Nonlinear Dyn (2015) 80:431–446
DOI 10.1007/s11071-014-1880-5

ORIGINAL PAPER

Nonlinear wave solutions and their relations for the modified Benjamin–Bona–Mahony equation

Ming Song

Received: 16 June 2014 / Accepted: 23 December 2014 / Published online: 8 January 2015

Abstract In this paper, we use the bifurcation method of dynamical systems to investigate the nonlinear wave solutions of the modified Benjamin–Bona–Mahony equation. These nonlinear wave solutions contain periodic wave solutions, solitary wave solutions, periodic blow-up wave solutions, kink wave solutions, unbounded wave solutions and blow-up wave solutions. Some previous results are extended.

Keywords Bifurcation method · MBBM equation · Phase portraits · Nonlinear wave solutions

Mathematics Subject Classification 35C07 · 34C25 · 76B25

1 Introduction

The Benjamin–Bona–Mahony (BBM) equation

$$u_t + u_x + uu_x - u_{xxt} = 0, \tag{1}$$

which was first derived to describe an approximation for surface long waves in nonlinear dispersive media [1]. The equation can also characterize the hydromagnetics waves in cold plasma, acoustic waves in enharmonic crystals and acoustic-gravity waves in compressible fluids [2,3].

Yusufoglu [4] investigate the modified Benjamin–Bona–Mahony (MBBM)

$$u_t + u_x + au^2u_x + u_{xxt} = 0. \tag{2}$$

Yusufoglu used the exp-function method to obtain generalized solitonary solutions of Eq. (2). When $a = 1$, Eq. (2) becomes the equation

$$u_t + u_x + u^2u_x + u_{xxt} = 0, \tag{3}$$

which was studied in [5–8]. Daghan et al. [5] obtained some traveling wave solutions of Eq. (3) by using $(\frac{G'}{G})$-expansion method. Abbasbandy and Shirzadi [6] used the first integral method to obtain two real exact solutions and two complex exact solutions of Eq. (3). Yusufoglu and Bekir [7] obtained the solitons solutions, periodic solutions and complex solutions of Eq. (3) by using the tanh and sine–cosine methods. Noor et al. [8] used the exp-function method to construct some soliton solutions of Eq. (3).

The aim of this paper is to investigate the nonlinear wave solutions and their phase portraits for Eq. (2) by using the bifurcation method and qualitative theory of dynamical systems [9–16]. Through some special phase orbits, we obtain many smooth periodic wave solutions, periodic blow-up solutions, solitary wave

Research is supported by the National Natural Science Foundation of China (No. 11361069).

M. Song (✉)
Department of Mathematics, Shaoxing University, Shaoxing 312000, People's Republic of China
e-mail: songming12_15@163.com

M. Song
Department of Mathematics, Yuxi Normal University, Yuxi 653100, People's Republic of China

solutions, kink wave solutions, unbounded wave solutions and blow-up wave solutions.

The remainder of this paper is organized as follows. In Sect. 2, we present our main results. Section 3 gives the derivation for our main results. A short conclusion will be given in Sect. 4.

2 Main results

In this section, we state our main results. To relate conveniently, let

$$\alpha = -\frac{a}{c}, \qquad \beta = -\frac{c-1}{c}, \tag{4}$$

$$g_0 = \frac{2|\beta|}{3}\sqrt{\frac{\beta}{\alpha}}, \tag{5}$$

$$\xi = x - ct. \tag{6}$$

Proposition 1 *For given positive constants c and g_0, (2) has the following periodic wave solutions when $\alpha > 0$.*

(1) *If $g = 0$, we get four periodic wave solutions*

$$u_1(x,t) = \frac{a_1 + b_1 \mathrm{sn}^2(\omega_1\xi, k_1)}{c_1 + d_1 \mathrm{sn}^2(\omega_1\xi, k_1)}, \tag{7}$$

$$u_2(x,t) = \sqrt{\varphi_4^2 - \left(2\varphi_4^2 - \frac{6\beta}{\alpha}\right)\mathrm{sn}^2\left(\varphi_4\sqrt{\frac{\alpha}{6}}\xi, k_2\right)}, \tag{8}$$

$$u_{3_\pm}(x,t) = \pm\varphi_6 \mathrm{cn}\left(\sqrt{\frac{\alpha}{3}\varphi_6^2 - \beta}\,\xi, \varphi_6\sqrt{\frac{\alpha}{2\alpha\varphi_6^2 - 6\beta}}\right), \tag{9}$$

where

$$a_1 = \varphi_1\left(-\varphi_1 + \sqrt{\frac{6\beta}{\alpha} - \varphi_1^2}\right),$$

$$b_1 = \varphi_1\left(\varphi_1 + \sqrt{\frac{6\beta}{\alpha} - \varphi_1^2}\right), \tag{10}$$

$$c_1 = -\varphi_1 + \sqrt{\frac{6\beta}{\alpha} - \varphi_1^2}, \qquad d_1 = -\varphi_1 - \sqrt{\frac{6\beta}{\alpha} - \varphi_1^2}, \tag{11}$$

$$\omega_1 = \frac{-\varphi_1\sqrt{\alpha} + \sqrt{6\beta - \alpha\varphi_1^2}}{2\sqrt{6}}, \qquad k_1 = \frac{\varphi_1 + \sqrt{\frac{6\beta}{\alpha} - \varphi_1^2}}{\varphi_1 - \sqrt{\frac{6\beta}{\alpha} - \varphi_1^2}}, \tag{12}$$

$$-\sqrt{\frac{6\beta}{\alpha}} < \varphi_1 < -\sqrt{\frac{3\beta}{\alpha}}, \qquad k_2 = \sqrt{2 - \frac{6\beta}{\alpha\varphi_4^2}}, \tag{13}$$

$$\sqrt{\frac{3\beta}{\alpha}} < \varphi_4 < \sqrt{\frac{6\beta}{\alpha}}, \qquad \varphi_5 < -\sqrt{\frac{6\beta}{\alpha}}, \qquad \varphi_6 > \sqrt{\frac{6\beta}{\alpha}}. \tag{14}$$

And two solitary wave solutions

$$u_{4_\pm}(x,t) = \pm\sqrt{\frac{6\beta}{\alpha}}\mathrm{sech}\sqrt{\beta}\xi, \tag{15}$$

(2) *If $-g_0 < g < 0$, we get six periodic wave solutions*

$$u_5(x,t) = \frac{A_1\varphi_{10} + \varphi_{11}B_1 + (A_1\varphi_{10} - \varphi_{11}B_1)\mathrm{cn}\left(\sqrt{\frac{\alpha A_1 B_1}{6}}\xi, k_3\right)}{A_1 + B_1 + (A_1 - B_1)\mathrm{cn}\left(\sqrt{\frac{\alpha A_1 B_1}{6}}\xi, k_3\right)}, \tag{16}$$

$$u_6(x,t) = \frac{-2\gamma_1 + \eta_1\delta_1 + \eta_1\sqrt{\mu_1}\cos\left(\sqrt{\frac{\alpha\gamma_1}{6}}\xi\right)}{\delta_1 + \sqrt{\mu_1}\cos\left(\sqrt{\frac{\alpha\gamma_1}{6}}\xi\right)}, \tag{17}$$

$$u_7(x,t) = \frac{\varphi_{14}(\varphi_{17} - \varphi_{15}) + \varphi_{17}(\varphi_{15} - \varphi_{14})\mathrm{sn}^2(\omega_2\xi, k_4)}{\varphi_{17} - \varphi_{15} + (\varphi_{15} - \varphi_{14})\mathrm{sn}^2(\omega_2\xi, k_4)}, \tag{18}$$

$$u_8(x,t) = \frac{\varphi_{17}(-\varphi_{16} + \varphi_{14}) - \varphi_{14}(\varphi_{17} - \varphi_{16})\mathrm{sn}^2(\omega_2\xi, k_4)}{-\varphi_{16} + \varphi_{14} - (\varphi_{17} - \varphi_{16})\mathrm{sn}^2(\omega_2\xi, k_4)}, \tag{19}$$

$$u_9(x,t) = \frac{A_2\varphi_{18} + \varphi_{19}B_2 + (A_2\varphi_{18} - \varphi_{19}B_2)\mathrm{cn}\left(\sqrt{\frac{\alpha A_2 B_2}{6}}\xi, k_5\right)}{A_2 + B_2 + (A_2 - B_2)\mathrm{cn}\left(\sqrt{\frac{\alpha A_2 B_2}{6}}\xi, k_5\right)}, \tag{20}$$

