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Madelung fluid description on a generalized mixed nonlinear Schrödinger equation

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Abstract Within the framework of the Madelung fluid description, in the present paper, we will derive *bright* and *dark* (including *gray-* and *black-soliton*) envelope solutions for a *generalized mixed nonlinear Schrödinger model* $\mathrm{i}\,\frac{\partial \Psi}{\partial t} = \frac{\partial^2 \Psi}{\partial x^2} + \mathrm{i}\,a\,|\Psi|^2\,\frac{\partial \Psi}{\partial x} + \mathrm{i}\,b\,\Psi^2\,\frac{\partial \Psi^*}{\partial x} + c\,|\Psi|^4\Psi + d\,|\Psi|^2\Psi$, by virtue of the corresponding solitary wave solutions for *the generalized stationary Gardner equations*. Via corresponding parametric constraints, our results are achieved under suitable assumptions for the current velocity associated with different boundary conditions of the fluid density ρ, while we have only considered the motion with stationary-profile current velocity case and excluded the motion with constant current velocity case. Note that our model is a generalized one with the inclusion of multiple coefficients (a, b, c and d).

Keywords Generalized mixed nonlinear Schrödinger equation · Madelung fluid description · Solitary wave · Envelope soliton

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1 Introduction

In virtue of the Madelung fluid description theory [1–5], families of generalized one-dimensional nonlinear Schrödinger equations (NLSEs) have been solved [6–11] with the achievement of *bright-*, *black-* and *gray-soliton*-type envelope solutions, which maybe illustrated in the fluid language with new insights. Within the framework of the Madelung fluid description, the complex wave function (say Ψ) is represented in terms of modulus and phase. Substitution of $\Psi(x,t) = \sqrt{\rho(x,t)}\, e^{\frac{\mathrm{i}}{\hbar}\Theta(x,t)}$ into the one-dimensional Schrödinger equation

$$\mathrm{i}\,\hbar\,\frac{\partial \Psi(x,t)}{\partial t} = -\frac{\hbar^2}{2\,m}\frac{\partial^2 \Psi(x,t)}{\partial x^2} + m\,U(x)\Psi(x,t), \tag{1}$$

leads to the following pair of coupled Madelung fluid equations

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho\, v) = 0, \tag{2a}$$

$$\frac{\partial v}{\partial t} + v\,\frac{\partial v}{\partial x} - \frac{\hbar^2}{2\,m^2}\frac{\partial}{\partial x}\left(\frac{1}{\sqrt{\rho}}\frac{\partial^2 \sqrt{\rho}}{\partial x^2}\right) + \frac{\partial U}{\partial x} = 0, \tag{2b}$$

where $\rho = |\Psi|^2$ is the fluid density and $v = \frac{1}{m}\frac{\partial \Theta}{\partial x}$ is the current velocity. Equation (2a) is a continuity equation for the fluid density, while Eq. (2b) is the equation of motion for the fluid velocity and contains a force term proportional to the gradient of the "quantum potential," $\frac{\hbar^2}{2\,m^2}\frac{\partial}{\partial x}\left(\frac{1}{\sqrt{\rho}}\frac{\partial^2\sqrt{\rho}}{\partial x^2}\right)$. Under suitable hypothesis for the current velocity, Eq. (2b) may be transformed into the stationary Korteweg-de Vries, modified Korteweg-de Vries or Gardner equations, which are directly solvable and possess abundant types of solitary wave solutions. Correspondingly, a number of envelope solitons can be constructed for the original equations.

Let us consider the following generalized mixed nonlinear Schrödinger equation (GMNLSE) in the form of [12–17]

$$\mathrm{i}\,\frac{\partial \Psi}{\partial t} = \frac{\partial^2 \Psi}{\partial x^2} + \mathrm{i}\,a\,|\Psi|^2\,\frac{\partial \Psi}{\partial x} + \mathrm{i}\,b\,\Psi^2\,\frac{\partial \Psi^*}{\partial x} + c\,|\Psi|^4\Psi + d\,|\Psi|^2\Psi, \tag{3}$$

where $*$ denotes the complex conjugation, $\Psi = \Psi(x,t)$ is the complex wave function, and a, b, c and d are all real constants. Equation (3) arises in several physical applications including quantum field theory, weakly nonlinear dispersive water waves, and nonlinear optics [12,13]. It is shown to enjoy the Painlevé property only if $c = \frac{1}{4}\,b\,(2\,b - a)$ holds, regardless of the value of d [14]. Based on its Lax pair, the systematic construction of infinitely many conservation laws has been given [15], and the soliton behavior has been studied with N-fold Darboux transformation [17]. With different choices of the multiple parameters a, b, c and d, a series of celebrated nonlinear evolution equations in mathematical physics are included by Eq. (3), as follows:

