

Fig. 4 Steady propagation of the two-soliton via (21) at $t = 0$, where $k_1 = 2, k_2 = 3, q_1 = 1, q_2 = 2, \phi(y) = \mathrm{sn}(y, 0.3)$ and **a** $p_1 = p_2 = 0$; **b** $p_1 = 6,\ p_2 = 0$

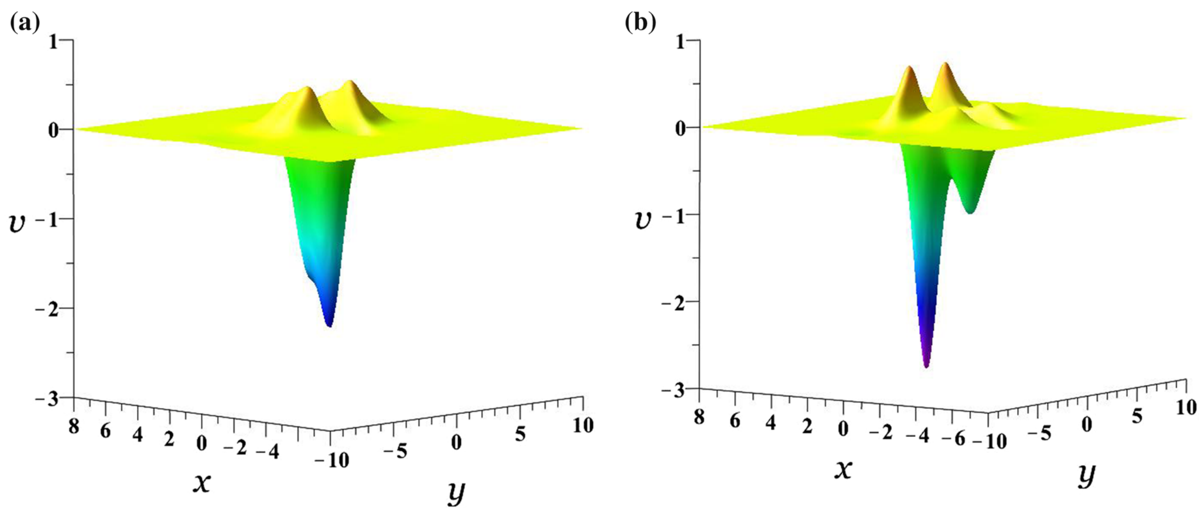

Fig. 5 Steady propagation of the two-soliton via (21) at $t = 0$, where $k_1 = 2, k_2 = 3, q_1 = 1, q_2 = 2, \phi(y) = \frac{\mathrm{sn}(y,0.6)}{1+y^2}$ and **a** $p_1 = p_2 = 0$; **b** $p_1 = 3,\ p_2 = 0$

Similar to the two-soliton, the breather-type three-soliton is composed of three breather-type solitons. The lump-type three-soliton is composed of three lump-type solitons. These phenomena are revealed in Figs. 6 and 7.

4 One-periodic waves and asymptotic properties

In order to obtain our results, we introduce the Riemann theta function

$$\vartheta(\xi, \tau) = \sum_{n \in Z^N} \mathrm{e}^{-\pi <\tau n, n> + 2\pi i <\xi, n>}, \tag{33}$$

here the integer value vector $n = (n_1, \ldots, n_N)^{\mathrm{T}}$, and complex phase variables $\xi = (\xi_1, \ldots, \xi_N)^{\mathrm{T}} \in C^N$. For two vectors $f = (f_1, \ldots, f_N)$ and $g = (g_1, \ldots, g_N)$, their inner product is defined by

$$\langle f, g \rangle = f_1 g_1 + f_2 g_2 + \cdots + f_N g_N. \tag{34}$$

$\tau = (\tau_{ij})$ is a positive definite and real-valued symmetric $N \times N$ matrix, which is called the period matrix of the theta function. The entries τ_{ij} of the period matrix τ can be considered as free parameters of the theta function.

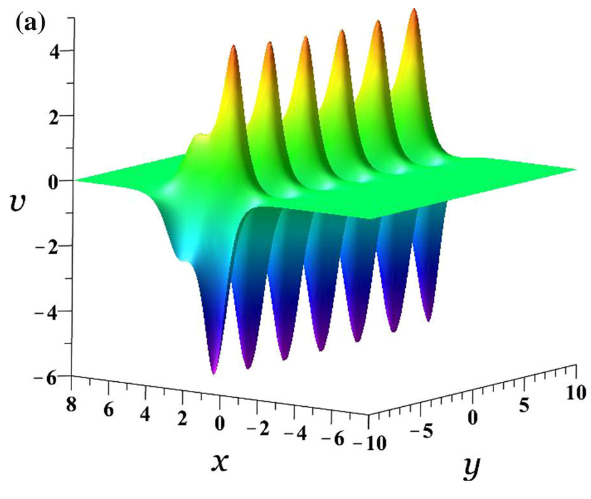

Fig. 6 Steady propagation of the three-soliton via (29) at $t = 0$, where $k_1 = 2, k_2 = 3, k_3 = 5, q_1 = 1, q_2 = 2, q_3 = 3, \phi(y) = \mathrm{sn}(y, 0.3)\,\mathrm{cn}(y, 0.6)$ and **a** $p_1 = p_2 = p_3 = 0$; **b** $p_1 = 6,\ p_2 = 3,\ p_3 = -6$

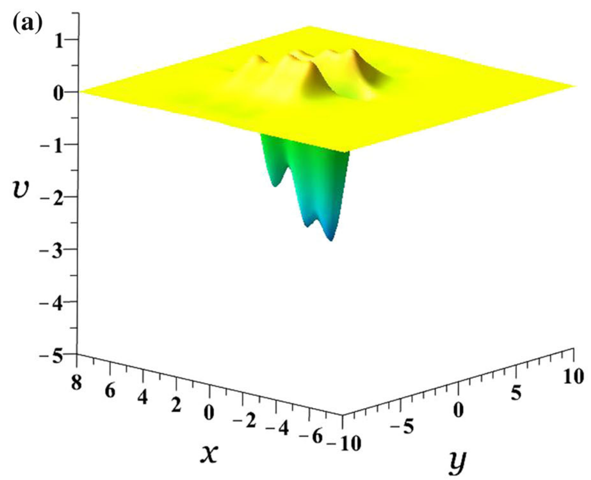

Fig. 7 Steady propagation of the three-soliton via (29) at $t = 0$, where $k_1 = 2, k_2 = 3, k_3 = 5, q_1 = 1, q_2 = 2, q_3 = 3, \phi(y) = \frac{\mathrm{sn}(y,0.6)}{1+y^2}$ and **a** $p_1 = p_2 = p_3 = 0$; **b** $p_1 = 6,\ p_2 = 6,\ p_3 = -6$

For the sake of quasi-periodic waves, we look for solution of Eq. (2) in the form

$$\begin{cases} v = \mu_0\phi'(y) - 2\partial^2_{xy}\ln\vartheta(\xi), \\ r = -2\partial^2_{xx}\ln\vartheta(\xi), \end{cases} \tag{35}$$

where μ_0 is a free constant and $\xi_j = \alpha_j x + \beta_j\phi(y) + \theta_j t + \sigma_j,\ \ j = 1, 2, \ldots, N$.

Substituting (35) into (2) and integrating with respect to x, we obtain the following bilinear form

$$\begin{aligned} &G(D_x, D_y, D_t)\vartheta(\xi)\vartheta(\xi) \\ &= \left(D_t D_y + D_x^3 D_y - 3\mu_0\phi'(y)D_x^2 + c\right)\vartheta(\xi)\vartheta(\xi) \\ &= 0, \end{aligned} \tag{36}$$

where $c = c(y, t)$.