$$u_{10}(x,t) = \frac{A_2\varphi_{18} + \varphi_{19}B_2 - (A_2\varphi_{18} - \varphi_{19}B_2)\mathrm{cn}\left(\sqrt{\frac{\alpha A_2 B_2}{6}}\xi, k_5\right)}{A_2 + B_2 - (A_2 - B_2)\mathrm{cn}\left(\sqrt{\frac{\alpha A_2 B_2}{6}}\xi, k_5\right)}, \tag{21}$$

where

$$A_1 = \sqrt{\left(\varphi_{11} - \frac{c_1 + \bar{c}_1}{2}\right)^2 - \frac{(c_1 - \bar{c}_1)^2}{4}},$$

$$B_1 = \sqrt{\left(\varphi_{10} - \frac{c_1 + \bar{c}_1}{2}\right)^2 - \frac{(c_1 - \bar{c}_1)^2}{4}}, \tag{22}$$

$$A_2 = \sqrt{\left(\varphi_{19} - \frac{c_2 + \bar{c}_2}{2}\right)^2 - \frac{(c_2 - \bar{c}_2)^2}{4}},$$

$$B_2=\sqrt{\left(\varphi_{18}-\frac{c_2+\overline{c}_2}{2}\right)^2-\frac{(c_2-\overline{c}_2)^2}{4}}, \quad (23)$$

$$k_3=\sqrt{\frac{(\varphi_{11}-\varphi_{10})^2-(A_1-B_1)^2}{4A_1B_1}},$$

$$k_4=\sqrt{\frac{(\varphi_{17}-\varphi_{16})(\varphi_{15}-\varphi_{14})}{(\varphi_{17}-\varphi_{15})(\varphi_{16}-\varphi_{14})}}, \quad (24)$$

$$\omega_2=\frac{\sqrt{\alpha(\varphi_{17}-\varphi_{15})(\varphi_{16}-\varphi_{14})}}{2\sqrt{6}},$$

$$k_5=\sqrt{\frac{(\varphi_{19}-\varphi_{18})^2-(A_2-B_2)^2}{4A_2B_2}}, \quad (25)$$

$$\gamma_1=\frac{1}{\alpha}\left(12\beta-3\alpha\varphi_9\left(\varphi_9+\sqrt{\frac{12\beta}{\alpha}-3\varphi_9^2}\right)\right),$$

$$\delta_1=-2\varphi_9+2\sqrt{\frac{12\beta}{\alpha}-3\varphi_9^2}, \quad (26)$$

$$\mu_1=4\varphi_9\left(\varphi_9+\sqrt{\frac{12\beta}{\alpha}-3\varphi_9^2}\right),$$

$$\eta_1=\frac{1}{2}\left(-\varphi_9+\sqrt{\frac{12\beta}{\alpha}-3\varphi_9^2}\right), \quad (27)$$

c_1, $\overline{c}_1$, c_2 and $\overline{c}_2$ are complex numbers. And two solitary wave solutions

$$u_{11_\pm}(x,t)=\varphi_9+\frac{6\beta-\alpha\varphi_9^2}{2\alpha\varphi_9\pm\sqrt{6\alpha\beta-2\alpha^2\varphi_9^2}\cosh\left(\sqrt{\beta-\alpha\varphi_9^2}\,\xi\right)}, \quad (28)$$

(3) *If $g=-g_0$, we get two periodic wave solutions as follows*

$$u_{12}(x,t)=\frac{A_3\varphi_{22}+\varphi_{23}B_3+(A_3\varphi_{22}-\varphi_{23}B_3)\mathrm{cn}\left(\sqrt{\frac{\alpha A_3B_3}{6}}\xi,k_6\right)}{A_3+B_3+(A_3-B_3)\mathrm{cn}\left(\sqrt{\frac{\alpha A_3B_3}{6}}\xi,k_6\right)}, \quad (29)$$

$$u_{13}(x,t)=\frac{A_4\varphi_{24}+\varphi_{25}B_4+(A_4\varphi_{24}-\varphi_{25}B_4)\mathrm{cn}\left(\sqrt{\frac{\alpha A_4B_4}{6}}\xi,k_7\right)}{A_4+B_4+(A_4-B_4)\mathrm{cn}\left(\sqrt{\frac{\alpha A_4B_4}{6}}\xi,k_7\right)}, \quad (30)$$

where

$$A_3=\sqrt{\left(\varphi_{23}-\frac{c_3+\overline{c}_3}{2}\right)^2-\frac{(c_3-\overline{c}_3)^2}{4}},$$

$$B_3=\sqrt{\left(\varphi_{22}-\frac{c_3+\overline{c}_3}{2}\right)^2-\frac{(c_3-\overline{c}_3)^2}{4}}, \quad (31)$$

$$A_4=\sqrt{\left(\varphi_{25}-\frac{c_4+\overline{c}_4}{2}\right)^2-\frac{(c_4-\overline{c}_4)^2}{4}},$$

$$B_4=\sqrt{\left(\varphi_{24}-\frac{c_4+\overline{c}_4}{2}\right)^2-\frac{(c_4-\overline{c}_4)^2}{4}}, \quad (32)$$

$$k_6=\sqrt{\frac{(\varphi_{23}-\varphi_{22})^2-(A_3-B_3)^2}{4A_3B_3}},$$

$$k_7=\sqrt{\frac{(\varphi_{25}-\varphi_{24})^2-(A_4-B_4)^2}{4A_4B_4}}, \quad (33)$$

c_3, $\overline{c}_3$, c_4 and $\overline{c}_4$ are complex numbers. And a solitary wave solution

$$u_{14}(x,t)=\frac{\sqrt{\beta}(-9+2\beta\xi^2)}{\sqrt{\alpha}(3+2\beta\xi^2)}. \quad (34)$$

Proposition 2 *For given positive constants c and g_0, (2) has the following periodic wave solution when $\alpha<0$.*

(1) *$g=0$, we get two periodic wave solutions*

$$u_{15_\pm}(x,t)=\pm\sqrt{\frac{6\beta}{\alpha}-\widetilde{\varphi}_4^2}\,\mathrm{sn}\left(\widetilde{\varphi}_4\sqrt{-\frac{\alpha}{6}}\xi,\frac{1}{\widetilde{\varphi}_4}\sqrt{\frac{6\beta}{\alpha}-\widetilde{\varphi}_4^2}\right), \quad (35)$$

six periodic blow-up wave solutions

$$u_{16_\pm}(x,t)=\pm\frac{\widetilde{\varphi}_4}{\mathrm{sn}\left(\widetilde{\varphi}_4\sqrt{-\frac{\alpha}{6}}\xi,\frac{1}{\widetilde{\varphi}_4}\sqrt{\frac{6\beta}{\alpha}-\widetilde{\varphi}_4^2}\right)}, \quad (36)$$

$$u_{17_\pm}(x,t)=\pm\sqrt{\frac{6\beta}{\alpha}}\sec\left(\sqrt{-\beta}\xi\right), \quad (37)$$

$$u_{18_\pm}(x,t)=\pm\sqrt{\frac{6\beta}{\alpha}}\csc\left(\sqrt{-\beta}\xi\right), \quad (38)$$

two kink wave solutions

$$u_{19_\pm}(x,t)=\pm\sqrt{\frac{3\beta}{\alpha}}\tanh\left(\sqrt{-\frac{\beta}{2}}\xi\right), \quad (39)$$

and two unbounded wave solutions

$$u_{20_\pm}(x,t)=\pm\sqrt{\frac{3\beta}{\alpha}}\coth\left(\sqrt{-\frac{\beta}{2}}\xi\right), \quad (40)$$

