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The adaptive synchronization of fractional-order chaotic system with fractional-order $1 < q < 2$ via linear parameter update law

Ping Zhou · Rongji Bai

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Abstract This paper presents an adaptive synchronization approach for fractional-order chaotic systems with fractional-order $1 < q < 2$ and unknown system parameters based on the Mittag–Leffler function and the generalized Gronwall inequality. A sufficient condition is obtained. The numerical simulations are given to verify the effectiveness of this synchronization scheme.

Keywords Mittag–Leffler function · Generalized Gronwall inequality · Fractional-order chaotic systems · Adaptive synchronization · Linear parameter update law

1 Introduction

Recently, synchronization of chaotic systems has been focused on more attentions in nonlinear science due to its potential applications in science and engineering. Up to now, various synchronization schemes have been proposed, such as complete synchronization (CS) [1], generalization synchronization (GS) [2], projective synchronization (PS) [3], impulsive synchronization (IS) [4], lag synchronization (LS) [5,6], outer synchronization [7] and so on. Nevertheless, many chaotic systems in practical situations are usually with fully or partially unknown parameters. Such unknown parameters usually appear in the control input law for chaos control and synchronization. Therefore, how to estimate the unknown parameters is a key issue and prerequisite in chaos control and synchronization. Adaptive control theory [8] is an effective tool to this problem. Thus, adaptive synchronization of chaotic systems has been attracted more and more attentions in practical chaos applications.

As physical interpretation of the fractional derivative becomes clear, a large number of real-world physical systems, such as viscoelasticity, dielectric polarization, electromagnetic waves and fractional kinetics, can be more accurately described by fractional-order differential equations [9–11]. In recent years, chaos has been observed in many physical fractional-order systems, e.g., electronic circuits [12,13], micro-electro-mechanical systems [14] and gyroscopes [15], and many modified fractional-order systems, e.g., the fractional-order Lorenz chaotic system [16], the fractional-order Chua's circuit [13], the fractional-order Arneodo chaotic system [17], the fractional-order Duffing chaotic system [18], the fractional-order Rossler chaotic system [17], the fractional-order Sprott chaotic system [19], the fractional-order Chen system [17] and the fractional-order Lu system [17]. Similarly,

P. Zhou (✉) · R. Bai
Center of System Theory and Its Applications,
Chongqing University of Posts and Telecommunications,
Chongqing 400065, China
e-mail: zhouping@cqupt.edu.cn

P. Zhou · R. Bai
Key Laboratory of Network Control and Intelligent Instrument of Ministry of Education, Chongqing University of Posts and Telecommunications, Chongqing 400065, China

synchronization for fractional-order chaotic systems has been also attracted much attention in both theoretical and applied perspectives, since it is usually a prerequisite of practical application in chaos engineering, such as chaotic communications [20] and authenticated encryption schemes [21].

Up to now, many synchronization approaches on fractional-order chaotic systems have been reported for fractional-order $0 < q < 1$ [17,19,22–25]. However, there are many fractional-order systems with fractional-order $1 < q < 2$ in the real world, for example, the fractional diffusion-wave equation [26], the space-time fractional diffusion equation [27], the fractional telegraph equation [28], the super-diffusion systems [29], the time fractional reaction-diffusion systems [30] and the time fractional heat conduction equation [31]. Nevertheless, a few results on synchronization of fractional-order chaotic system with fractional-order $1 < q < 2$ have been considered. Hence, how to synchronize fractional-order chaotic systems with fractional-order $1 < q < 2$ is an opening problem.

Motivated by the above-mentioned discussions, an adaptive synchronization scheme for fractional-order chaotic systems with fractional-order $1 < q < 2$ and with unknown parameters is presented in this paper. The linear parameter update laws are used in our scheme. Based on the Mittag–Leffler function and the generalized Gronwall inequality, a sufficient condition is obtained. This adaptive synchronization approach is applied to a fractional-order Lorenz chaotic system and a modified fractional-order Chua's chaotic system with fractional-order $1 < q < 2$ and with partially or fully unknown parameters.

2 Preliminaries and main results

The Caputo fractional derivative is widely used in engineering applications due to its initial conditions have the same form as integer-order differential equations. The Caputo fractional derivative is defined as follows:

$$D^q h(t) = \frac{1}{\Gamma(n-q)} \int_0^t \frac{h^{(n)}(\tau)}{(t-\tau)^{q+1-n}} \mathrm{d}\tau, \ n-1 \leq q < n,$$

where D^q is called the Caputo derivative of fractional-order q of function $h(t)$, n is the first integer that is not less than q, $h^{(n)}(t) = \mathrm{d}^n h(t)/\mathrm{d}t^n$, and $\Gamma(n-q) = \int_0^{+\infty} t^{(n-q-1)} \mathrm{e}^{-t} \mathrm{d}t$.

In this paper, the fractional-order chaotic systems are described as

$$D^q x = f(x, a), \tag{1}$$

where $x \in R^{n\times 1}$, $f(x,a) \in R^{n\times 1}$. $a \in R^{m\times 1}$ is the vector of unknown parameters, and its estimation is denoted by a_0. We assume that system (1) displays a chaotic attractor for $a = a_0$ in the following paper.

Now, the fractional-order chaotic system (1) can be modified as follows,

$$D^q x = P_l x + P_n(x, a), \tag{2}$$

where $P_n(x,a) \in R^{n\times 1}$ is the nonlinear part including all unknown parameters, and $P_l \in R^{n\times n}$ is a constant matrix.

In this paper, we assume that

$$P_n(y, a) - P_n(x, a) = \Psi_l(x, a_0)e + \Psi_n(e, x, a_0), \tag{3}$$

where $y \in R^{n\times 1}$, $e = (\, y - x \;\; a - a_0\,)^{\mathbf{T}}$. $\Psi_l(x, a_0) \in R^{n\times(n+m)}$ and $\Psi_n(e, x, a_0) \in R^{n\times 1}$ are real matrices. $\Psi_n(e, x, a_0)$ and $\Psi_l(x, a_0)e$ represent the nonlinear and linear parts with respect to e, respectively. In fact, there are many fractional-order chaotic systems such as the Lorenz chaotic system [16], the modified Chua's chaotic system [13], the Arneodo chaotic system [17], the Rossler chaotic system [17], the Sprott chaotic system [19], the Chen chaotic system [17], the Lu chaotic system [17], the stretch-twist-fold (STF) flow chaotic system [21] and so on. In these systems, the nonlinear terms [denoted by $P_n(x, a)$] are all polynomials, which can be easily implemented with electronic circuits. It is not hard to see that all these systems satisfy Eq. (3).

In order to realize chaotic synchronization for the fractional-order chaotic system (2) with unknown parameters, system (2) is chosen as drive system, and the response system with linear parameter update law is designed as

$$\begin{cases} D^q y = P_l y + P_n(y, a) + \kappa(x, y, a) \\ D^q a = \Lambda e \end{cases}, \tag{4}$$

where $y \in R^{n\times 1}$ is the state vector. $\kappa(x, y, a) \in R^{n\times 1}$ is a controller. $\Lambda \in R^{m\times(n+m)}$ is a constant matrix. The linear parameter update law is $D^q a = \Lambda e$. The adaptive synchronization errors are denoted as

$e = (e^{(s)}, e^{(a)})^{\mathbf{T}} \in R^{(n+m)\times 1}, e^{(s)} = (y - x) \in R^{n\times 1}, e^{(a)} = (a - a_0) \in R^{m\times 1}, e_i^{(s)} = (y_i - x_i) \in R(i = 1, 2, \ldots, n)$, and $e_j^{(a)} = (a_j - a_{j0}) \in R(j = 1, 2, \ldots, m)$.