- With $d = 0$, Eq. (3) reduces to the generalized derivative NLSE, i.e., Eq. (1.1) in Ref. [14].
- With $a = b = c = 0$ and $d = \pm\frac{1}{2}$, Eq. (3) reduces to the celebrated standard cubic NLSE [7,18–21].
- With $a = d = 0$, Eq. (3) reduces to the celebrated Gerdjikov-Ivanov equation [22–24].
- With $c = d = 0$ and $a = 2\,b$, Eq. (3) reduces to the celebrated Kaup-Newell or named derivative NLSI equation [25].
- With $b = c = d = 0$, Eq. (3) reduces to the celebrated Chen-Lee-Liu or named derivative NLSII equation [26].
- With $d = 0$ and $c = \frac{1}{4}\,b\,(2\,b - a)$, Eq. (3) reduces to the celebrated higher-order NLSE named by Kundu (see, e.g., Eq. (4) in Ref. [27]) [14,16,27–29].
- With $c = 0$ and $a = 2\,b$, Eq. (3) reduces to the celebrated mixed NLSE of Wadati et al [11,14,15,30,31].
- With $a = b = \pm 4\,\beta$, $c = 4\,\beta^2$ and $d = \mp 2$, Eq. (3) reduces to the generalized mixed NLSE of Geng et al (namely Eq. (1.2) in Ref. [32]) [15,16,32,33].

It is worthy of paying attention to the generalized Eq. (3) in that it covers abundant nonlinear models of physical and/or mathematical interest. The results for Eq. (3) can be correspondingly cast to these special cases with suitable parametric choices. In the present paper, without considering Painlevé integrable condition, we will directly investigate Eq. (3) within the context of the Madelung fluid description. Soliton solutions obtained hereby can be reduced to the above cases by different parameter choices. Based on the basic equations of Madelung fluid: the continuity equation for the fluid density and motion equation for the fluid velocity, it is found that, for a motion with stationary-profile current velocity, the fluid density satisfies a generalized stationary Gardner equation. Under suitable hypothesis for the current velocity and due to corresponding parametric constraints, we will derive *bright* and *dark* (including *gray* and *black*) solitary waves for the stationary Gardner equation, and finally associated envelope solitons will be found for Eq. (3).

2 Basic equations and motion with stationary-profile current velocity

By setting $\Psi(x,t) = \sqrt{\rho(x,t)}\,e^{\frac{\mathrm{i}}{k}\Theta(x,t)}$ with $k \neq 0$ in Eq. (3), the continuity equation for the fluid density ρ can be computed as

$$\frac{\partial \rho}{\partial t} - \frac{\partial}{\partial x}\left(\frac{2}{k}\,\rho\,v + \frac{1}{2}\,(a+b)\,\rho^2\right) = 0, \tag{4a}$$

and the equation of motion for the fluid velocity $v = \frac{\partial \Theta}{\partial x}$ as

$$\frac{\partial v}{\partial t} - \frac{2}{k}\,v\,\frac{\partial v}{\partial x} + k\,\frac{\partial}{\partial x}\left(\frac{1}{\sqrt{\rho}}\,\frac{\partial^2\sqrt{\rho}}{\partial x^2}\right) - \frac{\partial}{\partial x}\left[(a-b)\,\rho\,v - c\,k\,\rho^2 - d\,k\,\rho\right] = 0. \tag{4b}$$

With symbolic computation, Eq. (4b) is transformed into

$$\rho\,\frac{\partial v}{\partial t}-v\,\frac{\partial \rho}{\partial t}+2\,\frac{\partial \rho}{\partial x}\int \frac{\partial v}{\partial t}\,\mathrm{d}x+\frac{k}{2}\,\frac{\partial^3 \rho}{\partial x^3}+2\,c_0(t)\,\frac{\partial \rho}{\partial x}$$
$$+(4\,b-2\,a)\,\rho\,v\,\frac{\partial \rho}{\partial x}+3\,d\,k\,\rho\,\frac{\partial \rho}{\partial x}-(a-b)\,\rho^2\,\frac{\partial v}{\partial x}$$
$$+4\,c\,k\,\rho^2\,\frac{\partial \rho}{\partial x}=0, \tag{4c}$$

where $c_0(t)$ is an arbitrary function of t.

Equations (4a) and (4c) constitute the basic equations for the subsequent discussion, based on which we will discuss envelope solitons for Eq. (3) within the framework of the Madelung fluid description.

