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Nonlinear dynamics for local fractional Burgers' equation arising in fractal flow

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Abstract The local fractional Burgers' equation (LFBE) is investigated from the point of view of local fractional conservation laws envisaging a nonlinear local fractional transport equation with a linear non-differentiable diffusion term. The local fractional derivative transformations and the LFBE conversion to a linear local fractional diffusion equation are analyzed.

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1 Introduction

The Burgers' equation (BE) [1–3] is the simplest nonlinear diffusion equation arising in the fluid mechanics. The BE can be transformed into the diffusion equation by means of the Hopf–Cole transformation as shown in [4,5]. On the other hand, the conservation laws for BE were discussed in [6]. The BE was analyzed in a broad perspective, namely with singular data [7], for a non-commutative form [8], in lattice gas problems [9], and by means of a stochastic approach [10]. The BE has been successfully applied to turbulence problems [11], traffic flow [12] and plane waves [13]. The numerical solution of the BE was developed by finite element method [14], generalized boundary element method [15], tanh-coth method [16] and other methods (see also cited references therein).

In view of the fractional calculus theory [17–21], applicable to nonlinear problems on science and engineering, the adoption of fractional BEs was suggested [22] and several solutions were developed [23,24] and analyzed [25,26]. The solution strategies employed homotopy analysis [27] and Adomian decomposition methods [28] to solve the space- and time-fractional versions of the fractional BE. In this context, the classic finite difference method was proposed to solve the generalized FBE [29]. Furthermore, the variational iteration method (VIM) was successfully applied for taking the Burgers' flows with fractional derivatives [30]. The coupled BEs within time- and space-fractional derivatives were solved by the Adomian decomposition

method in [31]. We can mention also the generalized differential transformation and homotopy perturbation methods that were adopted to solve the time-fractional BEs [32].

Recently, the local fractional calculus was successfully applied to non-differentiable problems arising in the areas of solid mechanics [33], heat transfer and wave propagation [34], diffusion [35], hydrodynamics [36], vehicular traffic flow [37] and other topics [38–42] (see also references therein).

The present manuscript focuses on the LFBE arising from the nonlinear local fractional transport equation involving a linear non-differentiable diffusion term with the local fractional conservation laws. This article is structured as it follows. In Sect. 2, the nonlinear local fractional transport equation from the local fractional conservation laws is introduced. In Sect. 3, the LFBE arising in fractal flow is discussed. In Sect. 4, the local fractional derivative transformation is suggested. In Sect. 5, the results are discussed. Finally, Sect. 6 outlines the main conclusions.

2 The nonlinear local fractional transport equation via local fractional conservation laws

Let us consider a nonlinear local fractional transport equation from the local fractional conservation laws point of view. In this context, the local fractional partial derivative of the non-differentiable function $f(x, y)$ with respect to $x = x_0$ $(0 < \alpha < 1)$ is defined as [33,35]:

$$\frac{\partial^\alpha f(x_0, y)}{\partial x^\alpha} = \lim_{x \to x_0} \frac{\Delta^\alpha (f(x, y) - f(x_0, y))}{(x - x_0)^\alpha}, \quad (1)$$

where

$$\Delta^\alpha (f(x, y) - f(x_0, y)) \\ \cong \Gamma(1 + \alpha) [f(x, y) - f(x_0, y)]. \quad (2)$$

If $Q(x, t)$ denotes a fractal flow and its conserved density is $\phi(x, t)$, then we have [37]

$$Q(t) = \frac{1}{\Gamma(1 + \alpha)} \int_{x_1}^{x_2} \frac{\partial^\alpha \phi(x, t)}{\partial t^\alpha} (dx)^\alpha, \quad (3)$$

$$Q(t) = \frac{1}{\Gamma(1 + \alpha)} \int_{x_1}^{x_2} \frac{\partial^\alpha Q(x, t)}{\partial x^\alpha} (dx)^\alpha. \quad (4)$$