(2) *If* $0 < g < g_0$, *we get four periodic blow-up wave solutions*

$$u_{21}(x,t) = \frac{(-\widetilde{\varphi}_{11}+\widetilde{\varphi}_9)\widetilde{\varphi}_8+(\widetilde{\varphi}_{11}-\widetilde{\varphi}_8)\widetilde{\varphi}_9\mathrm{sn}^2(\omega_3\xi,k_8)}{-\widetilde{\varphi}_{11}+\widetilde{\varphi}_9+(\widetilde{\varphi}_{11}-\widetilde{\varphi}_8)\mathrm{sn}^2(\omega_3\xi,k_8)}, \tag{41}$$

$$u_{22}(x,t) = \frac{(\widetilde{\varphi}_{10}-\widetilde{\varphi}_8)\widetilde{\varphi}_{11}-(\widetilde{\varphi}_{11}-\widetilde{\varphi}_8)\widetilde{\varphi}_{10}\mathrm{sn}^2(\omega_3\xi,k_8)}{\widetilde{\varphi}_{10}-\widetilde{\varphi}_8-(\widetilde{\varphi}_{11}-\widetilde{\varphi}_8)\mathrm{sn}^2(\omega_3\xi,k_8)}, \tag{42}$$

$$u_{23_\pm}(x,t) = \frac{2\gamma_2+\eta_2\delta_2\pm\eta_2\sqrt{\mu_2}\cos(\sqrt{\frac{\alpha\gamma_2}{6}}\xi)}{\delta_2\pm\sqrt{\mu_2}\cos(\sqrt{\frac{\alpha\gamma_2}{6}}\xi)}, \tag{43}$$

a periodic wave solution

$$u_{24}(x,t) = \frac{(-\widetilde{\varphi}_{10}+\widetilde{\varphi}_8)\widetilde{\varphi}_9+(\widetilde{\varphi}_{10}-\widetilde{\varphi}_9)\widetilde{\varphi}_8\mathrm{sn}^2(\omega_3\xi,k_8)}{-\widetilde{\varphi}_{10}+\widetilde{\varphi}_8+(\widetilde{\varphi}_{10}-\widetilde{\varphi}_9)\mathrm{sn}^2(\omega_3\xi,k_8)}, \tag{44}$$

where

$$\omega_3 = \sqrt{\frac{-\alpha(\widetilde{\varphi}_{11}-\widetilde{\varphi}_9)(\widetilde{\varphi}_{10}-\widetilde{\varphi}_8)}{2\sqrt{6}}},$$

$$k_8 = \sqrt{\frac{(\widetilde{\varphi}_{10}-\widetilde{\varphi}_9)(\widetilde{\varphi}_{11}-\widetilde{\varphi}_8)}{(\widetilde{\varphi}_{11}-\widetilde{\varphi}_9)(\widetilde{\varphi}_{10}-\widetilde{\varphi}_8)}}, \tag{45}$$

$$\gamma_2 = \frac{12\beta-3\alpha\widetilde{\varphi}_7^2-3\sqrt{3}\widetilde{\varphi}_7\sqrt{4\alpha\beta-\alpha^2\widetilde{\varphi}_7^2}}{\alpha},$$

$$\delta_2 = \frac{2\left(\alpha\widetilde{\varphi}_7+\sqrt{12\alpha\beta-3\alpha^2\widetilde{\varphi}_7^2}\right)}{\alpha}, \tag{46}$$

$$\mu_2 = \frac{4\widetilde{\varphi}_7\left(\alpha\widetilde{\varphi}_7-\sqrt{12\alpha\beta-3\alpha^2\widetilde{\varphi}_7^2}\right)}{\alpha},$$

$$\eta_2 = \frac{1}{2\alpha}\left(-\alpha\widetilde{\varphi}_7-\sqrt{12\alpha\beta-3\alpha^2\widetilde{\varphi}_7^2}\right). \tag{47}$$

a blow-up wave solution

$$u_{25}(x,t) = \widetilde{\varphi}_7+\frac{6\beta-6\alpha\widetilde{\varphi}_7^2}{2\alpha\widetilde{\varphi}_7+\sqrt{6\alpha\beta-2\alpha^2\widetilde{\varphi}_7^2}\cosh\left(\sqrt{\beta-\alpha\widetilde{\varphi}_7^2}\xi\right)}, \tag{48}$$

and a solitary wave solution

$$u_{26}(x,t) = \widetilde{\varphi}_7+\frac{6\beta-6\alpha\widetilde{\varphi}_7^2}{2\alpha\widetilde{\varphi}_7-\sqrt{6\alpha\beta-2\alpha^2\widetilde{\varphi}_7^2}\cosh\left(\sqrt{\beta-\alpha\widetilde{\varphi}_7^2}\xi\right)}. \tag{49}$$

(3) *If* $g = g_0$, *we get three blow-up wave solutions*

$$u_{27}(x,t) = -\sqrt{\frac{\beta}{\alpha}}\frac{9-2\beta\xi^2}{3+2\beta\xi^2}, \tag{50}$$

$$u_{28}(x,t) = \frac{6\sqrt{-6\alpha}-\beta\sqrt{\alpha\beta}\xi^3}{6\alpha\xi+\alpha\beta\xi^3}, \tag{51}$$

$$u_{29}(x,t) = -\frac{6\sqrt{-6\alpha}+\beta\sqrt{\alpha\beta}\xi^3}{6\alpha\xi+\alpha\beta\xi^3}, \tag{52}$$

and a periodic wave solution

$$u_{30}(x,t) = \frac{-A_5\widetilde{\varphi}_{17}+B_5\widetilde{\varphi}_{18}+(A_5\widetilde{\varphi}_{17}+\widetilde{\varphi}_{18}B_5)\mathrm{cn}\left(\sqrt{\frac{-\alpha A_5B_5}{6}}\xi,k_9\right)}{-A_5+B_5+(A_5+B_5)\mathrm{cn}\left(\sqrt{\frac{-\alpha A_5B_5}{6}}\xi,k_9\right)}, \tag{53}$$

where

$$A_5 = \sqrt{\left(\widetilde{\varphi}_{18}-\frac{c_5+\bar{c}_5}{2}\right)^2-\frac{(c_5-\bar{c}_5)^2}{4}},$$

$$B_5 = \sqrt{\left(\widetilde{\varphi}_{17}-\frac{c_5+\bar{c}_5}{2}\right)^2-\frac{(c_5-\bar{c}_5)^2}{4}}, \tag{54}$$

$$k_9 = \sqrt{\frac{(A_5+B_5)^2-(\widetilde{\varphi}_{18}-\widetilde{\varphi}_{17})^2}{4A_5B_5}}, \tag{55}$$

c_5 *and* $\bar{c}_5$ *are conjugate complex numbers.*

Proposition 3 *For these solutions, the following are their relations.*

(1) *When* φ_1 *and* φ_5 *tend to* φ_7, *the periodic wave solutions* u_1 *and* u_{3_-} *tend to solitary wave solution* u_{4_-}, *that is*

$$\lim_{\varphi_1\to\varphi_7} u_1(x,t) = \lim_{\varphi_5\to\varphi_7} u_{3_-}(x,t) = u_{4_-}(x,t). \tag{56}$$

(2) *When* φ_4 *and* φ_6 *tend to* φ_8, *the periodic wave solutions* u_2 *and* u_{3_+} *tend to solitary wave solution* u_{4_+}, *that is*

$$\lim_{\varphi_4\to\varphi_8} u_2(x,t) = \lim_{\varphi_6\to\varphi_8} u_{3_+}(x,t) = u_{4_+}(x,t). \tag{57}$$

(3) *When φ_{11} tends to φ_{13}, the periodic wave solutions u_5 and u_7 tend to periodic wave solution u_6, that is*

$$\lim_{\varphi_{11}\to\varphi_{13}} u_5(x,t) = \lim_{\varphi_{11}\to\varphi_{13}} u_7(x,t) = u_6(x,t). \tag{58}$$

(4) *When φ_{17} and φ_{19} tends to φ_{21}, the periodic wave solution u_7 and u_9 tend to solitary wave solution u_{11_-}, that is*

$$\lim_{\varphi_{17}\to\varphi_{21}} u_7(x,t) = \lim_{\varphi_{19}\to\varphi_{21}} u_9(x,t) = u_{11_-}(x,t). \tag{59}$$

(5) *When φ_{17} and φ_{19} tends to φ_{21}, the periodic wave solution u_8 and u_{10} tend to solitary wave solution u_{11_+}, that is*

$$\lim_{\varphi_{17}\to\varphi_{21}} u_8(x,t) = \lim_{\varphi_{19}\to\varphi_{21}} u_{10}(x,t) = u_{11_+}(x,t). \tag{60}$$

(6) *When φ_{22} and φ_{24} tends to φ_{26}, the periodic wave solution u_{12} and u_{13} tend to solitary wave solution u_{14}, that is*

$$\lim_{\varphi_{22}\to\varphi_{26}} u_{12}(x,t) = \lim_{\varphi_{24}\to\varphi_{26}} u_{13}(x,t) = u_{14}(x,t). \tag{61}$$