In the following, we consider one-periodic wave solutions of Eq. (2). Firstly, we take $N = 1$, and then Riemann theta function reduces to the following Fourier series in n

$$\vartheta(\xi, \tau) = \sum_{n=-\infty}^{+\infty} \mathrm{e}^{-\pi n^2\tau + 2\pi i n\xi}, \tag{37}$$

where the phase variable $\xi = \alpha x + \beta\phi(y) + \theta t + \sigma$ and the parameter $\tau > 0$. The special case $\phi(y) = y$ has been considered in [50].

In order to make the theta function (37) satisfies the bilinear Eq. (36), we substitute function (37) into the left of Eq. (36), and it follows that

$$\begin{aligned} & G(D_x, D_y, D_t)\vartheta(\xi)\vartheta(\xi) \\ &= \sum_{m=-\infty}^{+\infty}\sum_{n=-\infty}^{+\infty} G(D_x, D_y, D_t) \\ &\quad \times \mathrm{e}^{-\pi m^2\tau+2\pi i m\xi}\mathrm{e}^{-\pi n^2\tau+2\pi i n\xi} \\ &= \sum_{m=-\infty}^{+\infty}\sum_{n=-\infty}^{+\infty} G[2\pi i(n-m)\alpha, 2\pi i(n-m) \\ &\quad \times \beta\phi'(y), 2\pi i(n-m)\theta] \\ &\quad \times \mathrm{e}^{-\pi m^2\tau+2\pi i m\xi}\mathrm{e}^{-\pi n^2\tau+2\pi i n\xi} \\ &= \sum_{m'=-\infty}^{+\infty}\left\{\sum_{n=-\infty}^{+\infty} G[2\pi i(2n-m')\alpha, 2\pi i(2n-m') \right. \\ &\quad \left. \times \beta\phi'(y), 2\pi i(2n-m')\theta] \times \mathrm{e}^{-\pi[n^2+(n-m')^2]\tau}\right\} \\ &\quad \times \mathrm{e}^{2\pi i m'\xi} \qquad (38) \\ &\triangleq \sum_{m'=-\infty}^{+\infty} \overline{G}(m')\mathrm{e}^{2\pi i m'\xi}, \qquad m' = m + n. \end{aligned}$$

In the following, we compute each series $\overline{G}(m')$ for $m' \in Z$. By shifting summation index by $n = n' + 1$, we have the following fact

$$\begin{aligned} \overline{G}(m') &= \overline{G}(m'-2)\mathrm{e}^{-2\pi(m'-1)\tau} \\ &= \cdots = \begin{cases} \overline{G}(0)\mathrm{e}^{-\pi m'^2\frac{\tau}{2}}, & m' \text{ is even}, \\ \overline{G}(1)\mathrm{e}^{-\pi(m'^2-1)\frac{\tau}{2}}, & m' \text{ is odd}, \end{cases} \end{aligned} \qquad (39)$$

which implies that $\overline{G}(m'), m' \in Z$ are completely dominated by two function $\overline{G}(0)$ and $\overline{G}(1)$. If $\overline{G}(0) = \overline{G}(1) = 0$, then it follows that $\overline{G}(m') = 0, m' \in Z$, and thus the theta function (37) is an exact solution to Eq. (36), namely $G(D_x, D_y, D_t)\vartheta(\xi)\vartheta(\xi) = 0$. And we have

$$\begin{aligned} \overline{G}(0) &= \sum_{n=-\infty}^{+\infty}[-16\pi^2 n^2\beta\theta\phi'(y) + 256\pi^4 n^4\beta\alpha^3\phi'(y) \\ &\quad + 48\mu_0\pi^2 n^2\alpha^2\phi'(y) + c]\mathrm{e}^{-2\pi n^2\tau} = 0, \end{aligned}$$

$$\begin{aligned} \overline{G}(1) &= \sum_{n=-\infty}^{+\infty}[-4\pi^2(2n-1)^2\beta\theta\phi'(y) \\ &\quad + 16\pi^4(2n-1)^4\beta\alpha^3\phi'(y) \\ &\quad + 12\mu_0\pi^2(2n-1)^2\alpha^2\phi'(y) + c] \\ &\quad \times \mathrm{e}^{-\pi(2n^2-2n+1)\tau} = 0. \end{aligned} \qquad (40)$$

By introducing the notations as

$$\begin{aligned} a_{11} &= \sum_{n=-\infty}^{+\infty}(-16\pi^2 n^2\beta)\phi'(y)\wp^{2n^2}, \\ a_{12} &= \sum_{n=-\infty}^{+\infty}\wp^{2n^2}, \\ a_{21} &= \sum_{n=-\infty}^{+\infty}[-4\pi^2(2n-1)^2\beta]\phi'(y)\wp^{2n^2-2n+1}, \\ a_{22} &= \wp^{2n^2-2n+1}, \\ b_1 &= -\sum_{n=-\infty}^{+\infty}(256\pi^4 n^4\beta\alpha^3 + 48\mu_0\pi^2 n^2\alpha^2) \\ &\quad \times \phi'(y)\wp^{2n^2}, \\ b_2 &= -\sum_{n=-\infty}^{+\infty}[16\pi^4(2n-1)^4\beta\alpha^3 \\ &\quad + 12\mu_0\pi^2(2n-1)^2\alpha^2]\phi'(y)\wp^{2n^2-2n+1}, \\ \wp &= \mathrm{e}^{-\pi\tau}, \end{aligned} \qquad (41)$$

we simply change Eqs. (40) into a linear system about the frequency θ and c, namely

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} \theta \\ c \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}. \qquad (42)$$

Then, we get a one-periodic wave solution of Eq. (2)

$$v = \mu_0\phi'(y) - 2\partial_{xy}^2 \ln\vartheta(\xi), \qquad r = -2\partial_{xx}^2 \ln\vartheta(\xi), \qquad (43)$$

which provided the vector $(\theta, c)^{\mathrm{T}}$ solves Eq. (42) with the theta function $\vartheta(\xi)$ given by Eq. (37) and parameters θ, c by (42). The other parameters $\alpha, \beta, \tau, \sigma$ and μ_0 are free. Figs. 8, 9 and 10 show one-periodic waves for some choice of the parameters and special $\phi(y)$.

Interestingly, we further consider asymptotic properties of the one-periodic wave solution, and the relation between the one-periodic solution (43) and the

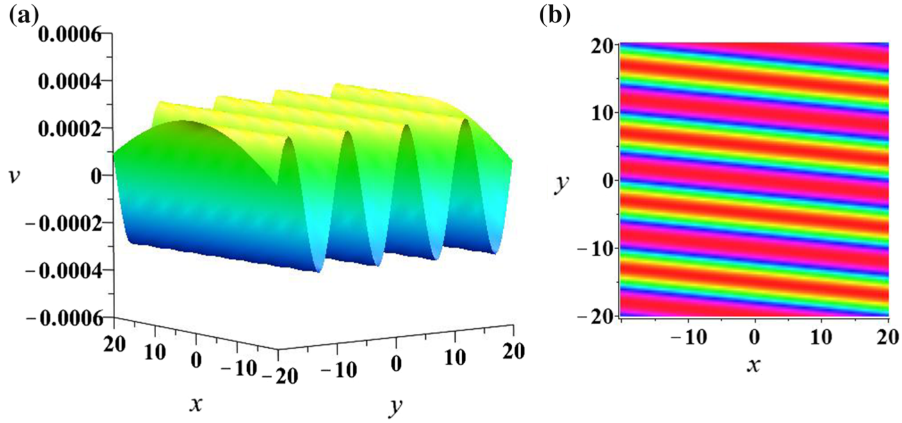

Fig. 8 An one-periodic wave for the GSWW equation, where $\mu_0 = \sigma = 0, \alpha = 0.1, \beta = 0.2, \tau = 0.2$, and $\phi(y) = \sin(y)$. **a** *Perspective view* of the wave and **b** *overhead view* of the wave