Remark Since Λ is a constant matrix, the parameter update law is a linear update law.

Definition Giving the drive system (2) and the response system (4) with linear parameter update law, it is said to be adaptive synchronization if there exist a controller $\kappa(x, y, a) \in R^{n\times 1}$ and a constant matrix $\Lambda \in R^{m\times(n+m)}$ such that

$$\lim_{t\to+\infty} \|y - x\| = 0, \quad \lim_{t\to+\infty} (a - a_0) = 0.$$

Lemma 1 *[32] If fractional-order $q \geq 1$ and $p = 1, 2, q$, then*

$$\left\|M_{q,p}(Lt^q)\right\| \leq \left\|\mathrm{e}^{Lt^q}\right\|$$

where $L \in R^{n\times n}$, $M_{q,p}(\bullet)$ denotes the two-parameter function of Mittag–Leffler type, i.e., $M_{q,p}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(qn+p)} (q > 0, p > 0, z \in C)$. $\|\bullet\|$ denotes the induced matrix norm.

Lemma 2 *[33]* (generalized Gronwall inequality) *Giving a interval $[0, T)$ (some $T \leq +\infty$), if fractional-order $q > 0$, $0 < b_1(t) \leq K$ (K is a constant) is a non-decreasing continuous function in $[0, T)$, $b_2(t) > 0$ is a non-decreasing function locally integrable in $[0, T)$, and if $b(t) > 0$ is locally integrable in $[0, T)$ with*

$$b(t) \leq b_2(t) + b_1(t) \int_0^t (t - \tau)^{q-1} b(\tau)\mathrm{d}\tau$$

in this interval $[0, T)$, then $b(t) \leq b_2(t)M_{q,1}[b_1(t)\Gamma(q)t^q]$.

Lemma 3 *[34]: If p is a real number, $l_i > 0 (i = 1, 2)$ is positive real constant, $0 < q < 2$ and $0.5\pi q < \beta < \min(\pi, q\pi)$, then*

$$\left|M_{q,p}(z)\right| \leq l_1(1 + |z|)^{(1-p)/q}\mathrm{e}^{\mathrm{Re}(z^{1/q})} + l_2(1 + |z|)^{-1}$$

where $|\arg(z)| \leq \beta$ and $|z| \geq 0$.

Now, we are in the position to state the main result of this paper.

Theorem *Let $\kappa(x, y, a) = [Q - \Psi_l(x, a_0)]e$. The fractional-order chaotic system (2) and the fractional-order system (4) are said to be adaptive synchronization, if the next two conditions are fulfilled,*

(i) $\left.\begin{pmatrix}\Psi_n(e, x, a_0)\\ 0\end{pmatrix}\right|_{e=0} = 0, \lim_{e\to 0} \frac{\left\|\begin{pmatrix}\Psi_n(e, x, a_0)\\ 0\end{pmatrix}\right\|}{\|e\|} = 0$ *for any x,*

(ii) $\mathrm{Re}[\lambda \begin{pmatrix}P+Q\\ \Lambda\end{pmatrix}] < 0, \omega = -\max\left[\mathrm{Re}\lambda\begin{pmatrix}P+Q\\ \Lambda\end{pmatrix}\right] > [\Gamma(q)]^{1/q}$,

where $Q \in R^{n\times(n+m)}$ is a suitable constant matrix, $P \in R^{n\times(n+m)}$ is a constant matrix satisfying $P(i, j) = P_l(i, j)(1 \leq j \leq n)$, and $P(i, j) = 0(n + 1 \leq j \leq n + m)$. Here, $P(i, j)$ and $P_l(i, j)$ are the elements of matrix P and P_l, respectively. $\begin{pmatrix}\Psi_n(e, x, a_0)\\ 0\end{pmatrix} \in R^{(n+m)\times 1}$, $\begin{pmatrix}P + Q\\ \Lambda\end{pmatrix} \in R^{(n+m)\times(n+m)}$. $\omega = -\max\left[\mathrm{Re}\lambda\begin{pmatrix}P + Q\\ \Lambda\end{pmatrix}\right] = \min\left|\mathrm{Re}\lambda\begin{pmatrix}P + Q\\ \Lambda\end{pmatrix}\right|$ is the minimum absolute value of the real part of the eigenvalue of matrix $\begin{pmatrix}P + Q\\ \Lambda\end{pmatrix}$.

Proof According to system (2) and (4), the adaptive synchronization error system can be shown as

$$\begin{cases} D^q e^{(s)} = P_l e^{(s)} + P_n(y, a) - P_n(x, a) + \kappa(x, y, a) \\ D^q a = \Lambda e \end{cases}. \tag{5}$$

Since $P_n(y, a) - P_n(x, a) = \Psi_l(x, a_0)e + \Psi_n(e, x, a_0)$, the system (5) can be rewritten as

$$\begin{cases} D^q e^{(s)} = P_l e^{(s)} + \Psi_l(x, a_0)e \\ \qquad\qquad + \Psi_n(e, x, a_0) + \kappa(x, y, a)\,. \\ D^q a = \Lambda e \end{cases} \tag{6}$$

It follows from $D^q a_0 = 0$ and $D^q a = D^q(a - a_0) = D^q e^{(a)}$ that

$$\begin{cases} D^q e^{(s)} = P_l e^{(s)} + \Psi_l(x, a_0)e + \Psi_n(e, x, a_0) \\ \qquad\qquad + \kappa(x, y, a) \\ D^q e^{(a)} = \Lambda e \end{cases}. \tag{7}$$

Invoking $\kappa(x, y, a) = [Q - \Psi_l(x, a_0)]e$ and $e = (e^{(s)}, e^{(a)})^{\mathbf{T}}$, one can derive that,

$$\begin{cases} D^q e^{(s)} = P_l e^{(s)} + Qe + \Psi_n(e, x, a_0) \\ D^q e^{(a)} = \Lambda e \end{cases}. \tag{8}$$

Using $P(i, j) = P_l(i, j)(1 \le j \le n)$ and $P(i, j) = 0(n+1 \le j \le n+m)$, system (8) can be rewritten as

$$\begin{cases} D^q e^{(s)} = (P+Q)e + \Psi_n(e, x, a_0) \\ D^q e^{(a)} = \Lambda e \end{cases}.$$

Therefore,

$$D^q e = \begin{pmatrix} P+Q \\ \Lambda \end{pmatrix} e + \begin{pmatrix} \Psi_n(e, x, a_0) \\ 0 \end{pmatrix}, \quad (9)$$

where matrix $\begin{pmatrix} \Psi_n(e, x, a_0) \\ 0 \end{pmatrix} \in R^{(n+m)\times 1}$.