(7) *When $\widetilde{\varphi}_4$ tends to φ_+, the periodic wave solution $u_{15_\pm}$ tends to kink wave solution $u_{19_\pm}$, that is*

$$\lim_{\widetilde{\varphi}_4\to\varphi_+} u_{15_\pm}(x,t) = u_{19_\pm}(x,t). \tag{62}$$

(8) *When $\widetilde{\varphi}_4$ tends to φ_+, the periodic wave solution $u_{16_\pm}$ tends to unbounded wave solution $u_{20_\pm}$, that is*

$$\lim_{\widetilde{\varphi}_4\to\varphi_+} u_{16_\pm}(x,t) = u_{20_\pm}(x,t). \tag{63}$$

(9) *When $\widetilde{\varphi}_{11}$ tends to $\widetilde{\varphi}_7$, the periodic wave solution u_{21} tends to blow-up wave solution u_{25}, that is*

$$\lim_{\widetilde{\varphi}_{11}\to\widetilde{\varphi}_7} u_{21}(x,t) = u_{25}(x,t). \tag{64}$$

(10) *When $\widetilde{\varphi}_{10}$ tends to $\widetilde{\varphi}_7$, the periodic wave solution u_{24} tends to solitary wave solution u_{26}, that is*

$$\lim_{\widetilde{\varphi}_{10}\to\widetilde{\varphi}_7} u_{24}(x,t) = u_{26}(x,t). \tag{65}$$

3 The derivation of main results

In this section, we will give the derivations for our main results.

3.1 Planar system and phase portraits

For given positive constant wave speed c, substituting $u = \varphi(\xi)$ with $\xi = x - ct$ into the MBBM equation (2), it follows that

$$-c\varphi' + \varphi' + a\varphi^2\varphi' - c\varphi''' = 0. \tag{66}$$

Integrating (66) once, we have

$$(-c+1)\varphi + \frac{a}{3}\varphi^3 - c\varphi'' = g_1, \tag{67}$$

where g_1 is integral constant.

Letting $\phi = \varphi'$, we get the following planar system

$$\begin{cases} \frac{d\varphi}{d\xi} = \phi, \\ \frac{d\phi}{d\xi} = -\frac{\alpha}{3}\varphi^3 + \beta\varphi + g, \end{cases} \tag{68}$$

where $\alpha = -\frac{a}{c}$, $\beta = -\frac{c-1}{c}$ and $g = -\frac{g_1}{c}$.

Obviously, the above system (68) is a Hamiltonian system with Hamiltonian function

$$H(\varphi,\phi) = \phi^2 + \frac{\alpha}{6}\varphi^4 - \beta\varphi^2 - 2g\varphi. \tag{69}$$

Now, we consider the phase portraits of system (68). Set

$$f_0(\varphi) = -\frac{\alpha}{3}\varphi^3 + \beta\varphi, \tag{70}$$

$$f(\varphi) = -\frac{\alpha}{3}\varphi^3 + \beta\varphi + g. \tag{71}$$

Obviously, $f_0(\varphi)$ has three zero points, φ_-, φ_0 and φ_+, which are given as follows

$$\varphi_- = -\sqrt{\frac{3\beta}{\alpha}}, \qquad \varphi_0 = 0, \qquad \varphi_+ = \sqrt{\frac{3\beta}{\alpha}}. \tag{72}$$

It is easy to obtain two extreme points of $f_0(\varphi)$ as follows:

$$\varphi_-^* = -\sqrt{\frac{\beta}{\alpha}}, \qquad \varphi_+^* = \sqrt{\frac{\beta}{\alpha}}. \tag{73}$$

Letting

$$g_0 = |f_0(\varphi_-^*)| = |f_0(\varphi_+^*)| = \frac{2|\beta|}{3}\sqrt{\frac{\beta}{\alpha}}, \tag{74}$$

then it is easily seen that g_0 is the extreme values of $f_0(\varphi)$.

Let $(\varphi_i, 0)$ be one of the singular points of system (68), then the characteristic values of the linearized system of system (68) at the singular points $(\varphi_i, 0)$ are

$$\lambda_\pm = \pm\sqrt{f'(\varphi_i)}. \tag{75}$$

From the qualitative theory of dynamical systems, we therefore know that

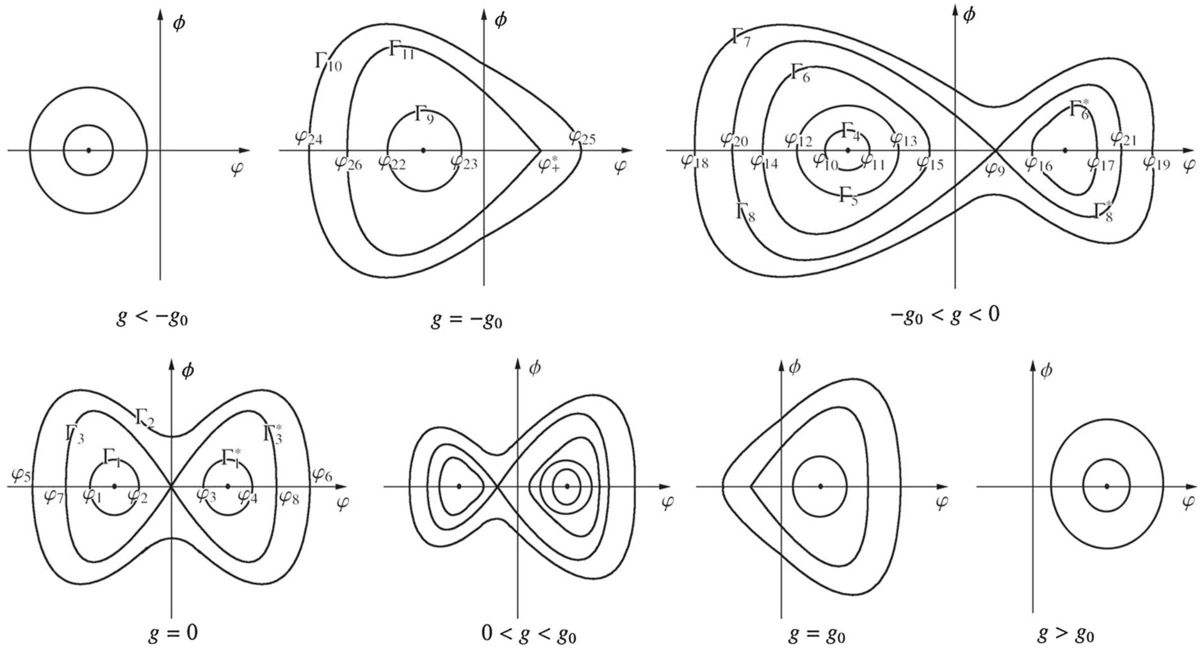

Fig. 1 The phase portraits of system (68) when $\alpha > 0$

(i) If $f'(\varphi_i) > 0$, $(\varphi_i, 0)$ is a saddle point.
(ii) If $f'(\varphi_i) < 0$, $(\varphi_i, 0)$ is a center point.
(iii) If $f'(\varphi_i) = 0$, $(\varphi_i, 0)$ is a degenerate saddle point.

Therefore, we obtain the phase portraits of system (68) in Figs. 1 and 2.

3.2 The derivation of Proposition 1

In this section, we will obtain the explicit expressions of solutions for the MBBM equation (2) when $\alpha > 0$.

(1) If $g = 0$, we set

$$\sqrt{\frac{3\beta}{\alpha}} < \varphi_4 < \sqrt{\frac{6\beta}{\alpha}}, \qquad \varphi_6 > \sqrt{\frac{6\beta}{\alpha}}. \tag{76}$$

(i) From the phase portrait, we see that there are two closed orbits Γ_1 and Γ_1^* passing the points $(\varphi_1, 0)$, $(\varphi_2, 0)$, $(\varphi_3, 0)$ and $(\varphi_4, 0)$. In (φ, ϕ)-plane the expressions of the closed orbits are given as

$$\phi = \pm\sqrt{\frac{\alpha}{6}}\sqrt{(\varphi - \varphi_1)(\varphi - \varphi_2)(\varphi - \varphi_3)(\varphi_4 - \varphi)}, \tag{77}$$

where $\varphi_1 = -\varphi_4$, $\varphi_2 = -\sqrt{\frac{6\beta}{\alpha} - \varphi_4^2}$ and $\varphi_3 = \sqrt{\frac{6\beta}{\alpha} - \varphi_4^2}$.