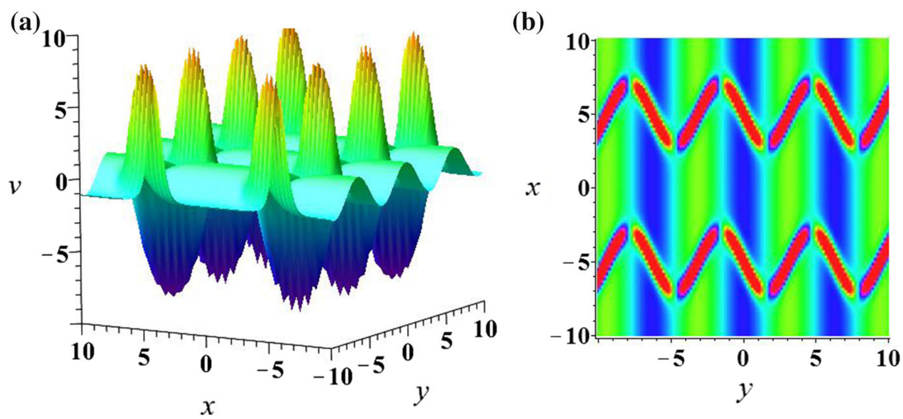

Fig. 9 An one-periodic wave for the GSWW equation with a interesting phenomenon, where $\mu_0 = \sigma = 0, \alpha = 0.1, \beta = 0.2, \tau = 0.2$, and $\phi(y) = \sin(y)$. **a** *Perspective view* of the wave and **b** *overhead view* of the wave

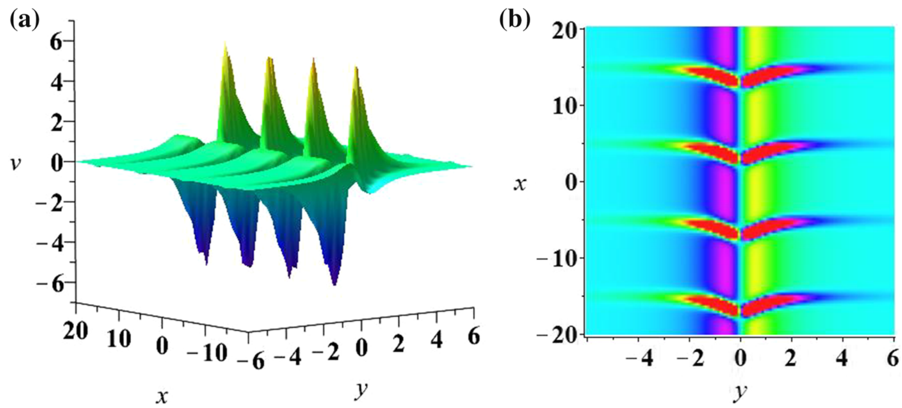

Fig. 10 An one-periodic wave for the GSWW equation with a interesting phenomenon, where $\mu_0 = \sigma = 0, \alpha = 0.1, \beta = 0.2, \tau = 0.2$, and $\phi(y) = \frac{1}{1+y^2}$. **a** *Perspective view* of the wave and **b** *overhead view* of the wave

one-soliton solution (11, 12) can be established as follows. In order to obtain our conclusion clearly, we rewrite the one-soliton solution as following form

$$v = -2\left[\ln\left(1+\mathrm{e}^{\omega}\right)\right]_{xy}, \qquad r = -2\left[\ln\left(1+\mathrm{e}^{\omega}\right)\right]_{xx}, \tag{44}$$

where $\omega = kx + q\phi(y) - k^3 t + p$.

Firstly, we let $\mu_0 = 0$ and write functions $a_{ij}, b_i, i, j = 1, 2$ as the series about $\wp$. We write the coefficient matrix and the right-side vector of system (42) into power series of $\wp$ as

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ -8\pi^2\beta\phi'(y) & 2 \end{pmatrix}\wp + \begin{pmatrix} -32\pi^2\beta\phi'(y) & 2 \\ 0 & 0 \end{pmatrix}\wp^2 + o(\wp^2), \tag{45}$$

$$\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} 0 \\ -32\pi^4\alpha^3\beta\phi'(y) \end{pmatrix} \wp + \begin{pmatrix} -512\pi^4\alpha^3\beta\phi'(y) \\ 0 \end{pmatrix} \wp^2 + o(\wp^2). \tag{46}$$

Substituting (45) and (46) into (42) and comparing the same order of $\wp$, we obtain

$$\theta = 4\pi^2\alpha^3 + o(\wp) \to 4\pi^2\alpha^3, \tag{47}$$

$$c = o(\wp) \to 0, \quad \text{as} \quad \wp \to 0, \tag{48}$$

which implies that

$$2\pi i\theta \to 8\pi^3 i\alpha^3 = -k^3, \quad c \to 0, \quad \text{as } \wp \to 0. \tag{49}$$

To show the one-periodic wave (43) degenerates to the one-soliton solution (44) under the limit $\wp \to 0$, we first expand the periodic function $\vartheta(\xi)$ in the form

$$\vartheta(\xi, \tau) = 1 + (e^{2\pi i\xi} + e^{-2\pi i\xi})\wp + (e^{4\pi i\xi} + e^{-4\pi i\xi})\wp^4 + \cdots \tag{50}$$

By using the following transformation

$$\mu_0 = 0, \quad \alpha = \frac{k}{2\pi i}, \quad \beta = \frac{q}{2\pi i}, \quad \sigma = \frac{p + \pi\tau}{2\pi i}, \tag{51}$$

we have

$$\vartheta(\xi, \tau) = 1 + e^{\widetilde{\xi}} + (e^{-\widetilde{\xi}} + e^{2\widetilde{\xi}})\wp^2 + (e^{-2\widetilde{\xi}} + e^{3\widetilde{\xi}})\wp^6 + \cdots \to 1 + e^{\widetilde{\xi}}, \text{as } \wp \to 0, \tag{52}$$

where

$$\widetilde{\xi} = 2\pi i\xi - \pi\tau. \tag{53}$$

Combining (49) and (53) deduces that

$$\widetilde{\xi} \to \omega, \quad \text{as} \quad \wp \to 0, \tag{54}$$

$$\xi \to \frac{\omega + \pi\tau}{2\pi i}, \quad \text{as} \quad \wp \to 0. \tag{55}$$

So we can acquire

$$\vartheta(\xi, \tau) \to 1 + e^{\omega}, \quad \text{as} \quad \wp \to 0. \tag{56}$$

From above, we conclude that the one-periodic solution (43) just goes to the one-soliton solution (44) as the amplitude $\wp \to 0$.