The local fractional integral operator of $f(x)$ of order α in the interval $[a, b]$ is defined as [33,35]

$${}_a I_b^{(\alpha)} f(x) = \frac{1}{\Gamma(1 + \alpha)} \int_a^b f(t) (dt)^\alpha \\ = \frac{1}{\Gamma(1 + \alpha)} \lim_{\Delta t \to 0} \sum_{j=0}^{j=N-1} f(t_j) (\Delta t_j)^\alpha, \quad (5)$$

with the partitions of the interval $[a, b]$, (t_j, t_{j+1}), $j = 0, ..., N - 1$, $t_0 = a$ and $t_N = b$, for $\Delta t_j = t_{j+1} - t_j$ and $\Delta t = \max \{\Delta t_0, \Delta t_1, \Delta t_j, ...\}$.

Employing expressions (3) and (4), we have [37]:

$$\frac{1}{\Gamma(1+\alpha)} \int_{x_1}^{x_2} \left\{ \frac{\partial^\alpha \phi(x, t)}{\partial t^\alpha} + \frac{\partial^\alpha Q(x, t)}{\partial x^\alpha} \right\} (dx)^\alpha = 0. \quad (6)$$

Hence, the local fractional conservation law reads as [37]

$$\frac{\partial^\alpha \phi(x, t)}{\partial t^\alpha} + \frac{\partial^\alpha Q(x, t)}{\partial x^\alpha} = 0, \quad (7)$$

where the functions $\phi(x, t)$ and $Q(x, t)$ are the conserved density and fractal flow, respectively.

If the fractal flow can be represented as

$$Q = \frac{\phi^2}{2}, \quad (8)$$

then the nonlinear transport equation in the local fractional conservation law becomes

$$\frac{\partial^\alpha \phi}{\partial t^\alpha} + \frac{\partial^\alpha}{\partial x^\alpha} \left(\frac{\phi^2}{2} \right) = 0. \quad (9)$$

Therefore, the nonlinear local fractional transport equation (also known as the local fractional inviscid BE) takes the form

$$\frac{\partial^\alpha \phi}{\partial t^\alpha} + \phi \frac{\partial^\alpha \phi}{\partial x^\alpha} = 0. \quad (10)$$

The linear form of the local fractional transport equation was discussed in [37].

3 Local fractional Burgers' equation

If the fractal flow is expressed as

$$Q = \frac{\phi^2}{2} - k \frac{\partial^\alpha \phi}{\partial x^\alpha}, \quad (11)$$

then, using (7), we arrive to the nonlinear local fractional transport equation with a linear non-differentiable diffusion term

$$\frac{\partial^\alpha \phi}{\partial t^\alpha} + \frac{\partial^\alpha}{\partial x^\alpha}\left(\frac{\phi^2}{2} - \kappa\frac{\partial^\alpha \phi}{\partial x^\alpha}\right) = 0, \tag{12}$$

where κ is a diffusion coefficient and $\phi(x,t)$ is a non-differentiable function.

Equation (12) can be expressed as

$$\frac{\partial^\alpha \phi}{\partial t^\alpha} + \phi\frac{\partial^\alpha \phi}{\partial x^\alpha} = \kappa\frac{\partial^{2\alpha} \phi}{\partial x^{2\alpha}}. \tag{13}$$

Equation (13) is the LFBE, and the diffusion coefficient κ denotes the fluid kinematic viscosity.

If the quadratic term in (13) is neglected, then we obtain the local fractional diffusion equation (i.e. the LFBE) [35]:

$$\frac{\partial^\alpha \phi}{\partial t^\alpha} - \kappa\frac{\partial^{2\alpha} \phi}{\partial x^{2\alpha}} = 0. \tag{14}$$

From (13), the local fractional forced BE, involving an external force $g_1(x,t)$, can be obtained in the form

$$\frac{\partial^\alpha \phi}{\partial t^\alpha} + \phi\frac{\partial^\alpha \phi}{\partial x^\alpha} = \kappa\frac{\partial^{2\alpha} \phi}{\partial x^{2\alpha}} + g_1. \tag{15}$$

Using (15), the nonlinear local fractional transport equation with a source term $g_2(x,t)$ is

$$\frac{\partial^\alpha \phi}{\partial t^\alpha} + \phi\frac{\partial^\alpha \phi}{\partial x^\alpha} = g_2. \tag{16}$$

In (16), the function $g_2(x,t)$ is a non-differentiable source term.