Let $e(1)$ and $e(2)$ be the initial conditions for system (9). Taking Laplace transform on Eq. (9), we obtain

$$s^q L(e(t)) - s^{q-1} e(1) - s^{q-2} e(2) = GL(e(t)) + L(\Phi_n(e(t), x(t))) \quad (10)$$

where $G = \begin{pmatrix} P+Q \\ \Lambda \end{pmatrix}$, $\Phi_n[e(t), x(t)] = \begin{pmatrix} \Psi_n[e(t), x(t), a_0] \\ 0 \end{pmatrix}$, and $L(.)$ denotes the Laplace transform. So, we can derive that

$$L(e(t)) = \frac{s^{q-1}}{s^q - G} e(1) + \frac{s^{q-2}}{s^q - G} e(2) + \frac{L(\Phi_n(e(t), x(t))}{s^q - G} \quad (11)$$

Taking Laplace inverse transform on Eq. (11) via the Mittag-Leffler function in two-parameter, the solution $e(t)$ of fractional-order system (9) can be shown as

$$e(t) = M_{q,1}(Gt^q)e(1) + tM_{q,2}(Gt^q)e(2) + \int_0^t (t-\tau)^{q-1} M_{q,q}(G(t-\tau)^q)\Phi_n[e(\tau), x(\tau)]\mathrm{d}\tau \quad (12)$$

where $\Phi_n[e(\tau), x(\tau)] = \begin{pmatrix} \Psi_n[e(\tau), x(\tau), a_0] \\ 0 \end{pmatrix}$. The matrix $M_{q,1}(Gt^q)$, $M_{q,2}(Gt^q)$ and $M_{q,q}(G(t-\tau)^q)$ denote the two-parameter function of Mittag-Leffler type.

According to Lemma 1 and Eq. (12), the following result can be yielded,

$$\begin{aligned} \|e(t)\| &\le \left\| M_{q,1}(Gt^q)e(1) \right\| + \left\| tM_{q,2}(Gt^q)e(2) \right\| \\ &\quad + \left\| \int_0^t (t-\tau)^{q-1} M_{q,q}(G(t-\tau)^q)\Phi_n[e(\tau), x(\tau)]\mathrm{d}\tau \right\| \\ &\le \left\| \mathrm{e}^{Gt^q} e(1) \right\| + \left\| \mathrm{e}^{Gt^q} e(2) \right\| t \\ &\quad + \int_0^t (t-\tau)^{q-1} \left\| \mathrm{e}^{G(t-\tau)^q} \Phi_n[e(\tau), x(\tau)] \right\| \mathrm{d}\tau. \end{aligned} \quad (13)$$

Due to $\mathrm{Re}[\lambda \begin{pmatrix} P+Q \\ \Lambda \end{pmatrix}] < 0$, $G = \begin{pmatrix} P+Q \\ \Lambda \end{pmatrix}$ is a stable matrix. So,

$$\left\| \mathrm{e}^{Gt} \right\| \le l_0 \mathrm{e}^{-\omega t}, \quad \left\| \mathrm{e}^{Gt^q} \right\| \le l_0 \mathrm{e}^{-\omega t^q} \le l_0 \mathrm{e}^{-\omega t},$$

where $l_0 > 0$.

According to the inequality (13), the following result can be obtained

$$\|e(t)\| \le l_0 \mathrm{e}^{-\omega t} \|e(1)\| + l_0 \mathrm{e}^{-\omega t} \|e(2)\| t + l_0 \int_0^t (t-\tau)^{q-1} \mathrm{e}^{-\omega(t-\tau)} \|\Phi_n[e(\tau), x(\tau)\| \mathrm{d}\tau, \quad (14)$$

Since $\left.\begin{pmatrix} \Psi_n(e,x,a_0) \\ 0 \end{pmatrix}\right|_{e=0} = 0$, $\lim\limits_{e \to 0} \frac{\left\| \begin{pmatrix} \Psi_n(e,x,a_0) \\ 0 \end{pmatrix} \right\|}{\|e\|} = 0$ for any x. So, $\Phi_n(e, x)|_{e=0} = 0$, $\lim\limits_{e \to 0} \frac{\|\Phi_n(e,x)\|}{\|e\|} = 0$ for any x. Hence, there exists a positive constant β such that

$$\|\Phi_n[e(t), x(t)]\| \le \|e(t)\| / l_0 \text{ as } \|e(t)\| < \beta.$$

Thus, the inequality (14) can be rewritten as

$$\|e(t)\| \le l_0 \mathrm{e}^{-\omega t} \|e(1)\| + l_0 \mathrm{e}^{-\omega t} \|e(2)\| t + \int_0^t (t-\tau)^{q-1} \mathrm{e}^{-\omega(t-\tau)} \|e(\tau)\| \mathrm{d}\tau. \quad (15)$$

From the inequality (15), one has the follows,

$$\|e(t)\| \mathrm{e}^{\omega t} \le l_0 \|e(1)\| + l_0 \|e(2)\| t + \int_0^t (t-\tau)^{q-1} \mathrm{e}^{\omega \tau} \|e(\tau)\| \mathrm{d}\tau. \quad (16)$$

According to Lemma 2 (generalized Gronwall inequality), the inequality (16) can be turned to

$$\|e(t)\| \mathrm{e}^{\omega t} \le (l_0 \|e(1)\| + l_0 \|e(2)\| t) M_{q,1}[\Gamma(q) t^q] \quad (17)$$

According to Lemma 3, the inequality (17) can be turned to

$$\|e(t)\| \mathrm{e}^{\omega t} \le (l_0 \|e(1)\| + l_0 \|e(2)\| t)[l_1 \mathrm{e}^{t(\Gamma(q))^{1/q}} + l_2/(1 + \Gamma(q) t^q)], \quad (18)$$

that is

$$\|e(t)\| \le (l_0\,\|e(1)\| + l_0\,\|e(2)\|\,t) l_1 e^{t[(\Gamma(q))^{1/q}-\omega]} \quad (19)$$
$$+ (l_0\,\|e(1)\| + l_0\,\|e(2)\|\,t) l_2 \{[1+\Gamma(q)t^q]e^{\omega t}\}^{-1}$$

Due to $\omega = -\max\left[\mathrm{Re}\lambda\begin{pmatrix} P+Q \\ \Lambda \end{pmatrix}\right] > [\Gamma(q)]^{1/q}$, therefore $\omega > 0$ and $[\Gamma(q)]^{1/q} - \omega < 0$. Then,

$$\lim_{t\to+\infty}(l_0\,\|e(1)\| + l_0\,\|e(2)\|\,t) l_1 e^{t[(\Gamma(q))^{1/q}-\omega]} = 0,$$
$$\lim_{t\to+\infty}(l_0\,\|e(1)\| + l_0\,\|e(2)\|\,t) l_2 \{[1+\Gamma(q)t^q]e^{\omega t}\}^{-1} = 0$$

From the inequality (19), the following result is derived,

$$\lim_{t\to+\infty}\|e(t)\| = 0. \quad (20)$$

So, the zero solution in the error system (9) is asymptotically stable. It implies,

$$\lim_{t\to+\infty}\|y-x\| = 0, \quad \lim_{t\to+\infty}(a-a_0) = 0.$$

Therefore, the fractional-order system (4) and the fractional-order chaotic system (2) can be arrived to synchronization. The proof is completed. □

3 Illustrative example

In order to verify the effectiveness of the above synchronization scheme, we illustrate two examples: (1) the fractional-order Lorenz chaotic system with fractional-order $q = 1.08$ and with partially or fully unknown parameters [16]; (2) the modified fractional-order Chua's chaotic system with fractional-order $q = 1.1$ and with fully unknown parameters [35]. The numerical results are in agreement with the theoretical analysis.