5 Two-periodic waves and asymptotic properties

In this section, we consider two-periodic wave solutions of Eq. (2) and their asymptotic property. In order to obtain two-periodic solution, we consider the case of $N = 2$ and the Riemann theta function (37) takes the form

$$\vartheta(\xi, r) = \vartheta(\xi_1, \xi_2, r) = \sum_{n\in Z^2} e^{-\pi\langle\tau n, n\rangle + 2\pi i\langle\xi, n\rangle}, \tag{57}$$

where $n = (n_1, n_2)^{\mathrm{T}} \in Z^2$, $\xi = (\xi_1, \xi_2)^{\mathrm{T}} \in C^2$, $\xi_j = \alpha_j x + \beta_j\phi(y) + \theta_j t + \sigma_j$, $j = 1, 2$. Here, τ is a positive definite and real-valued symmetric 2×2 matrix, which can be taken of the form

$$\tau = \begin{pmatrix} \tau_{11} & \tau_{12} \\ \tau_{12} & \tau_{22} \end{pmatrix}, \quad \tau_{11} > 0, \quad \tau_{22} > 0, \quad \tau_{11}\tau_{22} - \tau_{12}^2 > 0. \tag{58}$$

To make the theta function (57) satisfies the bilinear Eq. (36), we substitute function (57) into the left of Eq. (36) and obtain that

$$\begin{aligned} &G(D_x, D_y, D_t)\vartheta(\xi_1, \xi_2, \tau)\vartheta(\xi_1, \xi_2, \tau) \\ &= \sum_{m,n\in Z^2} G[2\pi i\langle n - m, \alpha\rangle, 2\pi i\langle n - m, \beta\rangle\phi'(y), \\ &\quad \times 2\pi i\langle n - m, \theta\rangle] \\ &\quad \times e^{-\pi(\langle\tau m, m\rangle + \langle\tau n, n\rangle) + 2\pi i\langle\xi, n+m\rangle} \\ &= \sum_{m'\in Z^2} \left\{ \sum_{n\in Z^2} G[2\pi i\langle 2n - m', \alpha\rangle, \right. \\ &\quad \times 2\pi i\langle 2n - m', \beta\rangle\phi'(y), 2\pi i\langle 2n - m', \theta\rangle] \\ &\quad \left. \times e^{-\pi[\langle\tau n, n\rangle + \langle\tau(n-m'), (n-m')\rangle]} \right\} \times e^{2\pi i\langle\xi, m'\rangle} \\ &\triangleq \sum_{m'\in Z^2} \overline{G}(m_1', m_2') e^{2\pi i\langle\xi, m'\rangle}, \quad m' = m + n. \end{aligned} \tag{59}$$

In the following, we compute each series $\overline{G}(m_1', m_2')$ for $m_1', m_2' \in Z^2$. By shifting summation index by $n = n' + \delta_{jk}$, $k = 1, 2$, we obtain that

$$\begin{aligned} \overline{G}(m_1', m_2') = &\sum_{n\in Z^2} G[2\pi i\langle 2n - m', \alpha\rangle, \\ &\times 2\pi i\langle 2n - m', \beta\rangle\phi'(y), \\ &\times 2\pi i\langle 2n - m', \theta\rangle] \\ &\times e^{-\pi[\langle\tau n, n\rangle + \langle\tau(n-m'), (n-m')\rangle]} \end{aligned}$$

$$= \sum_{n\in Z^2} G\left[2\pi i \sum_{j=1}^{2}(2n'_j - (m'_j - 2\delta_{jk}))\alpha_j,\right.$$
$$\times 2\pi i \sum_{j=1}^{2}(2n'_j - (m'_j - 2\delta_{jk}))\beta_j \phi'(y),$$
$$\left.\times 2\pi i \sum_{j=1}^{2}(2n'_j - (m'_j - 2\delta_{jk}))\theta_j\right]$$
$$\times e^{-\pi\sum_{j,l=1}^{2}[(n'_j+\delta_{jk})(n'_l+\delta_{lk})+(m'_j-n'_j-\delta_{jk})(m'_l-n'_l-\delta_{lk})]\tau_{jl}}$$
$$= \begin{cases}\overline{G}(m'_1-2, m'_2)e^{2\pi(1-m'_1)\tau_{11}-2\pi m'_2\tau_{12}}, & k=1,\\ \overline{G}(m'_1, m'_2-2)e^{2\pi(1-m'_2)\tau_{22}-2\pi m'_1\tau_{12}}, & k=2,\end{cases} \quad (60)$$

where δ_{ij} representing Kronecker's delta. If

$$\overline{G}(0,0)=0,\quad \overline{G}(1,0)=0,\quad \overline{G}(0,1)=0,$$
$$\overline{G}(1,1)=0, \quad (61)$$

then it follows that $\overline{G}(m', m'') = 0, m', m'' \in Z$, and thus the theta function (57) is an exact solution to Eq. (36), namely $G(D_x, D_y, D_t)\vartheta(\xi_1,\xi_2)\vartheta(\xi_1,\xi_2) = 0$. So we get

$$\sum_{n\in Z^2} G(2\pi i\langle 2n-\varrho_1,\alpha\rangle, 2\pi i\langle 2n-\varrho_1,\beta\rangle\phi'(y),$$
$$\times 2\pi i\langle 2n-\varrho_1,\theta\rangle)$$
$$\times e^{-\pi[\langle\tau(n-\varrho_1),n-\varrho_1\rangle+\langle\tau n,n\rangle]} = 0, \quad (62)$$
$$\sum_{n\in Z^2} G(2\pi i\langle 2n-\varrho_2,\alpha\rangle, 2\pi i\langle 2n-\varrho_2,\beta\rangle\phi'(y),$$
$$\times 2\pi i\langle 2n-\varrho_2,\theta\rangle)$$
$$\times e^{-\pi[\langle\tau(n-\varrho_2),n-\varrho_2\rangle+\langle\tau n,n\rangle]} = 0, \quad (63)$$
$$\sum_{n\in Z^2} G(2\pi i\langle 2n-\varrho_3,\alpha\rangle, 2\pi i\langle 2n-\varrho_3,\beta\rangle\phi'(y),$$
$$\times 2\pi i\langle 2n-\varrho_3,\theta\rangle)$$
$$\times e^{-\pi[\langle\tau(n-\varrho_3),n-\varrho_3\rangle+\langle\tau n,n\rangle]} = 0, \quad (64)$$
$$\sum_{n\in Z^2} G(2\pi i\langle 2n-\varrho_4,\alpha\rangle, 2\pi i\langle 2n-\varrho_4,\beta\rangle\phi'(y),$$
$$\times 2\pi i\langle 2n-\varrho_4,\theta\rangle)$$
$$\times e^{-\pi[\langle\tau(n-\varrho_4),n-\varrho_4\rangle+\langle\tau n,n\rangle]} = 0, \quad (65)$$

where $\varrho_j = (\varrho_j^1, \varrho_j^2)^{\mathrm{T}}$, $\varrho_1 = (0,0)^{\mathrm{T}}$, $\varrho_2 = (1,0)^{\mathrm{T}}$, $\varrho_3 = (0,1)^{\mathrm{T}}$, $\varrho_4 = (1,1)^{\mathrm{T}}$, $j = 1, 2, 3, 4$. By introducing the notations as

$$H = (h_{jk})_{4\times 4},\qquad b = (b_1, b_2, b_3, b_4)^{\mathrm{T}},$$

$$h_{j1} = -4\pi^2 \sum_{(n_1,n_2)\in Z^2}\langle 2n-\delta^j,\beta\rangle(2n_1-\delta_1^j)$$
$$\times\phi'(y)\lambda_j(n),$$
$$h_{j2} = -4\pi^2 \sum_{(n_1,n_2)\in Z^2}\langle 2n-\delta^j,\beta\rangle(2n_1-\delta_2^j)$$
$$\times\phi'(y)\lambda_j(n),$$
$$h_{j3} = 12\pi^2 \sum_{(n_1,n_2)\in Z^2}\langle 2n-\delta^j,\alpha\rangle^2$$
$$\times\phi'(y)\lambda_j(n),$$
$$h_{j4} = \sum_{(n_1,n_2)\in Z^2}\lambda_j(n),$$
$$b_j = -16\pi^4 \sum_{(n_1,n_2)\in Z^2}\langle 2n-\delta^j,\alpha\rangle^3\langle 2n-\delta^j,\beta\rangle$$
$$\times\phi'(y)\lambda_j(n),$$

$$\lambda_j(n) = \wp_1^{n_1^2+(n_1-\delta_1^j)^2}\wp_2^{n_2^2+(n_2-\delta_2^j)^2}\wp_3^{n_1n_2+(n_1-\delta_1^j)(n_2-\delta_2^j)},$$
$$\wp_1 = e^{-\pi\tau_{11}},\qquad \wp_2 = e^{-\pi\tau_{22}},\qquad \wp_3 = e^{-2\pi\tau_{12}},$$
$$\text{where } j = 1,2,3,4. \quad (66)$$