Firstly, assuming that both the quantities ρ and v involved in Eqs. (4a) and (4c) are functions of the combined variable $\xi=x-u_0\,t$ with u_0 being a real constant, we cast Eq. (4a) into

$$u_0\,\frac{\mathrm{d}\rho}{\mathrm{d}\xi}+\frac{\mathrm{d}}{\mathrm{d}\xi}\left[\frac{2}{k}\,\rho\,v+\frac{1}{2}\left(a+b\right)\rho^2\right]=0. \tag{5}$$

Integrating once of Eq. (5) with respect to ξ, and taking the integration constant as A_0, we obtain

$$v=\frac{k}{2}\left[-u_0-\frac{1}{2}\left(a+b\right)\rho+\frac{A_0}{\rho}\right]. \tag{6}$$

Substitution of Eqs. (6) into (4c) leads to

$$\left[-\frac{k}{2}\,u_0^2+2\,c_0+\frac{k}{2}\,A_0\left(3\,b-a\right)\right]\frac{\mathrm{d}\rho}{\mathrm{d}\xi}$$
$$+\left[\frac{3}{2}\,k\left(a-b\right)u_0+3\,d\,k\right]\rho\,\frac{\mathrm{d}\rho}{\mathrm{d}\xi}$$
$$+\left[\frac{k}{4}\left(3\,a^2-5\,b^2-2\,a\,b\right)+4\,c\,k\right]\rho^2\,\frac{\mathrm{d}\rho}{\mathrm{d}\xi}$$
$$+\frac{k}{2}\,\frac{\mathrm{d}^3\rho}{\mathrm{d}\xi^3}=0. \tag{7}$$

NOTES: **(i)** Different from that in Refs. [6–9], the fluid density ρ here satisfies a *generalized stationary Gardner equation*, namely, Eq. (7), which is not the Korteweg-de Vries or modified Korteweg-de Vries equation.

(ii) Attention should be paid to the expression of the fluid velocity v, i.e., Eq. (6). In the case of $a+b=0$, Eq. (3) reduces to $\mathrm{i}\,\frac{\partial \Psi}{\partial t}=\frac{\partial^2 \Psi}{\partial x^2}+\mathrm{i}\,a\,|\Psi|^2\,\frac{\partial \Psi}{\partial x}-\mathrm{i}\,a\,\Psi^2\,\frac{\partial \Psi^*}{\partial x}+c\,|\Psi|^4\Psi+d\,|\Psi|^2\Psi$, then we can consider both the motion with constant current velocity case ($A_0=0$) and the motion with stationary-profile current velocity case ($A_0\neq 0$); When $a=2\,b$ with $c=0$, Eq. (3) reduces to the generalized derivative NLSE, namely, Eq. (1) in Ref. [11], and has been studied there; When $a+b\neq 0$, the fluid velocity $v\neq$ *constant* whether $A_0=0$ or not, then we will study Eq. (3) on two aspects (the motion with constant current velocity and stationary-profile current velocity cases) in the subsequent sections.

3 Solitary wave versus envelope soliton

To be followed, we will construct the envelope soliton solutions for the GMNLSE, our Eq. (3). Actually, with the solutions of ρ and v for Eqs. (4a) and (4c), or equivalently Eqs. (6) and (7), the envelope solitons for Eq. (3) can be given by the following transformation:

$$\Psi(x,t)=\sqrt{\rho(x,t)}\,e^{\frac{\mathrm{i}}{k}\Theta(x,t)}. \tag{8}$$

It is clear that ρ is positive ($\rho>0$), and $\Theta(x,t)$ can be obtained by integration of v ($v=\frac{\partial \Theta}{\partial x}$). Under suitable assumptions for the current velocity associated with corresponding boundary conditions of ρ, we will investigate different types of solitary waves for Eq. (7). Then, fruitful envelope solitons can be correspondingly given for the GMNLSE, namely, Eq. (3).

3.1 Solitary waves under the boundary conditions: $\lim_{\xi\to\pm\infty}\rho(\xi)=0$

Assume ρ satisfies the boundary conditions in the ξ-space as $\lim_{\xi\to\pm\infty}\rho(\xi)=0$, and it follows from Eq. (6) that $A_0=0$ and $v=\frac{k}{2}\left[-u_0-\frac{1}{2}\left(a+b\right)\rho\right]$.

Consequently, Eq. (7) becomes

$$\left[-\frac{k}{2}\,u_0^2+2\,c_0\right]\frac{\mathrm{d}\rho}{\mathrm{d}\xi}+\left[\frac{3}{2}\,k\left(a-b\right)u_0+3\,d\,k\right]\rho\,\frac{\mathrm{d}\rho}{\mathrm{d}\xi}$$
$$+\left[\frac{k}{4}\left(3\,a^2-5\,b^2-2\,a\,b\right)+4\,c\,k\right]\rho^2\,\frac{\mathrm{d}\rho}{\mathrm{d}\xi}$$
$$+\frac{k}{2}\,\frac{\mathrm{d}^3\rho}{\mathrm{d}\xi^3}=0, \tag{9}$$

which can be integrated twice with respect to ξ and take the integration constant as zero to give

$$\left(\frac{\mathrm{d}\rho}{\mathrm{d}\xi}\right)^2=\rho^2\left[\left(\frac{1}{4}\,a^2-\frac{5}{12}\,b^2-\frac{1}{6}\,a\,b+\frac{4}{3}\,c\right)\rho^2\right.$$
$$\left.-\left(a\,u_0-b\,u_0+2\,d\right)\rho-\left(\frac{4}{k}\,c_0-u_0^2\right)\right]. \tag{10}$$