3.1 Adaptive synchronization of the fractional-order Lorenz chaotic system with fractional-order $q = 1.08$

The fractional-order Lorenz system is described by

$$\begin{pmatrix} D^q x_1 \\ D^q x_2 \\ D^q x_3 \end{pmatrix} = \begin{pmatrix} a_1(x_2 - x_1) \\ a_2 x_1 - x_2 - x_1 x_3 \\ x_1 x_2 - a_3 x_3 \end{pmatrix}, \quad (21)$$

where $a_i (i = 1, 2, 3)$ is the system parameters [16]. The fractional-order Lorenz system displays a chaotic attractor for $a_1 = a_{10} = 10$, $a_2 = a_{20} = 28$, $a_3 = a_{30} = 8/3$ and $q = 1.08$, as shown in Fig. 1.

Case 1. Parameter a_1 is unknown in the fractional-order system

Let a_1 be the unknown parameter in the fractional-order Lorenz system (21), and its estimation be $a_{10} = 10$.

Now, the fractional-order system (21) can be modified as

$$\begin{pmatrix} D^q x_1 \\ D^q x_2 \\ D^q x_3 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 28 & -1 & 0 \\ 0 & 0 & -8/3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} a_1(x_2 - x_1) \\ -x_1 x_3 \\ x_1 x_2 \end{pmatrix} \quad (22)$$

So,

$$P_l = \begin{pmatrix} 0 & 0 & 0 \\ 28 & -1 & 0 \\ 0 & 0 & -8/3 \end{pmatrix}, \quad P_n(x,a) = \begin{pmatrix} a_1(x_2 - x_1) \\ -x_1 x_3 \\ x_1 x_2 \end{pmatrix}.$$

Therefore, the response system with linear parameter update laws is

$$\begin{cases} D^q y = P_l y + P_n(y,a) + \kappa(x,y,a) \\ D^q a = \Lambda e \end{cases}, \quad (23)$$

where $\Lambda \in R^{1\times 4}$, $\kappa(x,y,a) = [Q - \Psi_l(x, a_{10})]e$, and $a = a_1$.

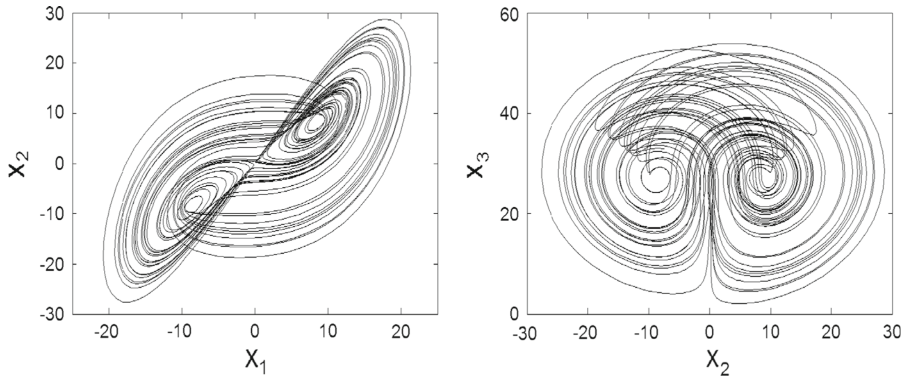

Fig. 1 The attractor of fractional-order Lorenz system for $a_1 = a_{10} = 10$, $a_2 = a_{20} = 28$, $a_3 = a_{30} = 8/3$, and $q = 1.08$

Now, we can obtain the follows,

$$P_n(y,a)-P_n(x,a)=\begin{pmatrix} a_1(y_2-y_1)\\ -y_1y_3\\ y_1y_2\end{pmatrix}-\begin{pmatrix} a_1(x_2-x_1)\\ -x_1x_3\\ x_1x_2\end{pmatrix}=\begin{pmatrix} -a_{10} & a_{10} & 0 & 0\\ -x_3 & 0 & -x_1 & 0\\ x_2 & x_1 & 0 & 0\end{pmatrix}\begin{pmatrix} e_1\\ e_2\\ e_3\\ e_{a_1}\end{pmatrix}+\begin{pmatrix} e_{a_1}(e_2-e_1)\\ -e_1e_3\\ e_1e_2\end{pmatrix}$$

and

$$\Psi_l(x,a_0)=\begin{pmatrix} -a_{10} & a_{10} & 0 & 0\\ -x_3 & 0 & -x_1 & 0\\ x_2 & x_1 & 0 & 0\end{pmatrix},$$

$$\Psi_n(e,x,a_0)=\begin{pmatrix} e_{a_1}(e_2-e_1)\\ -e_1e_3\\ e_1e_2\end{pmatrix}.$$

It is easily to obtain the follows,

$$\left.\begin{pmatrix}\Psi_n(e,x,a_0)\\ 0\end{pmatrix}\right|_{e=0}=0$$

and

$$\frac{\left\|\begin{pmatrix}\Psi_n(e,x,a_0)\\ 0\end{pmatrix}\right\|}{\|e\|}=\sqrt{\frac{e_{a_1}^2(e_2-e_1)^2+(e_1e_3)^2+(e_1e_2)^2}{e_1^2+e_2^2+e_3^2+e_{a_1}^2}}\le\sqrt{(e_2-e_1)^2+e_3^2+e_2^2}$$

So,

$$\lim_{e\to 0}\frac{\left\|\begin{pmatrix}\Psi_n(e,x,a_0)\\ 0\end{pmatrix}\right\|}{\|e\|}=\lim_{e\to 0}\sqrt{(e_2-e_1)^2+e_3^2+e_2^2}=0$$

Therefore, the first condition in the *Theorem* is satisfied.

Now, select suitable constant matrices $Q\in R^{3\times 4}$ and $\Lambda\in R^{1\times 4}$ such that

$$\mathrm{Re}\left[\lambda\begin{pmatrix}P+Q\\ \Lambda\end{pmatrix}\right]<0,\ \omega=-\max\left[\mathrm{Re}\lambda\begin{pmatrix}P+Q\\ \Lambda\end{pmatrix}\right]>[\Gamma(q)]^{1/q}$$

So, the second condition in the *Theorem* is true. These results indicate that the adaptive synchronization between drive system (22) and response system (23) with linear parameter update law can be arrived.

For example, let $Q=\begin{pmatrix}-1&0&0&0\\ 0&0&0&0\\ 0&0&0&0\end{pmatrix}$ and $\Lambda=\begin{pmatrix}0&0&0&-1\end{pmatrix}$. So,

$$\begin{pmatrix}P+Q\\ \Lambda\end{pmatrix}=\begin{pmatrix}-1&0&0&0\\ 28&-1&0&0\\ 0&0&-8/3&0\\ 0&0&0&-1\end{pmatrix}.$$

Therefore, $\lambda_i=-1(i=1,2,3)$, $\lambda_4=-8/3$, and $-\max\left[\mathrm{Re}\lambda\begin{pmatrix}P+Q\\ \Lambda\end{pmatrix}\right]=1>[\Gamma(q)]^{1/q}=0.9627$, respectively. Simulation results are displayed in Fig. 2. All the initial conditions in this paper are $(x_{10},x_{20},x_{30})=(3,4,5)$ and $(y_{10},y_{20},y_{30})=(10,20,30)$, respectively. Here $a_1(0)=15$.