Eqs. (62–65) can be written as a linear system

$$H(\theta_1,\theta_2,\mu_0,c)^{\mathrm{T}} = b. \quad (67)$$

Then, we get a two-periodic wave solution of Eq. (2)

$$\begin{cases} v = \mu_0\phi'(y) - 2\partial_{xy}^2\ln\vartheta(\xi_1,\xi_2),\\ r = -2\partial_{xx}^2\ln\vartheta(\xi_1,\xi_2),\end{cases} \quad (68)$$

where $\vartheta(\xi_1,\xi_2,\tau)$ and parameters θ_1, θ_2, μ_0, c are given by Eqs. (57) and (67), respectively. The other parameters α_1, α_2, β_1, β_2, σ_1, σ_2, τ_{11}, τ_{12} and τ_{22} are free. Figs. 11, 12, 13 and 14 show the two-periodic waves for different choice of the parameters and special $\phi(y)$.

Finally, we further consider asymptotic properties of the two-periodic wave solution. In a similar way to one-periodic solution, we rewrite the two-soliton solution as the following form

$$\begin{cases} v = -2\left[\ln\left(1+e^{\omega_1}+e^{\omega_2}+e^{\omega_1+\omega_2+A_{12}}\right)\right]_{xy},\\ r = -2\left[\ln\left(1+e^{\omega_1}+e^{\omega_2}+e^{\omega_1+\omega_2+A_{12}}\right)\right]_{xx},\end{cases} \quad (69)$$

where $A_{12} = \ln(a_{12})$ and $\omega_1, \omega_2, a_{12}$ are given in (7, 18) and (20).

Furthermore, we establish the relation between two-periodic solution (68) and the two-soliton solution (69) as follows.

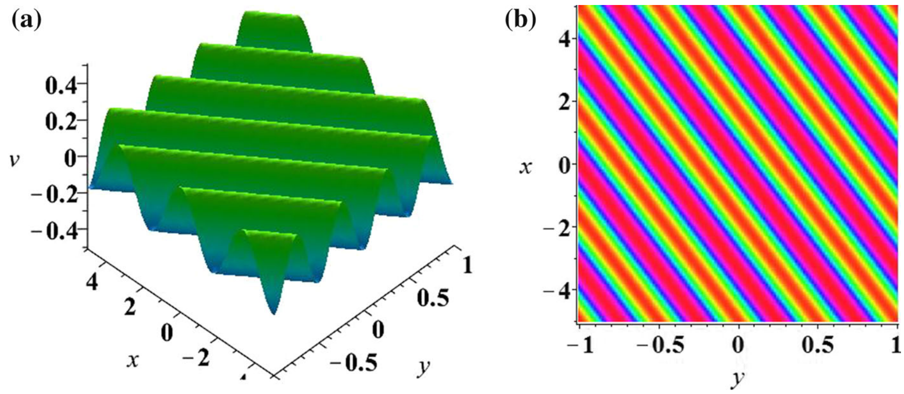

Fig. 11 A degenerate two-periodic wave for the GSWW equation, where $\frac{\alpha_1}{\alpha_2} = \frac{\beta_1}{\beta_2}$ and $\mu_0 = \sigma = 0, \alpha_1 = 0.3, \alpha_2 = 0.03, \beta_1 = 2, \beta_2 = 0.2, \tau_{11} = 2, \tau_{12} = 0.2, \tau_{22} = 2, \phi(y) = y$. This *figure* shows that the degenerate two-periodic wave is almost one dimensional. **a** *Perspective view* of the wave and **b** *overhead view* of the wave

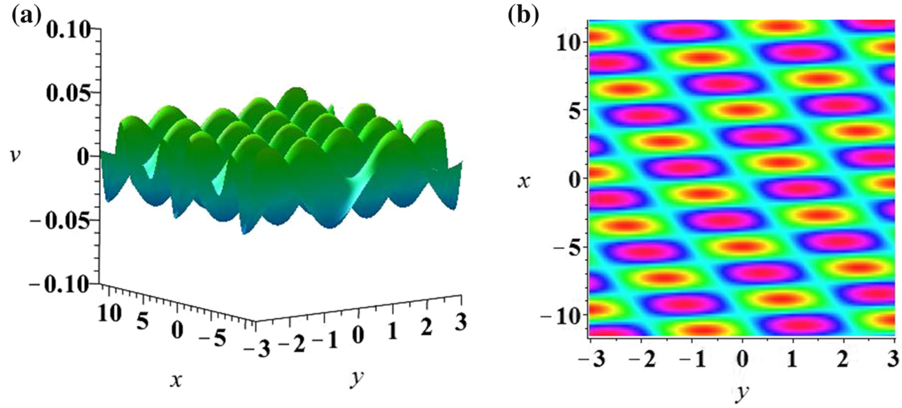

Fig. 12 An asymmetric two-periodic wave for the GSWW equation, where $\mu_0 = \sigma = 0, \alpha_1 = 0.3, \alpha_2 = 0.2, \beta_1 = 0.2, \beta_2 = -0.3, \tau_{11} = 2, \tau_{12} = 0.2, \tau_{22} = 2$ and $\phi(y) = y$. This *figure* shows that every asymmetric two-periodic wave is spatially periodic in two directions, but it need not be periodic in either the x- or y-direction. **a** *Perspective view* of the wave and **b** *overhead view* of the wave

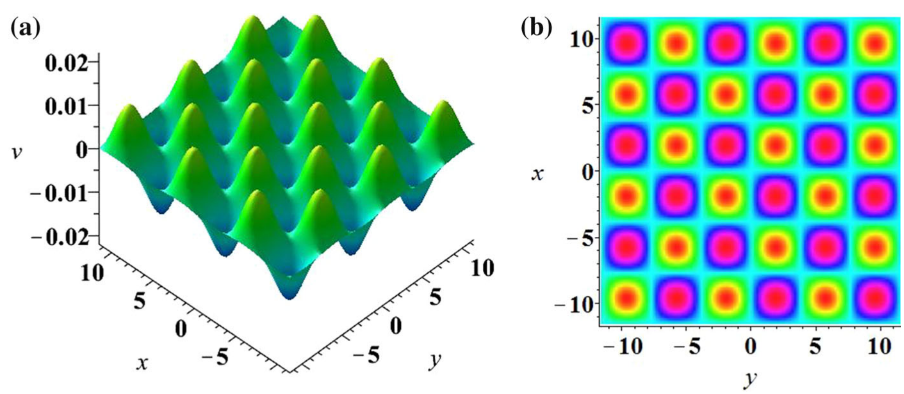

Fig. 13 An symmetric two-periodic wave for the GSWW equation, where $\mu_0 = \sigma = 0, \alpha_1 = 0.13, \alpha_2 = 0.13, \beta_1 = 0.13, \beta_2 = -0.13, \tau_{11} = 2, \tau_{12} = 0.2, \tau_{22} = 2$ and $\phi(y) = y$. This *figure* shows that the symmetric two-periodic wave is periodic both in x- or y-direction. **a** *Perspective view* of the wave and **b** *overhead view* of the wave