4 Local fractional derivative transformations

The local fractional derivative transformation can be used to convert the LFBE into a linear local fractional diffusion equation. The main idea and the transformation approach are explained in the sequel.

Let us define

$$\phi = \frac{\partial^\alpha \varphi}{\partial x^\alpha} \tag{17}$$

and

$$\frac{\partial^\alpha \varphi}{\partial t^\alpha} = \kappa\frac{\partial^\alpha \phi}{\partial x^\alpha} - \frac{\phi^2}{2}. \tag{18}$$

Then, from expressions (17) and (18), we obtain

$$\frac{\partial^\alpha \varphi}{\partial t^\alpha} = \kappa\frac{\partial^{2\alpha} \varphi}{\partial x^{2\alpha}} - \frac{1}{2}\left(\frac{\partial^\alpha \varphi}{\partial x^\alpha}\right)^2. \tag{19}$$

Let us define the function

$$\psi(x,t) = \sum_{i=0}^{\infty}\left(-\frac{1}{2\kappa}\right)^i\frac{\varphi^i(x,t)}{\Gamma(1+i\alpha)}. \tag{20}$$

This allows developing the following equations

$$\begin{aligned}
&\frac{\partial^\alpha \psi(x,t)}{\partial t^\alpha} \\
&= \left(-\frac{1}{2\kappa}\right)\frac{\partial^\alpha \varphi(x,t)}{\partial t^\alpha}\sum_{i=0}^{\infty}\left(-\frac{1}{2\kappa}\right)^i\frac{\varphi^i(x,t)}{\Gamma(1+i\alpha)} \\
&= -\frac{1}{2\kappa}\frac{\partial^\alpha \varphi(x,t)}{\partial t^\alpha}\psi(x,t),
\end{aligned} \tag{21}$$

and

$$\begin{aligned}
&\frac{\partial^\alpha \psi(x,t)}{\partial x^\alpha} \\
&= \left(-\frac{1}{2\kappa}\right)\frac{\partial^\alpha \varphi(x,t)}{\partial x^\alpha}\sum_{i=1}^{\infty}\left(-\frac{1}{2\kappa}\right)^i\frac{\varphi^i(x,t)}{\Gamma(1+i\alpha)} \\
&= -\frac{1}{2\kappa}\frac{\partial^\alpha \varphi(x,t)}{\partial x^\alpha}\psi(x,t).
\end{aligned} \tag{22}$$

In view of (21) and (22), we obtain the local fractional derivative transformation, which is given as follows:

$$\phi = \frac{\partial^\alpha \varphi(x,t)}{\partial t^\alpha} = -2\kappa\frac{\frac{\partial^\alpha \psi(x,t)}{\partial t^\alpha}}{\psi(x,t)}, \tag{23}$$

$$\frac{\partial^\alpha \varphi(x,t)}{\partial x^\alpha} = -2\kappa\frac{\frac{\partial^\alpha \psi(x,t)}{\partial x^\alpha}}{\psi(x,t)}, \tag{24}$$

$$\frac{\partial^{2\alpha} \varphi(x,t)}{\partial x^{2\alpha}} = 2\kappa\left(\frac{\frac{\partial^\alpha \psi(x,t)}{\partial x^\alpha}}{\psi(x,t)}\right)^2 - \frac{2\kappa}{\psi(x,t)}\frac{\partial^{2\alpha} \psi(x,t)}{\partial x^{2\alpha}}. \tag{25}$$

From Eqs. (23), (24) and (25), we convert eq. (19) into

$$\begin{aligned}
&-2\kappa\frac{\frac{\partial^\alpha \psi(x,t)}{\partial t^\alpha}}{\psi(x,t)} \\
&= \kappa\left(2\kappa\left(\frac{\frac{\partial^\alpha \psi(x,t)}{\partial x^\alpha}}{\psi(x,t)}\right)^2 - \frac{2\kappa}{\psi(x,t)}\frac{\partial^{2\alpha} \psi(x,t)}{\partial x^{2\alpha}}\right) \\
&\quad - 2\kappa^2\left(\frac{\frac{\partial^\alpha \psi(x,t)}{\partial x^\alpha}}{\psi(x,t)}\right)^2.
\end{aligned} \tag{26}$$