Substituting (77) into $\frac{d\varphi}{d\xi} = \phi$ and integrating them along Γ_1 and Γ_1^*, we have

$$\pm\int_{\varphi_1}^{\varphi} \frac{1}{\sqrt{(\varphi_4 - s)(\varphi_3 - s)(\varphi_2 - s)(s - \varphi_1)}} ds = \sqrt{\frac{\alpha}{6}} \int_0^{\xi} ds, \tag{78}$$

$$\pm\int_{\varphi_4}^{\varphi} \frac{1}{\sqrt{(s - \varphi_1)(s - \varphi_2)(s - \varphi_3)(\varphi_4 - s)}} ds = \sqrt{\frac{\alpha}{6}} \int_0^{\xi} ds. \tag{79}$$

From (78), (79) and noting that $u = \varphi(\xi)$ and $\xi = x - ct$, we obtain the periodic wave solutions $u_1(x, t)$ as (7) and $u_2(x, t)$ as (8).

(ii) From the phase portrait, we see that there are a closed orbit Γ_2 passing the points $(\varphi_5, 0)$ and $(\varphi_6, 0)$. In (φ, ϕ)-plane the expressions of the closed orbits are given as

$$\phi = \pm\sqrt{\frac{\alpha}{6}}\sqrt{(\varphi_6 - \varphi)(\varphi - \varphi_5)(\varphi - \varphi_5^*)(\varphi - \overline{\varphi}_5^*)}, \tag{80}$$

where $\varphi_5 = -\varphi_6$, $\varphi_5^* = i\sqrt{\varphi_6^2 - \frac{6\beta}{\alpha}}$ and $\overline{\varphi}_5^* = -i\sqrt{\varphi_6^2 - \frac{6\beta}{\alpha}}$.

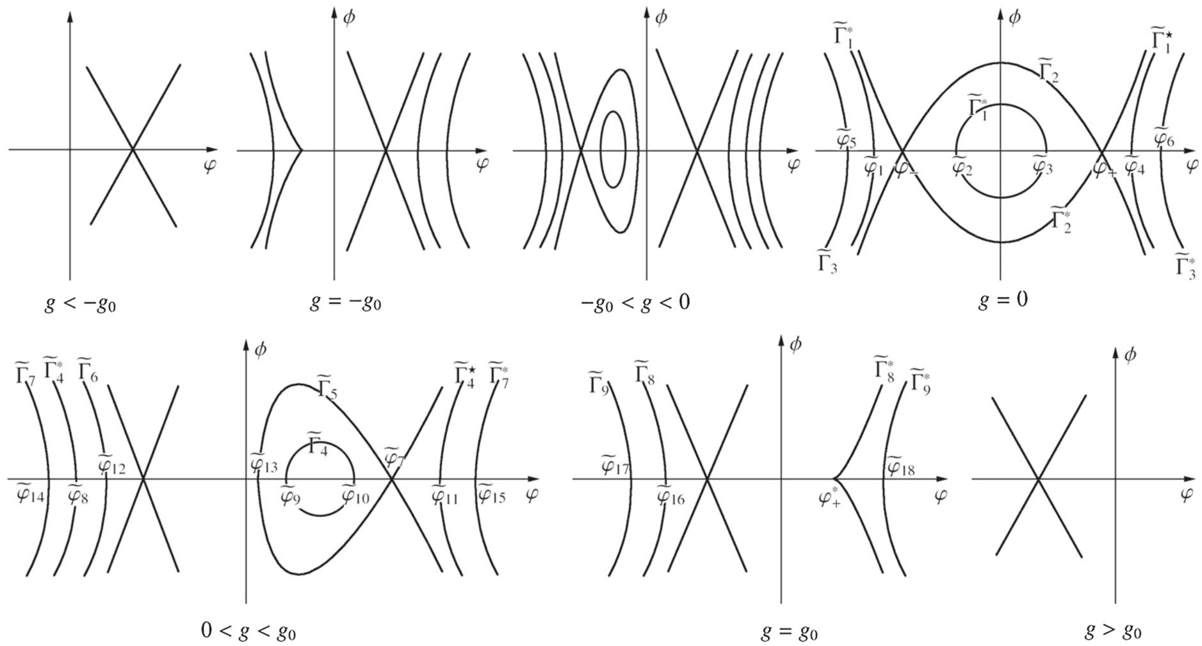

Fig. 2 The phase portraits of system (68) when $\alpha<0$

Substituting (80) into $\frac{d\varphi}{d\xi}=\phi$ and integrating them along the orbit Γ_2, we have

$$\pm\int_{\varphi_5}^{\varphi}\frac{1}{\sqrt{(\varphi_6-s)(s-\varphi_5)(s-\varphi_5^*)(s-\overline{\varphi}_5^*)}}ds=\sqrt{\frac{\alpha}{6}}\int_0^{\xi}ds, \tag{81}$$

$$\pm\int_{\varphi_6}^{\varphi}\frac{1}{\sqrt{(\varphi_6-s)(s-\varphi_5)(s-\varphi_5^*)(s-\overline{\varphi}_5^*)}}ds=\sqrt{\frac{\alpha}{6}}\int_0^{\xi}ds. \tag{82}$$

From (81), (82) and noting that $u=\varphi(\xi)$ and $\xi=x-ct$, we obtain the periodic wave solutions $u_{3_\pm}(x,t)$ as (9).

(iii) From the phase portrait, we see that there are two symmetric homoclinic orbits Γ_3 and Γ_3^* connected at the saddle point (0, 0). In (φ,ϕ)-plane, the expressions of the homoclinic orbits are given as

$$\phi=\pm\sqrt{\frac{\alpha}{6}}\varphi\sqrt{(\varphi-\varphi_7)(\varphi_8-\varphi)}, \tag{83}$$

where $\varphi_7=-\sqrt{\frac{6\beta}{\alpha}}$ and $\varphi_8=\sqrt{\frac{6\beta}{\alpha}}$.

Substituting (83) into $\frac{d\varphi}{d\xi}=\phi$ and integrating them along the orbits Γ_3 and Γ_3^*, we have

$$\pm\int_{\varphi_7}^{\varphi}\frac{1}{\varphi\sqrt{(s-\varphi_7)(\varphi_8-s)}}ds=\sqrt{\frac{\alpha}{6}}\int_0^{\xi}ds, \tag{84}$$

$$\pm\int_{\varphi_8}^{\varphi}\frac{1}{\varphi\sqrt{(s-\varphi_7)(\varphi_8-s)}}ds=\sqrt{\frac{\alpha}{6}}\int_0^{\xi}ds. \tag{85}$$

From (84), (85) and noting that $u=\varphi(\xi)$ and $\xi=x-ct$, we obtain the solitary wave solutions $u_{4_\pm}(x,t)$ as (15).

(2) If $-g_0<g<0$, we set the middle solution of $f(\varphi)=0$ be $\varphi_9(0<\varphi_9<\sqrt{\frac{\beta}{\alpha}})$. then we can get another two solutions of $f(\varphi)=0$ as follows:

$$\varphi_9^*=\frac{1}{2}\left(-\varphi_9-\sqrt{\frac{12\beta}{\alpha}-3\varphi_9^2}\right), \tag{86}$$

$$\varphi_9^\star=\frac{1}{2}\left(-\varphi_9+\sqrt{\frac{12\beta}{\alpha}-3\varphi_9^2}\right). \tag{87}$$

(i) From the phase portrait, we see that there are a closed orbit Γ_4 passing the points $(\varphi_{10},0)$ and $(\varphi_{11},0)$. In (φ,ϕ)-plane, the expressions of the closed orbits are given as

$$\phi=\pm\sqrt{\frac{\alpha}{6}}\sqrt{(\varphi_{11}-\varphi)(\varphi-\varphi_{10})(\varphi-c_1)(\varphi-\overline{c}_1)}, \tag{88}$$

where $\varphi_{12}<\varphi_{10}<\varphi_9^*<\varphi_{13}$, c_1 and $\overline{c}_1$ are conjugate complex numbers.