Firstly, we expand the periodic function $\vartheta(\xi_1, \xi_2)$ in the following form

$$\begin{aligned}\vartheta(\xi_1, \xi_2, \tau) = 1 &+ \left(e^{2\pi i \xi_1} + e^{-2\pi i \xi_1}\right) e^{-\pi \tau_{11}} \\ &+ \left(e^{2\pi i \xi_2} + e^{-2\pi i \xi_2}\right) e^{-\pi \tau_{22}} \\ &+ \left(e^{2\pi i(\xi_1+\xi_2)} + e^{-2\pi i(\xi_1+\xi_2)}\right) \\ &\times e^{-\pi(\tau_{11}+2\tau_{12}+\tau_{22})}.\end{aligned} \tag{70}$$

By using the following transformation

$$\begin{aligned}&\mu_0 = 0, \quad \alpha_j = \frac{k_j}{2\pi i}, \quad \beta_j = \frac{q_j}{2\pi i}, \\ &\sigma_j = \frac{p_j + \pi \tau_{jj}}{2\pi i}, \quad \tau_{12} = \frac{A_{12}}{2\pi i}, \quad j = 1, 2,\end{aligned} \tag{71}$$

we get

$$\begin{aligned}\vartheta(\xi_1, \xi_2, \tau) = 1 &+ e^{\widetilde{\xi}_1} + e^{\widetilde{\xi}_2} + e^{\widetilde{\xi}_1+\widetilde{\xi}_2-2\pi\tau_{12}} + \wp_1^2 e^{-\widetilde{\xi}_1} \\ &+ \wp_2^2 e^{-\widetilde{\xi}_2} + \wp_1^2 \wp_2^2 e^{-\widetilde{\xi}_1-\widetilde{\xi}_2-2\pi\tau_{12}}\end{aligned}$$

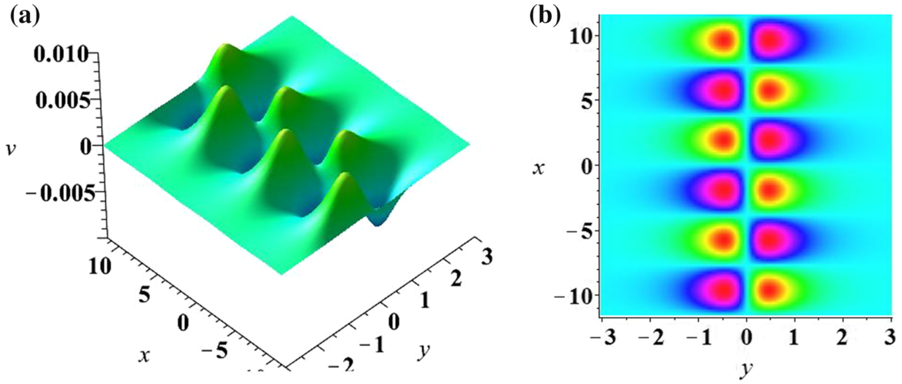

Fig. 14 An symmetric two-periodic wave for the GSWW equation with a interesting phenomenon, where $\mu_0 = \sigma = 0$, $\alpha_1 = 0.13, \alpha_2 = 0.13, \beta_1 = 0.13, \beta_2 = -0.13$, $\tau_{11} = 2, \tau_{12} = 0.2, \tau_{22} = 2$ and $\phi(y) = \frac{1}{1+y^2}$. **a** *Perspective view* of the wave and **b** *overhead view* of the wave

$$+\cdots \to 1 + e^{\widetilde{\xi}_1} + e^{\widetilde{\xi}_2} + e^{\widetilde{\xi}_1+\widetilde{\xi}_2+a_{12}},\quad \text{as } \wp_1,\ \wp_2 \to 0, \tag{72}$$

where

$$\widetilde{\xi}_j = 2\pi i \xi_j - \pi\tau_{jj} = k_j x + q_j\phi(y) + \widetilde{\theta}_j t + p_j, \tag{73}$$

$$\widetilde{\theta}_j = 2\pi i \theta_j, \quad j = 1, 2. \tag{74}$$

From above, we can expand each function in a_{jk}, b_k, $k = 1, 2, 3, 4$ into a series with $\wp_1, \wp_2$. Actually, we only need to make the first-order expansions of matrix M and vector b with $\wp_1, \wp_2$ to show the asymptotic relations. Here, we consider their second-order expansions to see relations among parameters for the two-periodic solution and the two-soliton solution (69). The expansions for the matrix M and the vector b are given by

$$H = \begin{pmatrix} 0&0&0&1\\ 0&0&0&0\\ 0&0&0&0\\ 0&0&0&0 \end{pmatrix} + \begin{pmatrix} 0&0&0&0\\ -8\pi^2\beta_1\phi'(y)&0&24\pi^2\alpha_1^2\phi'(y)&2\\ 0&0&0&0\\ 0&0&0&0 \end{pmatrix}\wp_1$$
$$+ \begin{pmatrix} 0&0&0&0\\ 0&0&0&0\\ 0&-8\pi^2\beta_2\phi'(y)&24\pi^2\alpha_2^2\phi'(y)&2\\ 0&0&0&0 \end{pmatrix}\wp_2$$
$$+ \begin{pmatrix} -32\pi^2\beta_1\phi'(y)&0&96\pi^2\alpha_1^2\phi'(y)&2\\ 0&0&0&0\\ 0&0&0&0\\ 0&0&0&0 \end{pmatrix}\wp_1^2$$
$$+ \begin{pmatrix} 0&-32\pi^2\beta_2\phi'(y)&96\pi^2\alpha_2^2\phi'(y)&2\\ 0&0&0&0\\ 0&0&0&0\\ 0&0&0&0 \end{pmatrix}\wp_2^2$$
$$+ \begin{pmatrix} 0&0&0&0\\ 0&0&0&0\\ 0&0&0&0\\ -8\pi^2(\beta_1-\beta_2)\phi'(y)&\Delta_1\phi'(y)&24\pi^2(\alpha_1-\alpha_2)^2\phi'(y)&2 \end{pmatrix}$$
$$\times \wp_1\wp_2 + o(\wp_1^i\wp_2^j), \tag{75}$$

where $\Delta_1 = 8\pi^2(\beta_1-\beta_2) - 8\pi^2(\beta_1+\beta_2)\lambda_3$, $i+j \geq 2$.