Case 2. Parameters a_2 and a_3 are unknown in the fractional-order system

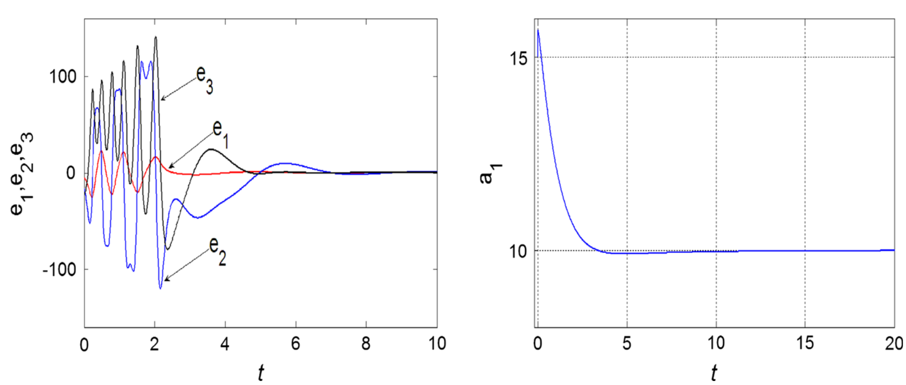

Fig. 2 Synchronization errors between systems (22) and (23)

Let a_2 and a_3 be the unknown parameters in the fractional-order Lorenz system (21), and their estimation be $a_{20} = 28$ and $a_{30} = 8/3$, respectively.

Now, the fractional-order system (21) can be adapted as

$$\begin{pmatrix} D^q x_1 \\ D^q x_2 \\ D^q x_3 \end{pmatrix} = \begin{pmatrix} -10 & 10 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} 0 \\ a_2 x_1 - x_1 x_3 \\ x_1 x_2 - a_3 x_3 \end{pmatrix} \tag{24}$$

So,

$$P_l = \begin{pmatrix} -10 & 10 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad P_n(x, a) = \begin{pmatrix} 0 \\ a_2 x_1 - x_1 x_3 \\ x_1 x_2 - a_3 x_3 \end{pmatrix}.$$

Therefore, the response system with linear parameter update laws is

$$\begin{cases} D^q y = P_l y + P_n(y, a) + \kappa(x, y, a) \\ D^q a = \Lambda e \end{cases}, \tag{25}$$

where $a = \begin{pmatrix} a_2 & a_3 \end{pmatrix}^{\mathbf{T}}$, $\Lambda \in R^{2\times 5}$, $\kappa(x, y, a) = [Q - \Psi_l(x, a_0)]e$ and $a_0 = \begin{pmatrix} a_{20} & a_{30} \end{pmatrix}^{\mathbf{T}}$.

Now, we have the follows,

$$P_n(y, a) - P_n(x, a) = \begin{pmatrix} 0 \\ a_2 y_1 - y_1 y_3 \\ y_1 y_2 - a_3 y_3 \end{pmatrix} - \begin{pmatrix} 0 \\ a_2 x_1 - x_1 x_3 \\ x_1 x_2 - a_3 x_3 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ a_{20} - x_3 & 0 & -x_1 & 0 & 0 \\ x_2 & x_1 & -a_{30} & 0 & 0 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \\ e_{a_2} \\ e_{a_3} \end{pmatrix} + \begin{pmatrix} 0 \\ e_{a_2} e_1 - e_1 e_3 \\ -e_{a_3} e_3 + e_1 e_2 \end{pmatrix}$$

and

$$\Psi_l(x, a_0) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ a_{20} - x_3 & 0 & -x_1 & 0 & 0 \\ x_2 & x_1 & -a_{30} & 0 & 0 \end{pmatrix},$$

$$\Psi_n(e, x, a_0) = \begin{pmatrix} 0 \\ e_{a_2} e_1 - e_1 e_3 \\ -e_{a_3} e_3 + e_1 e_2 \end{pmatrix}.$$

It is easily to derive the follows,

$$\left.\begin{pmatrix} \Psi_n(e, x, a_0) \\ 0 \end{pmatrix}\right|_{e=0} = 0$$

and

$$\frac{\left\| \begin{pmatrix} \Psi_n(e, x, a_0) \\ 0 \end{pmatrix} \right\|}{\|e\|}$$

$$= \sqrt{\frac{(e_{a_2} e_1 - e_1 e_3)^2 + (-e_{\alpha_3} e_3 + e_1 e_2)^2}{e_1^2 + e_2^2 + e_3^2 + e_{a_2}^2 + e_{a_3}^2}}$$

$$\le \sqrt{(e_{a_2} - e_3)^2 + \frac{(|e_{a_3} e_3| + |e_1 e_2|)^2}{e_1^2 + e_2^2 + e_3^2 + e_{a_2}^2 + e_{a_3}^2}}$$

$$\le \sqrt{(e_{a_2} - e_3)^2 + e_{a_3}^2 + e_1^2 + \frac{2\,|e_{a_3} e_3|\,|e_1 e_2|}{(e_1^2 + e_2^2) + (e_3^2 + e_{a_3}^2)}}$$

$$\le \sqrt{(e_{a_2} - e_3)^2 + e_{a_3}^2 + e_1^2 + \frac{2\,|e_{a_3} e_3|\,|e_1 e_2|}{2\,|e_1 e_2| + 2\,|e_{a_3} e_3|}}$$

$$\le \sqrt{(e_{a_2} - e_3)^2 + e_{a_3}^2 + e_1^2 + \left(\frac{1}{|e_{a_3} e_3|} + \frac{1}{|e_1 e_2|}\right)^{-1}}$$

$$\le \sqrt{(e_{a_2} - e_3)^2 + e_{a_3}^2 + e_1^2 + |e_{a_3} e_3| + |e_1 e_2|}.$$

Hence,

$$\lim_{e\to 0} \frac{\left\| \begin{pmatrix} \Psi_n(e, x, a_0) \\ 0 \end{pmatrix} \right\|}{\|e\|}$$

$$= \lim_{e\to 0} \sqrt{(e_{a_2} - e_3)^2 + e_{a_3}^2 + e_1^2 + |e_{a_3} e_3| + |e_1 e_2|} = 0$$

Therefore, the first condition of the *Theorem* is fulfilled.

Now, select suitable constant matrices $Q \in R^{3\times 5}$ and $\Lambda \in R^{2\times 5}$ such that

$$\mathrm{Re}\left[\lambda \begin{pmatrix} P + Q \\ \Lambda \end{pmatrix}\right] < 0, \quad \omega = -\max\left[\mathrm{Re}\lambda \begin{pmatrix} P + Q \\ \Lambda \end{pmatrix}\right] > [\Gamma(q)]^{1/q}$$

So, the second condition in the *Theorem* is fulfilled. These results imply that the adaptive synchronization between drive system (24) and response system (25) with linear parameter update law can be realized.

For example, let $Q = \begin{pmatrix} 0 & -10 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \end{pmatrix}$ and $\Lambda = \begin{pmatrix} 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 1 & -1 \end{pmatrix}$. So,

$$\begin{pmatrix} P + Q \\ \Lambda \end{pmatrix} = \begin{pmatrix} -10 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 1 & -1 \end{pmatrix}.$$

Therefore, $\lambda_1 = -10, \lambda_{2,3} = -1, \lambda_{4,5} = -1 \pm j$, and $-\max\left[\mathrm{Re}\lambda \begin{pmatrix} P + Q \\ \Lambda \end{pmatrix}\right] = 1 > [\Gamma(q)]^{1/q} = 0.9627$, respectively. Simulation results are shown in Fig. 3. Here $a_2(0) = 20$, $a_3(0) = 5$, respectively.