Finally, the linear local fractional diffusion equation reads as

$$\frac{\partial^\alpha \psi(x,t)}{\partial t^\alpha} = \kappa\frac{\partial^{2\alpha} \psi(x,t)}{\partial x^{2\alpha}}. \tag{27}$$

More details about linear local fractional diffusion equation are available in [35].

5 Discussion

If the fractal dimension varies from α to 1, then the *conserved density* changes into a *differentiable conserved density*. In this context, we transform (13) into the classical BE [2,3]:

$$\frac{\partial \phi}{\partial t} + \phi \frac{\partial \phi}{\partial x} = \kappa \frac{\partial^2 \phi}{\partial x^2}, \tag{28}$$

where $\phi(x, t)$ is a differentiable function.

For $\alpha = 1$, we have

$$\frac{\partial \varphi}{\partial t} = \kappa \frac{\partial^2 \varphi}{\partial x^2} - \frac{1}{2} \left(\frac{\partial \varphi}{\partial x}\right)^2. \tag{29}$$

With the Cole–Hopf transformation [3–5], we have

$$\phi = \frac{\partial \varphi(x, t)}{\partial t} = -2\kappa \frac{\frac{\partial \psi(x,t)}{\partial t}}{\psi(x, t)}, \tag{30}$$

$$\frac{\partial \varphi(x, t)}{\partial x} = -2\kappa \frac{\frac{\partial \psi(x,t)}{\partial x}}{\psi(x, t)}, \tag{31}$$

$$\frac{\partial^2 \varphi(x, t)}{\partial x^2} = 2\kappa \left(\frac{\frac{\partial \psi(x,t)}{\partial x}}{\psi(x, t)}\right)^2 - \frac{2\kappa}{\psi(x, t)} \frac{\partial^2 \psi(x, t)}{\partial x^2}. \tag{32}$$

In (20), we suggested a transformation function that in view the case at issue can be written as

$$\psi(x, t) = \sum_{i=0}^{\infty} \left(-\frac{1}{2\kappa}\right)^i \frac{\varphi^i(x, t)}{\Gamma(1+i)}. \tag{33}$$

This transformation function was developed in [5] as

$$\psi(x, t) = e^{-\frac{1}{2\kappa}\varphi(x,t)}, \tag{34}$$

which leads to

$$\varphi(x, t) = -2\kappa \ln \psi(x, t). \tag{35}$$

Hence, from (33) and (34), we obtain

$$e^{-\frac{1}{2\kappa}\varphi(x,t)} = \sum_{i=0}^{\infty} \left(-\frac{1}{2\kappa}\right)^i \frac{\varphi^i(x, t)}{\Gamma(1+i)}. \tag{36}$$

In fact, expression (36) is the Taylor expansion of $e^{-\frac{1}{2\kappa}\varphi(x,t)}$.

Moreover, using (20), we may develop the local fractional series expansion of $E_\alpha(-\varphi(x, t)/2\kappa)$ with a non-differentiable function $\varphi(x, t)$, namely

$$E_\alpha\left(-\frac{1}{2\kappa}\varphi(x, t)\right) = \sum_{i=0}^{\infty} \left(-\frac{1}{2\kappa}\right)^i \frac{\varphi^i(x, t)}{\Gamma(1+i\alpha)}. \tag{37}$$

6 Conclusions

The communication discussed the LFBE which can be developed on the basis of the nonlinear local fractional transport equation with a linear non-differentiable diffusion term. Consequently, the local fractional derivative transformation conceived by the presented analysis allowed transforming the LFBE into the local fractional diffusion equation. The classical BE emerges as a reasonable consequence from the LFBE when the fractal dimension α becomes equal to 1.

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