Substituting (88) into $\frac{d\varphi}{d\xi} = \phi$ and integrating them along Γ_4, we have

$$\pm \int_{\varphi_{10}}^{\varphi} \frac{1}{\sqrt{(\varphi_{11}-s)(s-\varphi_{10})(s-c_1)(s-\bar{c}_1)}} ds = \sqrt{\frac{\alpha}{6}} \int_0^{\xi} ds. \quad (89)$$

From (89) and noting that $u = \varphi(\xi)$ and $\xi = x - ct$, we get a periodic wave solution $u_5(x,t)$ as (16).

(ii) From the phase portrait, we note that there is a special orbit Γ_5, which has the same hamiltonian with that of $(\varphi_9^\star, 0)$. In (φ, ϕ)-plane the expressions of the orbits are given as

$$\phi = \pm\sqrt{\frac{\alpha}{6}}\sqrt{(\varphi-\varphi_9^\star)^2(\varphi-\varphi_{12})(\varphi_{13}-\varphi)}, \quad (90)$$

where

$$\varphi_{12} = \frac{1}{2}\left(\varphi_9 - \sqrt{\frac{12\beta}{\alpha} - 3\varphi_9^2} - 2\sqrt{\varphi_9\left(\varphi_9 + \sqrt{\frac{12\beta}{\alpha} - 3\varphi_9^2}\right)}\right), \quad (91)$$

$$\varphi_{13} = \frac{1}{2}\left(\varphi_9 - \sqrt{\frac{12\beta}{\alpha} - 3\varphi_9^2} + 2\sqrt{\varphi_9\left(\varphi_9 + \sqrt{\frac{12\beta}{\alpha} - 3\varphi_9^2}\right)}\right). \quad (92)$$

Substituting (90) into $\frac{d\varphi}{d\xi} = \phi$ and integrating them along Γ_5, it follows that

$$\pm \int_{\varphi_{12}}^{\varphi} \frac{1}{\sqrt{(\varphi_{13}-s)(s-\varphi_9^\star)^2(s-\varphi_{12})}} ds = \sqrt{\frac{\alpha}{6}} \int_0^{\xi} ds. \quad (93)$$

From (93) and noting that $u = \varphi(\xi)$ and $\xi = x - ct$, we get a periodic wave solution $u_6(x,t)$ as (17).

(iii) From the phase portrait, we note that there are two orbits Γ_6 and Γ_6^* passing the points $(\varphi_{14}, 0)$, $(\varphi_{15}, 0)$, $(\varphi_{16}, 0)$ and $(\varphi_{17}, 0)$. In (φ, ϕ)-plane, the expressions of the orbits are given as

$$\phi = \pm\sqrt{\frac{\alpha}{6}}\sqrt{(\varphi-\varphi_{14})(\varphi-\varphi_{15})(\varphi-\varphi_{16})(\varphi_{17}-\varphi)}, \quad (94)$$

where $\varphi_{20} < \varphi_{14} < \varphi_{12} < \varphi_{15} < \varphi_9 < \varphi_{16} < \varphi_9^\star < \varphi_{17} < \varphi_{21}$.

Substituting (94) into $\frac{d\varphi}{d\xi} = \phi$ and integrating them along Γ_6 and Γ_6^*, we have

$$\pm \int_{\varphi_{14}}^{\varphi} \frac{1}{\sqrt{(\varphi_{17}-s)(\varphi_{16}-s)(\varphi_{15}-s)(s-\varphi_{14})}} ds = \sqrt{\frac{\alpha}{6}} \int_0^{\xi} ds, \quad (95)$$

$$\pm \int_{\varphi_{17}}^{\varphi} \frac{1}{\sqrt{(\varphi_{17}-s)(s-\varphi_{16})(s-\varphi_{15})(s-\varphi_{14})}} ds = \sqrt{\frac{\alpha}{6}} \int_0^{\xi} ds. \quad (96)$$

From (95), (96) and noting that $u = \varphi(\xi)$ and $\xi = x - ct$, we get two periodic wave solutions $u_7(x,t)$ as (18) and $u_8(x,t)$ as (19).

(iv) From the phase portrait, we note that there is a special orbit Γ_7 passing the points $(\varphi_{18}, 0)$ and $(\varphi_{19}, 0)$. In (φ, ϕ)-plane, the expressions of the orbit are given as

$$\phi = \pm\sqrt{\frac{\alpha}{6}}\sqrt{(\varphi_{19}-\varphi)(\varphi-\varphi_{18})(\varphi-c_2)(\varphi-\bar{c}_2)}, \quad (97)$$

where $\varphi_{18} < \varphi_{20} < \varphi_{21} < \varphi_{19}$, c_2 and $\bar{c}_2$ are conjugate complex numbers.

Substituting (97) into $\frac{d\varphi}{d\xi} = \phi$ and integrating it along Γ_7, we have

$$\pm \int_{\varphi_{18}}^{\varphi} \frac{1}{\sqrt{(\varphi_{19}-s)(s-\varphi_{18})(s-c_2)(s-\bar{c}_2)}} ds = \sqrt{\frac{\alpha}{6}} \int_0^{\xi} ds. \quad (98)$$

From (98) and noting that $u = \varphi(\xi)$ and $\xi = x - ct$, we get a periodic wave solution $u_9(x,t)$ as (20).

If $\varphi(\xi)$ is a traveling wave solution, then $\varphi(\xi + q)$ is a traveling wave solution too. Taking $q = 2K$ and noting that $\mathrm{cn}(u + 2K) = -\mathrm{cn}u$, we get a periodic wave solution $u_{10}(x,t)$ as (21).

(v) From the phase portrait, we note that there are two homoclinic orbits Γ_8 and Γ_8^* connected at the saddle point $(\varphi_9, 0)$. In (φ, ϕ)-plane the expressions of the orbits are given as

$$\phi = \pm\sqrt{\frac{\alpha}{6}}\sqrt{(\varphi-\varphi_9)^2(\varphi-\varphi_{20})(\varphi_{21}-\varphi)}, \quad (99)$$

where

$$\varphi_{20} = -\varphi_9 - \sqrt{\frac{6\beta}{\alpha} - 2\varphi_9^2}, \quad (100)$$

$$\varphi_{21} = -\varphi_9 + \sqrt{\frac{6\beta}{\alpha} - 2\varphi_9^2}. \quad (101)$$

Substituting (99) into $\frac{d\varphi}{d\xi}=\phi$ and integrating them along $\varGamma_8$ and $\varGamma_8^*$, it follows that

$$\pm\int_{\varphi_{20}}^{\varphi}\frac{1}{\sqrt{(s-\varphi_9)^2(s-\varphi_{20})(\varphi_{21}-s)}}ds=\sqrt{\frac{\alpha}{6}}\int_0^{\xi}ds, \qquad (102)$$

$$\pm\int_{\varphi_{21}}^{\varphi}\frac{1}{\sqrt{(s-\varphi_9)^2(s-\varphi_{20})(\varphi_{21}-s)}}ds=\sqrt{\frac{\alpha}{6}}\int_0^{\xi}ds. \qquad (103)$$

From (102), (103) and noting that $u=\varphi(\xi)$ and $\xi=x-ct$, we get two solitary wave solutions $u_{11\pm}(x,t)$ as (28).

(3) If $g=-g_0$, we will consider two kinds of orbits.

(i) From the phase portrait, we note that there is a special orbit $\varGamma_9$ passing the points $(\varphi_{22},0)$ and $(\varphi_{23},0)$. In (φ,ϕ)-plane, the expressions of the orbit are given as

$$\phi=\pm\sqrt{\frac{\alpha}{6}}\sqrt{(\varphi_{23}-\varphi)(\varphi-\varphi_{22})(\varphi-c_3)(\varphi-\overline{c}_3)}, \qquad (104)$$

where $-\sqrt{3\alpha}<\varphi_{22}<-2\sqrt{\frac{\alpha}{3}}<\varphi_{23}<\sqrt{\frac{\alpha}{3}}$, c_3 and $\overline{c}_3$ are conjugate complex numbers.

Substituting (104) into $\frac{d\varphi}{d\xi}=\phi$ and integrating it along $\varGamma_9$, we have

$$\pm\int_{\varphi_{22}}^{\varphi}\frac{1}{\sqrt{(\varphi_{23}-s)(s-\varphi_{22})(s-c_3)(s-\overline{c}_3)}}ds=\sqrt{\frac{\alpha}{6}}\int_0^{\xi}ds. \qquad (105)$$

From (105) and noting that $u=\varphi(\xi)$ and $\xi=x-ct$, we get a periodic wave solutions $u_{12}(x,t)$ as (29).