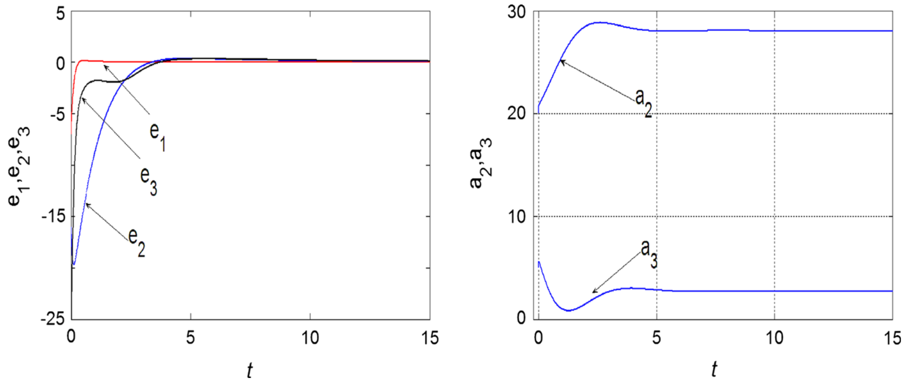

Fig. 3 Synchronization errors between systems (24) and (25)

Case 3. Parameters a_1, a_2, and a_3 are unknown in the fractional-order system

Let a_1, a_2 and a_3 be the unknown parameters in the fractional-order Lorenz system (21), and their estimation be $a_{10} = 10$, $a_{20} = 28$ and $a_{30} = 8/3$, respectively.

The fractional-order system (21) can be rewritten as

$$\begin{pmatrix} D^q x_1 \\ D^q x_2 \\ D^q x_3 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} a_1(x_2 - x_1) \\ a_2 x_1 - x_1 x_3 \\ x_1 x_2 - a_3 x_3 \end{pmatrix}. \tag{26}$$

So,

$$P_l = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad P_n(x, a) = \begin{pmatrix} a_1(x_2 - x_1) \\ a_2 x_1 - x_1 x_3 \\ x_1 x_2 - a_3 x_3 \end{pmatrix}.$$

Therefore, the response system with linear parameter update laws is

$$\begin{cases} D^q y = P_l y + P_n(y, a) + \kappa(x, y, a) \\ D^q a = \Lambda e \end{cases} \tag{27}$$

Where $a = \begin{pmatrix} a_1 & a_2 & a_3 \end{pmatrix}^{\mathbf{T}}$, $\Lambda \in R^{3\times 6}$, $\kappa(x, y, a) = [Q - \Psi_l(x, a_0)]e$ and $a_0 = \begin{pmatrix} a_{10} & a_{20} & a_{30} \end{pmatrix}^{\mathbf{T}}$.

Now, the following results can be gained,

$$P_n(y, a) - P_n(x, a) = \begin{pmatrix} a_1(y_2 - y_1) \\ a_2 y_1 - y_1 y_3 \\ y_1 y_2 - a_3 y_3 \end{pmatrix} - \begin{pmatrix} a_1(x_2 - x_1) \\ a_2 x_1 - x_1 x_3 \\ x_1 x_2 - a_3 x_3 \end{pmatrix} = \begin{pmatrix} -a_{10} & a_{10} & 0 & 0\;0\;0 \\ a_{20} - x_3 & 0 & -x_1 & 0\;0\;0 \\ x_2 & x_1 & -a_{30} & 0\;0\;0 \end{pmatrix} \times \begin{pmatrix} e_1 \\ e_2 \\ e_3 \\ e_{a_1} \\ e_{a_2} \\ e_{a_3} \end{pmatrix} + \begin{pmatrix} e_{a_1}(e_2 - e_1) \\ e_{a_2} e_1 - e_1 e_3 \\ -e_{a_3} e_3 + e_1 e_2 \end{pmatrix},$$

and

$$\Psi_l(x, a_0) = \begin{pmatrix} -a_{10} & a_{10} & 0 & 0\;0\;0 \\ a_{20} - x_3 & 0 & -x_1 & 0\;0\;0 \\ x_2 & x_1 & -a_{30} & 0\;0\;0 \end{pmatrix},$$

$$\Psi_n(e, x, a_0) = \begin{pmatrix} e_{a_1}(e_2 - e_1) \\ e_{a_2} e_1 - e_1 e_3 \\ -e_{a_3} e_3 + e_1 e_2 \end{pmatrix}.$$

It is easily to get the follows,

$$\left.\begin{pmatrix} \Psi_n(e, x, a_0) \\ 0 \end{pmatrix}\right|_{e=0} = 0$$

and

$$\frac{\left\| \begin{pmatrix} \Psi_n(e, x, a_0) \\ 0 \end{pmatrix} \right\|}{\|e\|} = \sqrt{\frac{e_{a_1}^2(e_2 - e_1)^2 + (e_{a_2} e_1 - e_1 e_3)^2 + (-e_{a_3} e_3 + e_1 e_2)^2}{e_1^2 + e_2^2 + e_3^2 + e_{a_1}^2 + e_{a_2}^2 + e_{a_3}^2}}$$
$$\leq \sqrt{(e_2 - e_1)^2 + (e_{a_2} - e_3)^2 + \frac{(|e_{a_3} e_3| + |e_1 e_2|)^2}{e_1^2 + e_2^2 + e_3^2 + e_{a_1}^2 + e_{a_2}^2 + e_{a_3}^2}}$$
$$\leq \sqrt{(e_2 - e_1)^2 + (e_{a_2} - e_3)^2 + e_{a_3}^2 + e_1^2 + |e_{a_3} e_3| + |e_1 e_2|}$$

So,

$$\lim_{e\to 0} \frac{\left\| \begin{pmatrix} \Psi_n(e, x, a_0) \\ 0 \end{pmatrix} \right\|}{\|e\|} = \lim_{e\to 0} \sqrt{(e_2 - e_1)^2 + (e_{a_2} - e_3)^2 + e_{a_3}^2 + e_1^2 + |e_{a_3} e_3| + |e_1 e_2|} = 0$$

Therefore, the first condition of the *Theorem* is fulfilled.