With the constraint $\frac{4}{k}\,c_0-u_0^2>0$ and $\frac{1}{4}\,a^2-\frac{5}{12}\,b^2-\frac{1}{6}\,a\,b+\frac{4}{3}\,c>0$, the positive solitary wave solution of Eq. (10) can be written as

$$\rho=\frac{2\,\alpha_0}{\alpha_1+\sqrt{\alpha_1^2+4\,\alpha_0\,\alpha_2}\,\mathrm{Cosh}\left[\sqrt{\alpha_0}\,(\xi+\xi_1)\right]},\tag{11}$$

with $\alpha_0=\frac{4}{k}\,c_0-u_0^2$, $\alpha_1=b\,u_0-a\,u_0-2\,d$, $\alpha_2=\frac{1}{4}\,a^2-\frac{5}{12}\,b^2-\frac{1}{6}\,a\,b+\frac{4}{3}\,c$ and ξ_1 as an integration constant.

According to Eq. (6), it is easy to find

$$v=\frac{\mathrm{d}\Theta}{\mathrm{d}\xi}=\frac{k}{2}\left(-u_0-\frac{(a+b)\,\alpha_0}{\alpha_1+\sqrt{\alpha_1^2+4\,\alpha_0\,\alpha_2}\,\mathrm{Cosh}\left[\sqrt{\alpha_0}\,(\xi+\xi_1)\right]}\right),\tag{12}$$

and

$$\begin{aligned}\Theta(x,t)&=-\frac{k}{2}\,u_0\,\xi-\frac{k}{4}\,(a+b)\int\rho\,\mathrm{d}\xi-2\,c_0\,t-\Theta_1\\&=-\frac{k}{2}\,u_0\,x+\left(\frac{k}{2}\,u_0^2-2\,c_0\right)t-\frac{k\,(a+b)}{2\,\sqrt{\alpha_2}}\\&\quad\times\mathrm{ArcTan}\left[\frac{\sqrt{\alpha_1^2+4\,\alpha_0\,\alpha_2}-\alpha_1}{2\,\sqrt{\alpha_0\,\alpha_2}}\mathrm{Tanh}\left(\frac{\sqrt{\alpha_0}}{2}\,x\right.\right.\\&\quad\left.\left.-\frac{\sqrt{\alpha_0}}{2}\,u_0\,t+\frac{\sqrt{\alpha_0}}{2}\,\xi_1\right)\right]-\Theta_1,\end{aligned}\tag{13}$$

where Θ_1 is an initial phase (integration constant).

Notice that, in this case, the solitary wave solution of the stationary Gardner equation (i.e., Eq. 7) possesses a *bright-soliton*-type profile (see Expression 26 above). When $\alpha_1=0$, Expression (26) turns to be the standard *Sech*-type bright soliton. Finally, the *bright-soliton*-type envelope solution of the GMNLSE (see Eq. 3 above) can be derived via Expression (8) with $\rho(x,t)$ and $\Theta(x,t)$ listed in Eqs. (26) and (27), respectively.

3.2 Solitary waves under the boundary conditions: $\lim_{\xi\to\pm\infty}\rho(\xi)\neq 0$

In the case of $\lim_{\xi\to\pm\infty}\rho(\xi)\neq 0$, we express $\rho(\xi)=\rho_0+\rho_1(\xi)$ with $\lim_{\xi\to\pm\infty}\rho_1(\xi)=0$ and $\rho_0=constant>0$, which determines that $v=\frac{k}{2}\left[-u_0-\frac{a+b}{2}\left(\rho_0+\rho_1(\xi)\right)+\frac{A_0}{\rho_0+\rho_1(\xi)}\right]$. Via Eq. (7) with $\rho(\xi)=\rho_0+\rho_1(\xi)$, we have

$$\begin{aligned}&\left\{\left[\frac{k}{4}\left(3\,a^2-5\,b^2-2\,a\,b\right)+4\,c\,k\right]\rho_0^2\right.\\&\quad-\left[\frac{3}{2}\,k\,u_0\left(a-b\right)+3\,k\,d\right]\rho_0\\&\quad\left.+\frac{k}{2}\,A_0\left(3\,b-a\right)-\frac{k}{2}\,u_0^2+2\,c_0\right\}\frac{\mathrm{d}\rho_1}{\mathrm{d}\xi}\\&\quad+\left\{\frac{3}{2}\,k\,u_0\left(a-b\right)+3\,k\,d\right.\\&\quad\left.+2\,\rho_0\left[\frac{k}{4}\left(3\,a^2-5\,b^2-2\,a\,b\right)+4\,c\,k\right]\right\}\rho_1\,\frac{\mathrm{d}\rho_1}{\mathrm{d}\xi}\\&\quad+\left[\frac{k}{4}\left(3\,a^2-5\,b^2-2\,a\,b\right)+4\,c\,k\right]\rho_1^2\,\frac{\mathrm{d}\rho_1}{\mathrm{d}\xi}\\&\quad+\frac{k}{2}\,\frac{\mathrm{d}^3\rho_1}{\mathrm{d}\xi^3}=0,\end{aligned}\tag{14}$$