(ii) From the phase portrait, we note that there is a special orbit $\varGamma_{10}$ passing the points $(\varphi_{24},0)$ and $(\varphi_{25},0)$. In (φ,ϕ)-plane, the expressions of the orbit are given as

$$\phi=\pm\sqrt{\frac{\alpha}{6}}\sqrt{(\varphi_{25}-\varphi)(\varphi-\varphi_{24})(\varphi-c_4)(\varphi-\overline{c}_4)}, \qquad (106)$$

where $\varphi_{24}<-\sqrt{3\alpha}<\sqrt{\frac{\alpha}{3}}<\varphi_{25}$, c_4 and $\overline{c}_4$ are conjugate complex numbers

Substituting (106) into $\frac{d\varphi}{d\xi}=\phi$ and integrating them along $\varGamma_{10}$, we have

$$\pm\int_{\varphi_{24}}^{\varphi}\frac{1}{\sqrt{(\varphi_{25}-s)(s-\varphi_{24})(s-c_4)(s-\overline{c}_4)}}ds=\sqrt{\frac{\alpha}{6}}\int_0^{\xi}ds. \qquad (107)$$

From (107) and noting that $u=\varphi(\xi)$ and $\xi=x-ct$, we get a periodic wave solutions $u_{13}(x,t)$ as (30).

(iii) From the phase portrait, we see that there is a homoclinic orbit $\varGamma_{11}$, which passes the degenerate saddle point $(\varphi_+^*,0)$ for system (68). In (φ,ϕ)-plane, the expressions of the homoclinic orbit are given as

$$\phi=\pm\sqrt{\frac{\alpha}{6}}\sqrt{(\varphi_+^*-\varphi)^3(\varphi-\varphi_{26})}, \qquad (108)$$

where

$$\varphi_{26}=-3\sqrt{\frac{\beta}{\alpha}}. \qquad (109)$$

Substituting (108) into $\frac{d\varphi}{d\xi}=\phi$ and integrating them along $\varGamma_{11}$, it follows that

$$\pm\int_{\varphi_{26}}^{\varphi}\frac{1}{(s-\varphi_+^*)\sqrt{(s-\varphi_+^*)(\varphi_{26}-s)}}ds=\sqrt{\frac{\alpha}{6}}\int_0^{\xi}ds. \qquad (110)$$

From (110) and noting that $u=\varphi(\xi)$ and $\xi=x-ct$, we get a solitary wave solution $u_{14}(x,t)$ as (34).

Thus, the derivation of Proposition 1 has been finished.

3.3 The derivation of Proposition 2

In this section, we will obtain the explicit expressions of solutions for the MBBM equation (2) when $\alpha<0$.

(1) If $g=0$, we will consider three kinds of orbits.

(i) From the phase portrait, we note that there are three special orbits $\widetilde{\varGamma}_1^*$, $\widetilde{\varGamma}_1$ and $\widetilde{\varGamma}_1^\star$ passing the points $(\widetilde{\varphi}_1,0)$, $(\widetilde{\varphi}_2,0)$, $(\widetilde{\varphi}_3,0)$ and $(\widetilde{\varphi}_4,0)$. In (φ,ϕ)-plane the expressions of the orbit are given as

$$\phi=\pm\sqrt{-\frac{\alpha}{6}}\sqrt{(\varphi-\widetilde{\varphi}_1)(\varphi-\widetilde{\varphi}_2)(\varphi-\widetilde{\varphi}_3)(\varphi-\widetilde{\varphi}_4)}, \qquad (111)$$

where $\widetilde{\varphi}_1=-\widetilde{\varphi}_4$, $\widetilde{\varphi}_2=-\sqrt{\frac{6\beta}{\alpha}-\widetilde{\varphi}_4^2}$, $\widetilde{\varphi}_3=\sqrt{\frac{6\beta}{\alpha}-\widetilde{\varphi}_4^2}$ and $\sqrt{\frac{3\beta}{\alpha}}<\widetilde{\varphi}_4<\sqrt{\frac{6\beta}{\alpha}}$.

Substituting (111) into $\frac{d\varphi}{d\xi}=\phi$ and integrating them along $\widetilde{\varGamma}_1^*$, $\widetilde{\varGamma}_1$ and $\widetilde{\varGamma}_1^\star$, we have

$$\pm\int_0^{\varphi}\frac{1}{\sqrt{(s-\widetilde{\varphi}_1)(s-\widetilde{\varphi}_2)(s-\widetilde{\varphi}_3)(s-\widetilde{\varphi}_4)}}\mathrm{d}s$$
$$=\sqrt{-\frac{\alpha}{6}}\int_0^{\xi}\mathrm{d}s, \tag{112}$$
$$\pm\int_{\varphi}^{\infty}\frac{1}{\sqrt{(s-\widetilde{\varphi}_1)(s-\widetilde{\varphi}_2)(s-\widetilde{\varphi}_3)(s-\widetilde{\varphi}_4)}}\mathrm{d}s$$
$$=\sqrt{-\frac{\alpha}{6}}\int_0^{\xi}\mathrm{d}s. \tag{113}$$

From (112), (113) and noting that $u=\varphi(\xi)$ and $\xi=x-ct$, we get two periodic wave solutions $u_{15_\pm}(x,t)$ as (35) and two periodic blow-up solutions $u_{16_\pm}(x,t)$ as (36).

(ii) From the phase portrait, we note that there are two special orbits $\widetilde{\Gamma}_3$ and $\widetilde{\Gamma}_3^*$, which have the same hamiltonian with that of the center point $(0,0)$. In (φ,ϕ)-plane, the expressions of these two orbits are given as

$$\phi=\pm\sqrt{-\frac{\alpha}{6}}\varphi\sqrt{(\varphi-\widetilde{\varphi}_5)(\varphi-\widetilde{\varphi}_6)}, \tag{114}$$

where $\widetilde{\varphi}_5=-\sqrt{\frac{6\beta}{\alpha}}$ and $\widetilde{\varphi}_6=\sqrt{\frac{6\beta}{\alpha}}$.

Substituting (114) into $\frac{\mathrm{d}\varphi}{\mathrm{d}\xi}=\phi$ and integrating them along the two orbits $\widetilde{\Gamma}_3$ and $\widetilde{\Gamma}_3^*$, it follows that

$$\pm\int_{\widetilde{\varphi}_6}^{\varphi}\frac{1}{s\sqrt{(s-\widetilde{\varphi}_5)(s-\widetilde{\varphi}_6)}}\mathrm{d}s=\sqrt{-\frac{\alpha}{6}}\int_0^{\xi}\mathrm{d}s, \tag{115}$$
$$\pm\int_{\varphi}^{+\infty}\frac{1}{s\sqrt{(s-\widetilde{\varphi}_5)(s-\widetilde{\varphi}_6)}}\mathrm{d}s=\sqrt{-\frac{\alpha}{6}}\int_0^{\xi}\mathrm{d}s. \tag{116}$$

From (115), (116) and noting that $u=\varphi(\xi)$ and $\xi=x-ct$, we get four periodic blow-up solutions $u_{17_\pm}(x,t)$ and $u_{18_\pm}(x,t)$ as (37) and (38).

(iii) From the phase portrait, we see that there are two heterclinic orbits $\widetilde{\Gamma}_2$ and $\widetilde{\Gamma}_2^*$ connected at saddle points $(\varphi_-,0)$ and $(\varphi_+,0)$. In (φ,ϕ)-plane, the expressions of the heterclinic orbits are given as

$$\phi=\pm\sqrt{-\frac{\alpha}{6}}\sqrt{(\varphi-\varphi_-)^2(\varphi-\varphi_+)^2}. \tag{117}$$

Substituting (117) into $\frac{\mathrm{d}\varphi}{\mathrm{d}\xi}=\phi$ and integrating them along the heterclinic orbits $\widetilde{\Gamma}_2$ and $\widetilde{\Gamma}_2^*$, it follows that

$$\pm\int_0^{\varphi}\frac{1}{(s-\varphi_-)(\varphi_+-s)}\mathrm{d}s=\sqrt{-\frac{\alpha}{6}}\int_0^{\xi}\mathrm{d}s, \tag{118}$$
$$\pm\int_{\varphi}^{+\infty}\frac{1}{(s-\varphi_-)(s-\varphi_+)}\mathrm{d}s=\sqrt{-\frac{\alpha}{6}}\int_0^{\xi}\mathrm{d}s. \tag{119}$$

From (118), (119) and noting that $u=\varphi(\xi)$ and $\xi=x-ct$, we get two kink wave solutions $u_{19_\pm}(x,t)$ as (39) and two unbounded solutions $u_{20_\pm}(x,t)$ as (40).