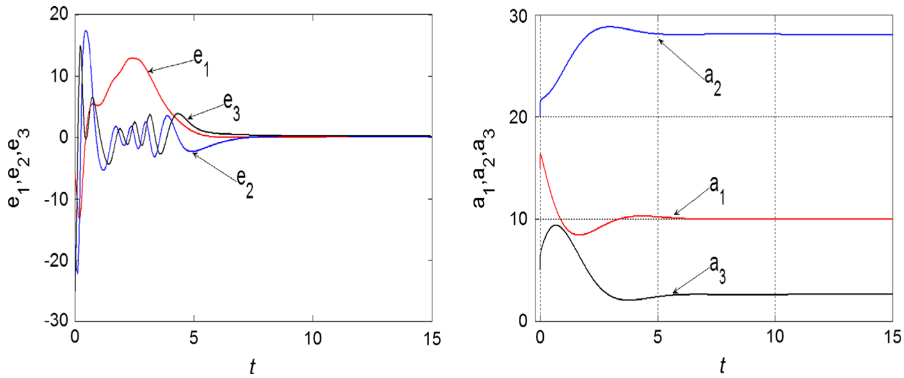

Fig. 4 Synchronization errors between systems (26) and (27)

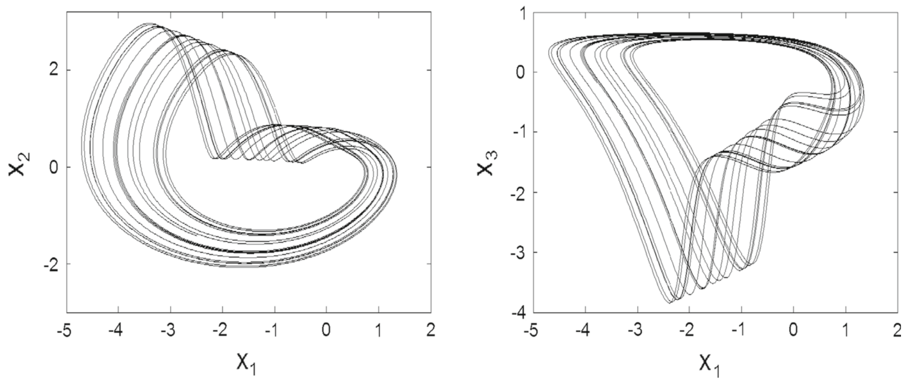

Fig. 5 The attractor of the modified fractional-order Chua's chaotic system for $a_1 = a_{10} = 1/2$, $a_2 = a_{20} = 1.2$, and $q = 1.08$

Now, select suitable constant matrices $Q \in R^{3\times 6}$ and $\Lambda \in R^{3\times 6}$ such that

$$\mathrm{Re}\left[\lambda\begin{pmatrix} P+Q \\ \Lambda \end{pmatrix}\right] < 0,$$

$$\omega = -\max\left[\mathrm{Re}\lambda\begin{pmatrix} P+Q \\ \Lambda \end{pmatrix}\right] > [\Gamma(q)]^{1/q}$$

So, the second condition in the *Theorem* is fulfilled. These results indicate that the adaptive synchronization between drive system (26) and response system (27) with linear parameter update law can be achieved.

For example, let $Q = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \end{pmatrix}$ and $\Lambda = \begin{pmatrix} 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & -1 & -1 & -1 \end{pmatrix}$. So,

$$\begin{pmatrix} P+Q \\ \Lambda \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & -1 & -1 & -1 \end{pmatrix}.$$

Therefore, $\lambda_i = -1 (i = 1, 2, 3)$, $\lambda_{4,5} = -1 \pm j$, $\lambda_6 = -1$, and $-\max\left[\mathrm{Re}\lambda\begin{pmatrix} P+Q \\ \Lambda \end{pmatrix}\right] = 1 > [\Gamma(q)]^{1/q} = 0.9627$, respectively. Simulation results are presented in Fig. 4. Here $a_1(0) = 15$, $a_2(0) = 20$, and $a_3(0) = 5$, respectively.

3.2 Adaptive synchronization of the modified fractional-order Chua's chaotic system with fractional-order $q = 1.1$

In 2010, the simplest modified Chua's chaotic circuit was built by Muthuswamy and Chua [35]. In this paper, based on this modified Chua's chaotic circuit, the modified fractional-order Chua's chaotic system can be shown as

$$\begin{cases} D^q x_1 = x_2 \\ D^q x_2 = a_1(x_2 - x_1) - 0.5x_2x_3^2 \\ D^q x_3 = -x_2 - a_2x_3 + x_2x_3 \end{cases} \tag{28}$$

where $a_i (i = 1, 2)$ is the system parameter. The modified fractional-order Chua's system (28) displays a chaotic attractor for $a_1 = a_{10} = 1/2$, $a_2 = a_{20} = 1.2$ and $q = 1.1$. Its chaotic attractor is displayed as Fig. 5.

Now, let a_1 and a_2 be the unknown parameters in the modified fractional-order Chua's system (28), and their estimation be $a_1 = a_{10} = 1/2$ and $a_2 = a_{20} = 1.2$, respectively.

Now, the fractional-order system (28) can be adapted as

$$\begin{pmatrix} D^q x_1 \\ D^q x_2 \\ D^q x_3 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} 0 \\ a_1(x_2 - x_1) - 0.5x_2x_3^2 \\ -a_2x_3 + x_2x_3 \end{pmatrix} \tag{29}$$

So,

$$P_l = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix},$$

$$P_n(x,a) = \begin{pmatrix} 0 \\ a_1(x_2 - x_1) - 0.5x_2x_3^2 \\ -a_2x_3 + x_2x_3 \end{pmatrix}.$$

Therefore, the response system with linear parameter update laws is

$$\begin{cases} D^q y = P_l y + P_n(y,a) + \kappa(x,y,a) \\ D^q a = \Lambda e \end{cases}, \tag{30}$$

where $a = (a_1\ a_2)^{\mathbf{T}}$, $\Lambda \in R^{2\times 5}$, $\kappa(x,y,a) = [Q - \Psi_l(x,a_0)]e$ and $a_0 = (a_{10}\ a_{20})^{\mathbf{T}}$.

Now, we have the follows,

$$P_n(y,a) - P_n(x,a) = \begin{pmatrix} 0 \\ a_1(y_2 - y_1) - 0.5y_2y_3^2 \\ -a_2y_3 + y_2y_3 \end{pmatrix} - \begin{pmatrix} 0 \\ a_1(x_2 - x_1) - 0.5x_2x_3^2 \\ -a_2x_3 + x_2x_3 \end{pmatrix}$$
$$= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ -a_{10} & a_{10} - 0.5x_3^2 & -x_2x_3 & 0 & 0 \\ 0 & x_3 & x_2 - a_{20} & 0 & 0 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \\ e_{a_1} \\ e_{a_2} \end{pmatrix} + \begin{pmatrix} 0 \\ e_{a_1}(e_2 - e_1) - 0.5e_2e_3^2 - e_2e_3x_3 - 0.5x_2e_3^2 \\ -e_{a_2}e_3 + e_2e_3 \end{pmatrix}$$

and

$$\Psi_l(x,a_0) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ -a_{10} & a_{10} - 0.5x_3^2 & -x_2x_3 & 0 & 0 \\ 0 & x_3 & x_2 - a_{20} & 0 & 0 \end{pmatrix},$$

$$\Psi_n(e,x,a_0) = \begin{pmatrix} 0 \\ e_{a_1}(e_2-e_1) - 0.5e_2e_3^2 - e_2e_3x_3 - 0.5x_2e_3^2 \\ -e_{a_2}e_3 + e_2e_3 \end{pmatrix}.$$