which is also a generalized stationary Gardner equation and can be similarly solved as Eq. (9). *The key point lies in that the sign of* ρ_1, *which can be negative or positive, is different from the sign of* ρ *in Sect.* 3.1. *Just this point inspires us to study other types of solutions.*

Integrating twice of Eq. (14) with respect to ξ and taking the integration constants as zero, we obtain

$$\begin{aligned}\left(\frac{\mathrm{d}\rho_1}{\mathrm{d}\xi}\right)^2&=\rho_1^2\left\{\rho_1^2\left[\frac{1}{12}\left(5\,b^2-3\,a^2+2\,a\,b\right)-\frac{4}{3}\,c\right]\right.\\&\quad-\rho_1\left[\frac{1}{3}\,\rho_0\left(3\,a^2-5\,b^2-2\,a\,b+16\,c\right)\right.\\&\quad\left.+\left(a-b\right)u_0+2\,d\right]+\left[u_0^2-\frac{4}{k}\,c_0\right.\\&\quad-A_0\left(3\,b-a\right)-6\,d\,\rho_0+3\left(b-a\right)u_0\,\rho_0\\&\quad\left.\left.-\frac{1}{2}\left(3\,a^2-5\,b^2-2\,a\,b+16\,c\right)\rho_0^2\right]\right\}.\end{aligned}\tag{15}$$

3.2.1 *Up-shifted bright- and upper-shifted bright-type solitary waves*

Under the constraint $u_0^2-\frac{4}{k}\,c_0-A_0\,(3\,b-a)-6\,d\,\rho_0+3\,(b-a)\,u_0\,\rho_0-\frac{1}{2}\,(3\,a^2-5\,b^2-2\,a\,b+16\,c)\,\rho_0^2<0$

and $\frac{1}{12}(5b^2-3a^2+2ab)-\frac{4}{3}c>0$, the positive solution of Eq. (15) is taken as

$$\rho_1=\frac{2\beta_0}{\beta_1+\sqrt{\beta_1^2+4\beta_0\beta_2}\,\mathrm{Cosh}\left[\sqrt{\beta_0}\,(\xi+\xi_{21})\right]}>0, \tag{16}$$

with $\beta_0=A_0(3b-a)-u_0^2+\frac{4}{k}c_0+6d\rho_0-3(b-a)u_0\rho_0+\frac{1}{2}(3a^2-5b^2-2ab+16c)\rho_0^2$, $\beta_1=-\frac{1}{3}\rho_0(3a^2-5b^2-2ab+16c)-(a-b)u_0-2d$, $\beta_2=\frac{1}{12}(5b^2-3a^2+2ab)-\frac{4}{3}c$ and ξ_{21} as an integration constant.

Hereby, the solitary wave solution of Eq. (7) is in the form of

$$\rho=\rho_0+\frac{2\beta_0}{\beta_1+\sqrt{\beta_1^2+4\beta_0\beta_2}\,\mathrm{Cosh}\left[\sqrt{\beta_0}\,(\xi+\xi_{21})\right]}, \tag{17}$$

and subsequently,

$$\begin{aligned}\Theta(x,t)=&\left[\frac{kA_0}{2\rho_0}-\frac{k}{2}u_0-\frac{k}{4}\left(a+b\right)\rho_0\right]x\\&+\left[\frac{k}{2}u_0^2+\frac{k}{4}\left(a+b\right)\rho_0u_0-\frac{kA_0u_0}{2\rho_0}-2c_0\right]t\\&+\frac{kA_0\xi_{21}}{2\rho_0}-\frac{k(a+b)}{2\sqrt{\beta_2}}\\&\times\mathrm{ArcTan}\left[\frac{\sqrt{\beta_1^2+4\beta_0\beta_2}-\beta_1}{2\sqrt{\beta_0\beta_2}}\mathrm{Tanh}\left(\frac{\sqrt{\beta_0}}{2}x\right.\right.\\&\left.\left.-\frac{\sqrt{\beta_0}}{2}u_0t+\frac{\sqrt{\beta_0}}{2}\xi_{21}\right)\right]\\&-\Theta_{21}-\frac{kA_0}{\rho_0\sqrt{\beta_0}\sqrt{1+A\rho_0}\sqrt{1+B\rho_0}}\\&\times\mathrm{ArcTanh}\left[\frac{\sqrt{1+B\rho_0}}{\sqrt{1+A\rho_0}}\right.\\&\left.\times\mathrm{Tanh}\left[\frac{\sqrt{\beta_0}}{2}(x-u_0t+\xi_{21})\right]\right],\end{aligned} \tag{18}$$