$$b = \begin{pmatrix} 0\\ -32\pi^4\alpha_1^3\beta_1\phi'(y)\\ 0\\ 0 \end{pmatrix}\wp_1 + \begin{pmatrix} 0\\ 0\\ -32\pi^4\alpha_2^3\beta_2\phi'(y)\\ 0 \end{pmatrix}\wp_2$$
$$+ \begin{pmatrix} -512\pi^4\alpha_1^3\beta_1\phi'(y)\\ 0\\ 0\\ 0 \end{pmatrix}\wp_1^2 + \begin{pmatrix} 0\\ -512\pi^4\alpha_2^3\beta_2\phi'(y)\\ 0\\ 0 \end{pmatrix}\wp_2^2$$
$$+ \begin{pmatrix} 0\\ 0\\ 0\\ \Delta_2\phi'(y) \end{pmatrix}\wp_1\wp_2 + o(\wp_1^i\wp_2^j), \tag{76}$$

where $\Delta_2 = -32\pi^4(\alpha_1+\alpha_2)^3(\beta_1+\beta_2)\lambda_3 - 32\pi^4(\alpha_1-\alpha_2)^3(\beta_1-\beta_2)$, $i+j \geq 2$. We also assume the solution of system (67) in the following form

$$\begin{pmatrix} \theta_1\\ \theta_2\\ \mu_0\\ c \end{pmatrix} = \begin{pmatrix} \theta_1^{(0)}\\ \theta_2^{(0)}\\ 0\\ c^{(0)} \end{pmatrix} + \begin{pmatrix} \theta_1^{(1)}\\ \theta_2^{(1)}\\ 0\\ c^{(1)} \end{pmatrix}\wp_1 + \begin{pmatrix} \theta_1^{(2)}\\ \theta_2^{(2)}\\ 0\\ c^{(2)} \end{pmatrix}\wp_2$$
$$+ \begin{pmatrix} \theta_1^{(11)}\\ \theta_2^{(11)}\\ 0\\ c^{(11)} \end{pmatrix}\wp_1^2 + \begin{pmatrix} \theta_1^{(22)}\\ \theta_2^{(22)}\\ 0\\ c^{(22)} \end{pmatrix}\wp_2^2 + \begin{pmatrix} \theta_1^{(12)}\\ \theta_2^{(12)}\\ 0\\ c^{(12)} \end{pmatrix}$$
$$\times \wp_1\wp_2 + o(\wp_1^i\wp_2^j), \quad i+j \geq 2. \tag{77}$$

Substituting (75–77) into (67) and comparing the same order of $\wp$, we obtain

$$c^{(0)} = c^{(1)} = c^{(2)} = c^{(12)} = 0,$$
$$\beta_1\theta_1^{(0)} = 4\pi^2\alpha_1^3\beta_1, \quad \beta_2\theta_3^{(0)} = 4\pi^2\alpha_2^3\beta_2,$$
$$\beta_1\theta_1^{(1)} = 0, \quad \beta_2\theta_2^{(1)} = 0,$$
$$c^{(11)} - 32\pi^2\beta_1\theta_1^{(0)} = -512\pi^4\alpha_1^3\beta_1,$$
$$c^{(22)} - 32\pi^2\beta_2\theta_2^{(0)} = -512\pi^4\alpha_2^3\beta_2,$$
$$\cdots \tag{78}$$

so we can obtain

$$
\begin{aligned}
c &= o(\wp_1, \wp_2) \to 0, \\
\theta_1 &= 4\pi^2\alpha_1^3 + o(\wp_1, \wp_2) \to 4\pi^2\alpha_1^3, \\
\theta_2 &= 4\pi^2\alpha_2^3 + o(\wp_1, \wp_2) \to 4\pi^2\alpha_2^3, \\
&\quad \text{as } \wp_1, \wp_2 \to 0.
\end{aligned} \tag{79}
$$

From above, we conclude that the two-periodic solution (68) tends to the two-soliton solution (69) as $\wp_1, \wp_2 \to 0$.

6 Conclusion

In this work, we conduct an analysis on a (2 + 1)-dimensional shallow water wave equation. In the light of the construction of the Eq. (4), the new special multiple-soliton solutions and the singular-soliton solutions are acquired by means of the Hirota method. Also, the rational function solution is obtained through the limit method. By means of the graphic analysis and asymptotic analysis, dynamic properties and interaction mechanisms for the solitons are revealed. Furthermore, the breather-type and lump-type solitons are obtained for the certain $\phi(y)$. The higher level soliton solutions, for $n \geq 4$ can be obtained in a parallel manner. And their dynamic properties and asymptotic analysis can be discussed similarly.

Using the Hirota method and Riemann theta function, we construct quasi-periodic wave solutions with a generalized form. Besides, the asymptotic analysis of the quasi-periodic(multi-periodic) wave solution is presented and the relation between the quasi-periodic solutions and soliton solutions acquired in this paper are rigorously established. Furthermore, all the solutions we obtained above can be verified by Maple. Also, the results perhaps can be extended to the case when $N > 2$, but there are still certain numerical difficulties in the calculation. We will consider it in our future work.

Acknowledgments The authors would like to express their sincere thanks to Prof. Liming Ling for his enthusiastic guidance. The work was supported by the National Natural Science foundation of China (Nos. 11171115 and 11361069).

References

1. Whitham, G.B.: Linear and Nonlinear Waves. Wiley, New York (1974)
2. Hirota, R.: Solitons. Springer, Berlin (1980)
3. Hietarinta, J.: Partially Integrable Evolution Equations in Physics. Kluwer, Dordrecht (1990)
4. Clarkson, P.A., Mansfield, E.L.: On a shallow water wave equation. Nonlinearity **7**, 975–1000 (1994)
5. Elwakil, S.A., El-Labany, S.K., Zahran, M.A., Sabry, R.: Exact travelling wave solutions for the generalized shallow water wave equation. Chaos Solitons Fractals **17**, 121–126 (2003)
6. Inc, M., Ergut, M.: Periodic wave solutions for the generalized shallow water wave equation by the improved Jacobi elliptic function method. Appl. Math. E-Notes **5**, 89–96 (2005)
7. Wazwaz, A.M.: Solitary wave solutions of the generalized shallow water wave (GSWW) equation by Hirota's method, tanh-coth method and Exp-function method. Appl. Math. Comput. **202**, 275–286 (2008)
8. Borhanifar, A., Zamiri, A., Kabir, M.M.: Exact traveling wave solution for the generalized shallow water wave (GSWW) equation. Middle East J. Sci. Res. **10**, 310–315 (2011)
9. Jiang, Y., Tian, B., Li, M., Wang, P.: Bilinearization and soliton solutions for some nonlinear evolution equations in fluids via the Bell polynomials and auxiliary functions. Phys. Scr. **88**, 025004 (2013)
10. Wen, X.Y.: Extended Jacobi elliptic function expansion solutions of variant Boussinesq equations. Appl. Math. Comput. **217**, 2808–2820 (2010)
11. Hong, B.J., Lu, D.C.: New Jacobi elliptic function-like solutions for the general KdV equation with variable coefficients. Math. Comput. Model. **55**, 1594–1600 (2012)
12. Bhrawy, A.H., Abdelkawy, M.A., Biswas, A.: Cnoidal and snoidal wave solutions to coupled nonlinear wave equations by the extended Jacobis elliptic function method. Commun. Nonlinear Sci. Numer. Simulat. **18**, 915–925 (2013)
13. Bhrawy, A.H., Abdelkawy, M.A., Hilal, E.M., Alshaery, A.A., Biswas, A.: Solitons, cnoidal waves, snoidal waves and other solutions to Whitham–Broer–Kaup system. Appl. Math. inf. sci. **8**, 2119–2128 (2014)
14. Constantin, A., Ivanov, R.I., Lenells, J.: Inverse scattering transform for the Degasperis–Procesi equation. Nonlinearity **23**, 2559–2575 (2010)
15. Ablowitz, M.J., Segur, H.: Solitons, nonlinear evolution equations and inverse scattering. J. Fluid Mech. **244**, 721–725 (1992)
16. Ma, W.X., Huang, T., Zhang, Y.: A multiple exp-function method for nonlinear differential equations and its application. Phys. Scr. **82**, 065003 (2010)
17. Ma, W.X., Zhu, Z.N.: Solving the (3 + 1)-dimensional generalized KP and BKP equations by the multiple exp-function algorithm. Appl. Math. Comput. **218**, 11871–11879 (2012)
18. Bhrawy, A.H., Abdelkawy, M.A., Kumar, S., Biswas, A.: Solitons and other solutions to Kadomtsev–Petviashvili equation of B-type. Rom. J. Phys. **58**, 729–748 (2013)
19. Hirota, R.: Exact solutions of the Korteweg–de Vries equation for multiple collisions of solitons. J. Phys. Soc. Jpn. **33**, 1456–1458 (1972)
20. Lü, X., Tian, B., Zhang, H.Q., Li, H.: Generalized (2 + 1)-dimensional Gardner model: bilinear equations, Bäcklund transformation, Lax representation and interaction mechanisms. Nonlinear Dyn. **67**, 2279–2290 (2012)