It is easily to derive the follows,

$$\left.\begin{pmatrix} \Psi_n(e,x,a_0) \\ 0 \end{pmatrix}\right|_{e=0} = 0$$

and

$$\frac{\left\| \begin{pmatrix} \Psi_n(e,x,a_0) \\ 0 \end{pmatrix} \right\|}{\|e\|} = \sqrt{\frac{[e_{a_1}(e_2 - e_1) - 0.5e_2e_3^2 - e_2e_3x_3 - 0.5x_2e_3^2]^2 + [e_2e_3 - e_{a_2}e_3]^2}{e_1^2 + e_2^2 + e_3^2 + e_{a_1}^2 + e_{a_2}^2}}$$
$$\le \sqrt{(e_2 - e_{a_2})^2 + \frac{[e_{a_1}(e_2 - e_1) - 0.5e_2e_3^2 - e_2e_3x_3 - 0.5x_2e_3^2]^2}{e_1^2 + e_2^2 + e_3^2 + e_{a_1}^2 + e_{a_2}^2}}$$
$$\le \sqrt{A + (e_2x_3 - 0.5x_2e_3)^2 + \frac{\left|2[e_{a_1}(e_2 - e_1) - 0.5e_2e_3^2][e_2e_3x_3 + 0.5x_2e_3^2]\right|}{e_1^2 + e_2^2 + e_3^2 + e_{a_1}^2 + e_{a_2}^2}}.$$

Here $A = (e_2 - e_{a_2})^2 + (e_2 - e_1)^2 + 0.25e_3^4 - e_{a_1}e_2(e_2 - e_1)$. Therefore,

$$\frac{\left\|\begin{pmatrix}\Psi_n(e,x,a_0)\\0\end{pmatrix}\right\|}{\|e\|} \le \sqrt{A + (e_2x_3 - 0.5x_2e_3)^2 + \frac{\left|2[e_{a_1}(e_2-e_1) - 0.5e_2e_3^2][e_2e_3x_3 + 0.5x_2e_3^2]\right|}{e_1^2+e_2^2+e_3^2+e_{a_1}^2+e_{a_2}^2}}$$

$$\le \sqrt{A + (e_2x_3 - 0.5x_2e_3)^2 + \left|[e_{a_1}(e_2-e_1) - 0.5e_2e_3^2]x_2\right| + \frac{\left|2[e_{a_1}(e_2-e_1) - 0.5e_2e_3^2]e_2e_3x_3\right|}{e_1^2+e_2^2+e_3^2+e_{a_1}^2+e_{a_2}^2}}$$

$$\le \sqrt{A + (e_2x_3 - 0.5x_2e_3)^2 + \left|[e_{a_1}(e_2-e_1) - 0.5e_2e_3^2]x_2\right| + \left|e_2^2e_3x_3\right| + \frac{\left|2e_{a_1}(e_2-e_1)e_2e_3x_3\right|}{e_1^2+e_2^2+e_3^2+e_{a_1}^2+e_{a_2}^2}}$$

$$\le \sqrt{A + (e_2x_3 - 0.5x_2e_3)^2 + \left|[e_{a_1}(e_2-e_1) - 0.5e_2e_3^2]x_2\right| + \left|e_2^2e_3x_3\right| + \left|2e_{a_1}e_3x_3\right| + \frac{\left|2e_{a_1}e_1e_2e_3x_3\right|}{e_1^2+e_2^2}}$$

$$\le \sqrt{A + (e_2x_3 - 0.5x_2e_3)^2 + \left|[e_{a_1}(e_2-e_1) - 0.5e_2e_3^2]x_2\right| + \left|e_2^2e_3x_3\right| + 3\left|e_{a_1}e_3x_3\right|}$$

According to the boundedness of the modified Chua's chaotic system (28), there exist a real positive constant N such that

$$N \ge \max(|x_2|, |x_3|)$$

Hence,

$$\lim_{e\to 0}\frac{\left\|\begin{pmatrix}\Psi_n(e,x,a_0)\\0\end{pmatrix}\right\|}{\|e\|} \le \lim_{e\to 0}\sqrt{A+(Ne_2-0.5Ne_3)^2+\left|[e_{a_1}(e_2-e_1)-0.5e_2e_3^2]N\right|+\left|e_2^2e_3N\right|+3\left|e_{a_1}e_3N\right|} = 0$$

Therefore, the first condition of the *Theorem* is fulfilled.

Now, select suitable constant matrices $Q \in R^{3\times 5}$ and $\Lambda \in R^{2\times 5}$ such that

$$\mathrm{Re}\left[\lambda\begin{pmatrix}P+Q\\\Lambda\end{pmatrix}\right] < 0,\ \omega = -\max\left[\mathrm{Re}\lambda\begin{pmatrix}P+Q\\\Lambda\end{pmatrix}\right] > [\Gamma(q)]^{1/q}$$

For example, let $Q = \begin{pmatrix}-1 & -1 & 0 & 0 & 0\\0 & -1 & 0 & 0 & 0\\0 & 1 & -1 & 0 & 0\end{pmatrix}$ and $\Lambda = \begin{pmatrix}0 & 0 & 0 & -1 & -1\\0 & 0 & 0 & 1 & -1\end{pmatrix}$. So, the second condition in the *Theorem* is fulfilled. These results imply that the adaptive synchronization between drive system (28) and response system (30) with linear parameter update law can be realized.

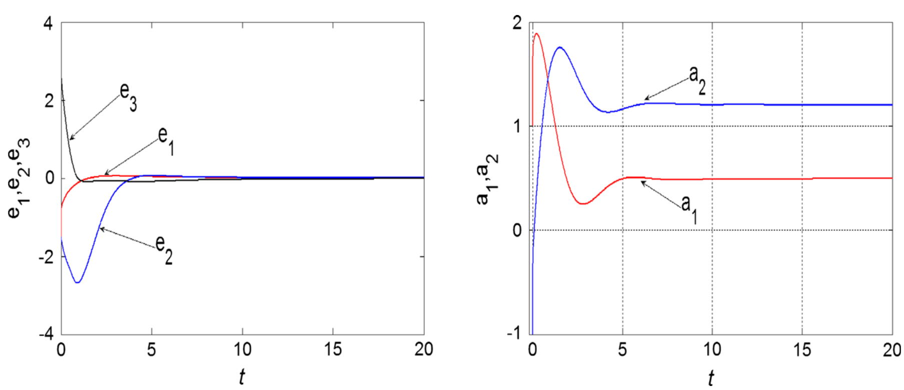

Fig. 6 Synchronization errors of adaptive synchronization for the modified fractional-order Chua's chaotic system

$$\begin{pmatrix} P+Q \\ \Lambda \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 1 & -1 \end{pmatrix}.$$

Therefore, $\lambda_{1,2,3} = -1, \lambda_{4,5} = -1 \pm j$, and $-\max\left[\mathrm{Re}\lambda \begin{pmatrix} P+Q \\ \Lambda \end{pmatrix}\right] = 1 > [\Gamma(q)]^{1/q} = 0.9557$, respectively. Simulation results are shown in Fig. 6. Here $a_1(0) = 1, a_2(0) = -1$, respectively.

4 Conclusions

The adaptive synchronization for fractional-order chaotic systems with fractional-order $1 < q < 2$ has been presented in this paper. A sufficient condition on synchronization of fractional-order chaotic systems with fractional-order $1 < q < 2$ and unknown parameters is obtained theoretically by using the Mittag–Leffler function and the generalized Gronwall inequality. Only the linear parameter update laws are used in our synchronization scheme. Furthermore, this adaptive synchronization approach is applied to the fractional-order Lorenz chaotic system with partially or fully unknown parameters for fractional-order $q = 1.08$ and the modified fractional-order Chua's chaotic system with partially or fully unknown parameters for fractional-order $q = 1.1$. The numerical results agree with the theoretical analysis well.

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