where $A=\frac{\beta_1+\sqrt{\beta_1^2+4\beta_0\beta_2}}{2\beta_0}$ and $B=\frac{\beta_1-\sqrt{\beta_1^2+4\beta_0\beta_2}}{2\beta_0}$ with Θ_{21} and ξ_{21} as two integration constants.

It is shown in Expression (17) that the solution ρ enjoys a *bright-soliton* profile. By means of Expressions (8), (17) and (18), we can correspondingly discuss different cases of solution for Eq. (3).

Under the condition $\frac{1}{3}\rho_0(3a^2+5b^2-2ab+16c)+(a-b)u_0+2d=0$, Expression (17) takes the form of

$$\rho=\rho_0\left\{1+\varepsilon\,\mathrm{Sech}\left[\sqrt{\beta_0}\left(x-u_0t-\xi_{01}\right)\right]\right\}, \tag{19}$$

with

$$\varepsilon=\sqrt{\frac{A_0\left(3b-a\right)-u_0^2+\frac{4}{k}c_0+6d\rho_0-3\left(b-a\right)u_0\rho_0+\frac{1}{2}\left(3a^2-5b^2-2ab+16c\right)\rho_0^2}{\frac{1}{12}\left(5b^2-3a^2+2ab\right)\rho_0^2-\frac{4}{3}c\rho_0^2}}.$$

- With $0<\varepsilon<1$, Expression (19) represents a *up-shifted-bright-type solitary wave*, whose maximum amplitude is $\rho_0(1+\varepsilon)$ and up-shifted by the quantity ρ_0.
- With $\varepsilon=1$, Expression (19) represents a *upper-shifted-bright-type solitary wave*, whose maximum amplitude is $2\rho_0$ and up-shifted by the quantity ρ_0.

Having obtained different forms and types of the solitary waves for stationary Gardner equation via Expression (19), we can correspondingly investigate the associated different types of envelope solitons for the original GMNLSE, i.e., Eq. (3) by means of Expression (8) (details ignored here).

3.2.2 Gray- and dark-type solitary waves

With the constraint $u_0^2-\frac{4}{k}c_0-A_0(3b-a)-6d\rho_0+3(b-a)u_0\rho_0-\frac{1}{2}(3a^2-5b^2-2ab+16c)\rho_0^2<0$ and $\frac{1}{12}(5b^2-3a^2+2ab)-\frac{4}{3}c>0$, the negative solution of Eq. (15) can be taken as

$$\rho_1=\frac{-2\beta_0}{\sqrt{\beta_1^2+4\beta_0\beta_2}\,\mathrm{Cosh}\left[\sqrt{\beta_0}\,(\xi+\xi_{02})\right]-\beta_1}<0\,, \tag{20}$$

After tedious calculations, we find

$$\rho=\rho_0+\frac{2\,\beta_0}{\beta_1-\sqrt{\beta_1^2+4\,\beta_0\,\beta_2}\,\mathrm{Cosh}\left[\sqrt{\beta_0}\,(\xi+\xi_{02})\right]}, \quad (21)$$

$$\begin{aligned}\Theta(x,t)=&\left[\frac{k\,A_0}{2\,\rho_0}-\frac{k}{2}\,u_0-\frac{k}{4}\left(a+b\right)\rho_0\right]x+\left[\frac{k}{2}\,u_0^2\right.\\&\left.+\frac{k}{4}\left(a+b\right)\rho_0\,u_0-\frac{k\,A_0\,u_0}{2\,\rho_0}-2\,c_0\right]t\\&+\frac{k\,A_0\,\xi_{22}}{2\,\rho_0}+\frac{k\,(a+b)}{2\,\sqrt{\beta_2}}\\&\times\mathrm{ArcTan}\Bigg[\frac{\sqrt{\beta_1^2+4\,\beta_0\,\beta_2}+\beta_1}{2\,\sqrt{\beta_0\,\beta_2}}\\&\times\mathrm{Tanh}\left(\frac{\sqrt{\beta_0}}{2}\,x-\frac{\sqrt{\beta_0}}{2}\,u_0\,t+\frac{\sqrt{\beta_0}}{2}\,\xi_{22}\right)\Bigg]\\&-\Theta_{22}-\frac{k\,A_0}{\rho_0\,\sqrt{\beta_0}\,\sqrt{1+A\,\rho_0}\,\sqrt{1+B\,\rho_0}}\\&\times\mathrm{ArcTanh}\Bigg[\frac{\sqrt{1+A\,\rho_0}}{\sqrt{1+B\,\rho_0}}\\&\times\mathrm{Tanh}\left[\frac{\sqrt{\beta_0}}{2}\,(x-u_0\,t+\xi_{22})\right]\Bigg],\end{aligned} \quad (22)$$

where Θ_{22} and ξ_{22} are two integration constants.
In Expression (20), notice that ρ_1, the solution of Eq. (15), enjoys a *non-bright-soliton* profile. By virtue of Expression (21), we can discuss different solution cases for Eq. (15) as follows, and along with Expression (22) to correspondingly discuss the original GMNLSE.