21. Hereman, W., Nuseir, A.: Symbolic methods to construct exact solutions of nonlinear partial differential equations. Math. Comput. Simulat. **43**, 13–27 (1997)
22. Wazwaz, A.M., Triki, H.: Soliton solutions for a generalized KdV and BBM equations with time-dependent coefficients. Commun. Nonlinear Sci. Numer. Simul. **16**, 1122–1126 (2011)
23. Wazwaz, A.M.: $(2+1)$-Dimensional Burgers equations BE $(m+n+1)$: using the recursion operator. Appl. Math. Comput. **219**, 9057–9068 (2013)
24. Wazwaz, A.M.: Kink solutions for three new fifth order nonlinear equations. Appl. Math. Model. **38**, 110–118 (2014)
25. Wazwaz, A.M.: A study on a $(2+1)$-dimensional and a $(3+1)$-dimensional generalized Burgers equation. Appl. Math. Lett. **31**, 41–45 (2014)
26. Ebadi, G., Fard, N.Y., Bhrawy, A.H., Kumar, S., Triki, H., Yildirim, A., Biswas, A.: Solitons and other solutions to the $(2+1)$-dimensional extended Kadomtsev–Petviashvili equation with power law nonlinearity. Rom. J. Phys. **65**, 27–62 (2013)
27. Biswas, A., Bhrawy, A.H., Abdelkawy, M.A., Alshaery, A.A., Hilal, E.M.: Symbolic computation of some nonlinear fractional differential equations. Rom. J. Phys. **59**, 433–442 (2014)
28. Shi, L.M., Zhang, L.F., Meng, H., Zhao, H.W., Zhou, S.P.: A method to construct Weierstrass elliptic function solution for nonlinear equations. Int. J. Mod. Phys. B **25**, 1931–1939 (2011)
29. Guo, Y.X., Wang, Y.: On Weierstrass elliptic function solutions for a $(N+1)$ dimensional potential KdV equation. Appl. Math. Comput. **217**, 8080–8092 (2011)
30. Ebaid, A., Aly, E.H.: Exact solutions for the transformed reduced Ostrovsky equation via the F-expansion method in terms of Weierstrass-elliptic and Jacobian-elliptic functions. Wave Motion **49**, 296–308 (2012)
31. Nakamura, A.: A direct method of calculating periodic wave solutions to nonlinear evolution equations. I. Exact two-periodic wave solution. J. Phys. Soc. Jpn. **47**, 1701–1705 (1979)
32. Nakamura, A.: A direct method of calculating periodic wave solutions to nonlinear evolution equations. II. Exact one-and two-periodic wave solution of the coupled bilinear equations. J. Phys. Soc. Jpn. **48**, 1365–1370 (1980)
33. Hon, Y.C., Fan, E.G., Qin, Z.: A kind of explicit quasi-periodic solution and its limit for the Toda lattice equation. Mod. Phys. Lett. B **22**, 547–553 (2008)
34. Fan, E.G., Hon, Y.C.: Quasiperiodic waves and asymptotic behavior for Bogoyavlenskii's breaking soliton equation in $(2+1)$ dimensions. Phys. Rev. E **78**, 036607–036619 (2008)
35. Fan, E.G., Chow, K.W.: On the periodic solutions for both nonlinear differential and difference equations: a unified approach. Phys. Lett. A **374**, 3629–3634 (2010)
36. Tian, S.F., Zhang, H.Q.: Riemann theta functions periodic wave solutions and rational characteristics for the nonlinear equations. J. Math. Anal. Appl. **371**, 585–608(2010)
37. Tian, S.F., Zhang, H.Q.: A kind of explicit Riemann theta functions periodic waves solutions for discrete soliton equations. Commun. Nonlinear Sci. Numer. Simul. **16**, 173–186 (2011)
38. Tian, S.F., Zhang, H.Q.: Riemann theta functions periodic wave solutions and rational characteristics for the $(1+1)$-dimensional and $(2+1)$-dimensional Ito equation. Chaos Solitons Fractals **47**, 27–41 (2013)
39. Boiti, M., Leon, J.P., Manna, M., Pempinelli, F.: On the spectral transform of a Korteweg–de Vries equation in two spatial dimensions. Inverse probl. **2**, 271 (1986)
40. Tian, B., Gao, Y.T.: Soliton-like solutions for a $(2+1)$-dimensional generalization of the shallow water wave equations. Chaos Solitons Fractals **7**, 1497–1499 (1996)
41. Gao, Y.T., Tian, B.: Generalized tanh method with symbolic computation and generalized shallow water wave equation. Comput. Math. Appl. **33**, 115–118 (1997)
42. Lou, S.Y.: Generalized dromion solutions of the $(2+1)$-dimensional KdV equation. J. Phys. A: Math. Gen. **28**, 7227 (1995)
43. Lou, S.Y.: Conformal invariance and integrable models. J. Phys. A: Math. Gen. **30**, 4803 (1997)
44. Lou, S.Y., Hu, X.B.: Infinitely many Lax pairs and symmetry constraints of the KP equation. J. Math. Phys. **38**, 6401–6427 (1997)
45. Lou, S.Y., Ruan, H.Y.: Revisitation of the localized excitations of the $(2+1)$-dimensional KdV equation. J. Phys. A: Math. Gen. **34**, 305 (2001)
46. Tang, X.Y., Lou, S.Y.: A variable separation approach to solve the integrable and nonintegrable models: coherent structures of the $(2+1)$-dimensional KdV equation. Commun. Theor. Phys. **38**, 1–8 (2002)
47. Chen, Y., Wang, Q., Li, B.: A series of soliton-like and double-like periodic solutions of a $(2+1)$-dimensional asymmetric Nizhnik–Novikov–Vesselov equation. Commun. Theor. Phys. **42**, 655–660 (2004)
48. Ma, S.H., Fang, J.P.: Multi dromion-solitoff and fractal excitations for $(2+1)$-dimensional Boiti–Leon–Manna–Pempinelli system. Commun. Theor. Phys. **52**, 641–645 (2009)
49. Fan, E.G.: Quasi-periodic waves and an asymptotic property for the asymmetrical Nizhnik–Novikov–Veselov equation. J. Phys. A **42**, 095206 (2009)
50. Luo, L.: Quasi-periodic waves and asymptotic property for Boiti–Leon–Manna–Pempinelli Equation. Commun. Theor. Phys. **54**, 208–214 (2010)
51. Luo, L.: New exact solutions and Bäcklund transformation for Boiti–Leon–Manna–Pempinelli equation. Phys. Lett. A **375**, 1059–1063 (2011)
52. Darvishi, M.T., Najafi, M., Kavitha, L., Venkatesh, M.: Stair and step soliton solutions of the integrable $(2+1)$ and $(3+1)$-dimensional Boiti–Leon–Manna–Pempinelli equations. Commun. Theor. Phys. **58**, 785–794 (2012)