Under the following condition

$$\begin{cases}\frac{1}{3}\,\rho_0\,(3\,a^2+5\,b^2-2\,a\,b+16\,c)+(a-b)\,u_0+2\,d=0,\\ \rho_0>\sqrt{\frac{\beta_0}{\beta_2}}\,,\end{cases} \quad (23)$$

Expression (21) reduces to

$$\rho=\rho_0\left\{1-\delta\,\mathrm{Sech}\left[\sqrt{\beta_0}\left(x-u_0\,t-\xi_{22}\right)\right]\right\}, \quad (24)$$

with

$$\delta=\sqrt{\frac{A_0\left(3\,b-a\right)-u_0^2+\frac{4}{k}\,c_0+6\,d\,\rho_0-3\left(b-a\right)u_0\,\rho_0+\frac{1}{2}\left(3\,a^2-5\,b^2-2\,a\,b+16\,c\right)\rho_0^2}{\frac{1}{12}\left(5\,b^2-3\,a^2+2\,a\,b\right)\rho_0^2-\frac{4}{3}\,c\,\rho_0^2}}.$$

- With $0<\delta<1$, Expression (24) denotes a *gray-type solitary wave*, whose minimum amplitude is $\rho_0(1-\delta)$ and reaches asymptotically the upper limit ρ_0.
- With $\delta=1$, Expression (24) denotes a *black-type solitary wave*, whose minimum amplitude is zero and reaches asymptotically the upper limit ρ_0.

Different forms and types of the solitary waves for stationary Gardner equation can be obtained via Expression (24), and as a result, we can investigate the associated different types of envelope solitons for the original GMNLSE by use of Expression (8) (details ignored here).

4 Example: bright-soliton solution reduction

Hereby, one can recover one-soliton solutions for all the known cases as tabulated in the Introduction, directly from the soliton solutions (of different nature) obtained here, when the Painlevé integrability conditions on the parameters are imposed. As an example, we show explicitly the bright-soliton solution reduction.

With $c=0$ and $a=2\,b$, Eq. (3) reduces to the celebrated mixed NLSE of Wadati et al. [11,14,15,30,31] as follows,

$$\begin{aligned}\mathrm{i}\,\frac{\partial\Psi}{\partial t}=&\frac{\partial^2\Psi}{\partial x^2}+2b\,\mathrm{i}\,|\Psi|^2\,\frac{\partial\Psi}{\partial x}+\mathrm{i}\,b\,\Psi^2\,\frac{\partial\Psi^*}{\partial x}\\&+d\,|\Psi|^2\Psi.\end{aligned} \quad (25)$$

With corresponding parameter reductions of Expressions (11) and (13), the bright-type one-soliton solution of Eq. (25) can be obtained as $\Psi(x,t)=\sqrt{\rho(x,t)}\,e^{\frac{\mathrm{i}}{k}\Theta(x,t)}$, where

$$\rho=\frac{2\,\alpha_0}{\alpha_1+\sqrt{\alpha_1^2+4\,\alpha_0\,\alpha_2}\,\mathrm{Cosh}\left[\sqrt{\alpha_0}\,(\xi+\xi_1)\right]}, \quad (26)$$

Table 1 Types of solitary waves

Solitary wave types for stationary Gardner Eq. (7)	Parametric constraints
bright-soliton-type	$A_0 = 0;$ $\frac{4}{k}\, c_0 - u_0^2 > 0;$ $\frac{1}{4}\, a^2 - \frac{5}{12}\, b^2 - \frac{1}{6}\, a\, b + \frac{4}{3}\, c > 0;$
bright-soliton-type	$\rho_0 > 0;$ $\frac{1}{12}\,(5\, b^2 - 3\, a^2 + 2\, a\, b) - \frac{4}{3}\, c > 0;$ $u_0^2 - \frac{4}{k}\, c_0 - A_0\,(3\, b - a) - 6\, d\, \rho_0 + 3\,(b - a)\, u_0\, \rho_0$ $-\frac{1}{2}\,(3\, a^2 - 5\, b^2 - 2\, a\, b + 16\, c)\, \rho_0^2 < 0;$
up-shifted-bright-type	$\rho_0 > 0;$ $0 < \varepsilon < 1;$ $\frac{1}{12}\,(5\, b^2 - 3\, a^2 + 2\, a\, b) - \frac{4}{3}\, c > 0;$ $\frac{1}{3}\, \rho_0\,(3\, a^2 + 5\, b^2 - 2\, a\, b + 16\, c) + (a - b)\, u_0$ $+2\, d = 0;$ $u_0^2 - \frac{4}{k}\, c_0 - A_0\,(3\, b - a) - 6\, d\, \rho_0 + 3\,(b - a)\, u_0\, \rho_0$ $-\frac{1}{2}\,(3\, a^2 - 5\, b^2 - 2\, a\, b + 16\, c)\, \rho_0^2 < 0;$
upper-shifted-bright-type	$\rho_0 > 0;$ $\varepsilon = 1;$ $\frac{1}{12}\,(5\, b^2 - 3\, a^2 + 2\, a\, b) - \frac{4}{3}\, c > 0;$ $\frac{1}{3}\, \rho_0\,(3\, a^2 + 5\, b^2 - 2\, a\, b + 16\, c) + (a - b)\, u_0$ $+2\, d = 0;$ $u_0^2 - \frac{4}{k}\, c_0 - A_0\,(3\, b - a) - 6\, d\, \rho_0 + 3\,(b - a)\, u_0\, \rho_0$ $-\frac{1}{2}\,(3\, a^2 - 5\, b^2 - 2\, a\, b + 16\, c)\, \rho_0^2 < 0;;$
gray-soliton-type	$0 < \delta < 1;$ Expression (23); $\frac{1}{12}\,(5\, b^2 - 3\, a^2 + 2\, a\, b) - \frac{4}{3}\, c > 0;$ $u_0^2 - \frac{4}{k}\, c_0 - A_0\,(3\, b - a) - 6\, d\, \rho_0 + 3\,(b - a)\, u_0\, \rho_0$ $-\frac{1}{2}\,(3\, a^2 - 5\, b^2 - 2\, a\, b + 16\, c)\, \rho_0^2 < 0;$
black-soliton-type	$\delta = 1;$ Expression (23); $\frac{1}{12}\,(5\, b^2 - 3\, a^2 + 2\, a\, b) - \frac{4}{3}\, c > 0;$ $u_0^2 - \frac{4}{k}\, c_0 - A_0\,(3\, b - a) - 6\, d\, \rho_0 + 3\,(b - a)\, u_0\, \rho_0$ $-\frac{1}{2}\,(3\, a^2 - 5\, b^2 - 2\, a\, b + 16\, c)\, \rho_0^2 < 0;$

$$\Theta(x,t) = -\frac{k}{2}u_0 x + \left(\frac{k}{2}u_0^2 - 2c_0\right)t - \frac{2kb}{\sqrt{\alpha_2}} \times \mathrm{ArcTan}\left[\frac{\sqrt{\alpha_1^2 + 4\alpha_0\alpha_2} - \alpha_1}{2\sqrt{\alpha_0\alpha_2}}\mathrm{Tanh}\left(\frac{\sqrt{\alpha_0}}{2}x - \frac{\sqrt{\alpha_0}}{2}u_0 t + \frac{\sqrt{\alpha_0}}{2}\xi_1\right)\right] - \Theta_1, \tag{27}$$

while $\alpha_0 = \frac{4}{k}c_0 - u_0^2, \alpha_1 = -b\,u_0 - 2\,d, \alpha_2 = \frac{1}{4}b^2$, and ξ_1 and Θ_1 are two integration constants.

5 Conclusions

In the present paper, we have derived *bright* and *dark* (including *gray* and *black*) solitary wave solutions for the generalized stationary Gardner equation, namely, Eq. (7), which is associated with the GMNLSE. Within the context of the Madelung fluid description, we have investigated the corresponding *bright*-, *black*- and *gray-soliton*-type envelope solutions for the GMNLSE via the transformation (8). Attention should be emphasized on that our model, i.e., Eq. (3) is a generalized one with the inclusion of multiple parameter coefficients (a, b, c and d). With suitable choice of these coefficients, abundant nonlinear models of physical and mathematical interest can be covered in Eq. (3). In this sense, many previous results in published papers maybe included here, e.g., envelope soliton solutions derived by using the Madelung fluid description approach in Refs. [7–11]. It should also be noted that our solutions are derived under suitable assumptions for the current velocity associated with corresponding boundary conditions of ρ, and under corresponding parametric constraints. The types of the solitary waves with associated parametric constraints have been listed in Table 1. Finally, all of those parametric constraints can be referred to the free unknown quantities k, A_0, ρ_0 with u_0, the arbitrary integration constant c_0, and the coefficient parameters a, b, c with d. We hope our investigation in this paper maybe useful for the study of other nonlinear models [34–37].

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