(2) If $0<g<g_0$, we set the largest solution of $f(\varphi)=0$ be $\widetilde{\varphi}_7$ $\left(\sqrt{\frac{\beta}{\alpha}}<\widetilde{\varphi}_7<\sqrt{\frac{3\beta}{\alpha}}\right)$, then we can get another two solutions of $f(\varphi)=0$ as follows:

$$\widetilde{\varphi}_7^*=\frac{1}{2\alpha}\left(-\alpha\widetilde{\varphi}_7+\sqrt{12\alpha\beta-3\alpha^2\widetilde{\varphi}_7^2}\right), \tag{120}$$
$$\widetilde{\varphi}_7^\star=\frac{1}{2\alpha}\left(-\alpha\widetilde{\varphi}_7-\sqrt{12\alpha\beta-3\alpha^2\widetilde{\varphi}_7^2}\right). \tag{121}$$

(i) From the phase portrait, we note that there are three special orbits $\widetilde{\Gamma}_4^*$, $\widetilde{\Gamma}_4$ and $\widetilde{\Gamma}_4^\star$ passing the points $(\widetilde{\varphi}_8,0)$, $(\widetilde{\varphi}_9,0)$, $(\widetilde{\varphi}_{10},0)$ and $(\widetilde{\varphi}_{11},0)$. In (φ,ϕ)-plane, the expressions of the orbit are given as

$$\phi=\pm\sqrt{-\frac{\alpha}{6}}\sqrt{(\varphi-\widetilde{\varphi}_8)(\varphi-\widetilde{\varphi}_9)(\varphi-\widetilde{\varphi}_{10})(\varphi-\widetilde{\varphi}_{11})}, \tag{122}$$

where $\widetilde{\varphi}_{14}<\widetilde{\varphi}_8<\widetilde{\varphi}_{12}<\widetilde{\varphi}_{13}<\widetilde{\varphi}_9<\widetilde{\varphi}_7^\star<\widetilde{\varphi}_{10}<\widetilde{\varphi}_7<\widetilde{\varphi}_{11}<\widetilde{\varphi}_{15}$.

Substituting (122) into $\frac{\mathrm{d}\varphi}{\mathrm{d}\xi}=\phi$ and integrating them along $\widetilde{\Gamma}_4^*$, $\widetilde{\Gamma}_4$ and $\widetilde{\Gamma}_4^\star$, we have

$$\int_{\varphi}^{\widetilde{\varphi}_8}\frac{1}{\sqrt{(\widetilde{\varphi}_{11}-s)(\widetilde{\varphi}_{10}-s)(\widetilde{\varphi}_9-s)(\widetilde{\varphi}_8-s)}}\mathrm{d}s$$
$$=\sqrt{-\frac{\alpha}{6}}\int_0^{\xi}, \tag{123}$$
$$\int_{\widetilde{\varphi}_{11}}^{\varphi}\frac{1}{\sqrt{(\widetilde{\varphi}_{11}-s)(\widetilde{\varphi}_{10}-s)(s-\widetilde{\varphi}_9)(s-\widetilde{\varphi}_8)}}\mathrm{d}s$$
$$=\sqrt{-\frac{\alpha}{6}}\int_0^{\xi}, \tag{124}$$
$$\int_{\widetilde{\varphi}_9}^{\varphi}\frac{1}{\sqrt{(s-\widetilde{\varphi}_{11})(s-\widetilde{\varphi}_{10})(s-\widetilde{\varphi}_9)(s-\widetilde{\varphi}_8)}}\mathrm{d}s$$
$$=\sqrt{-\frac{\alpha}{6}}\int_0^{\xi}. \tag{125}$$

From (123), (124), (125) and noting that $u=\varphi(\xi)$ and $\xi=x-ct$, we get two periodic blow-up wave solutions $u_{21}(x,t)$, $u_{22}(x,t)$ as (41), (42) and a periodic wave solution $u_{24}(x,t)$ as (44).

(ii) From the phase portrait, we see that there are a homoclinic orbit $\widetilde{\Gamma}_5$, which passes the saddle point $(\widetilde{\varphi}_7,0)$, and a spacial orbit $\widetilde{\Gamma}_6$ passing the point $(\widetilde{\varphi}_{12},0)$. In (φ,ϕ)-plane, the expressions of the orbits are given as

$$\phi = \pm\sqrt{-\frac{\alpha}{6}}\sqrt{(\varphi - \widetilde{\varphi}_7)^2(\varphi - \widetilde{\varphi}_{12})(\varphi - \widetilde{\varphi}_{13})}, \quad (126)$$

where

$$\widetilde{\varphi}_{12} = \frac{-\alpha\widetilde{\varphi}_7 + \sqrt{6\alpha\beta - 2\alpha^2\widetilde{\varphi}_7^2}}{\alpha}, \quad (127)$$

$$\widetilde{\varphi}_{13} = \frac{-\alpha\widetilde{\varphi}_7 - \sqrt{6\alpha\beta - 2\alpha^2\widetilde{\varphi}_7^2}}{\alpha}. \quad (128)$$

Substituting (126) into $\frac{d\varphi}{d\xi} = \phi$ and integrating them along the orbits, it follows that

$$\pm\int_{\varphi}^{\infty} \frac{1}{\sqrt{(s - \widetilde{\varphi}_7)^2(s - \widetilde{\varphi}_{12})(s - \widetilde{\varphi}_{13})}} ds = \sqrt{-\frac{\alpha}{6}} \int_0^{\xi} ds. \quad (129)$$

$$\pm\int_{\widetilde{\varphi}_{13}}^{\varphi} \frac{1}{\sqrt{(s - \widetilde{\varphi}_7)^2(s - \widetilde{\varphi}_{12})(s - \widetilde{\varphi}_{13})}} ds = \sqrt{-\frac{\alpha}{6}} \int_0^{\xi} ds. \quad (130)$$

From (129), (130) and noting that $u = \varphi(\xi)$ and $\xi = x - ct$, we get a blow-up solution $u_{25}(x,t)$ as (48) and a solitary wave solution $u_{26}(x,t)$ as (49).

(iii) From the phase portrait, we see that there are two special orbits $\widetilde{\Gamma}_7$ and $\widetilde{\Gamma}_7^*$, which have the same hamiltonian with that of the center point $(\widetilde{\varphi}_7^{\star}, 0)$. In (φ, ϕ)-plane, the expressions of the orbits are given as

$$\phi = \pm\sqrt{-\frac{\alpha}{6}}\sqrt{(\varphi - \widetilde{\varphi}_7^{\star})^2(\varphi - \widetilde{\varphi}_{14})(\varphi - \widetilde{\varphi}_{15})}, \quad (131)$$

where

$$\widetilde{\varphi}_{14} = \frac{1}{2\alpha}\left(\alpha\widetilde{\varphi}_7 + \sqrt{12\alpha\beta - 3\alpha^2\widetilde{\varphi}_7^2} + 2\sqrt{\alpha\widetilde{\varphi}_7(\alpha\widetilde{\varphi}_7 - \sqrt{12\alpha\beta - 3\alpha^2\widetilde{\varphi}_7^2})}\right), \quad (132)$$

$$\widetilde{\varphi}_{15} = \frac{1}{2\alpha}\left(\alpha\widetilde{\varphi}_7 + \sqrt{12\alpha\beta - 3\alpha^2\widetilde{\varphi}_7^2} - 2\sqrt{\alpha\widetilde{\varphi}_7(\alpha\widetilde{\varphi}_7 - \sqrt{12\alpha\beta - 3\alpha^2\widetilde{\varphi}_7^2})}\right). \quad (133)$$

Substituting (131) into $\frac{d\varphi}{d\xi} = \phi$ and integrating them along the orbits, it follows that

$$\pm\int_{\widetilde{\varphi}_{14}}^{\varphi} \frac{1}{\sqrt{(s - \widetilde{\varphi}_7^{\star})^2(s - \widetilde{\varphi}_{14})(s - \widetilde{\varphi}_{15})}} ds = \sqrt{-\frac{\alpha}{6}} \int_0^{\xi} ds, \quad (134)$$

$$\pm\int_{\widetilde{\varphi}_{15}}^{\varphi} \frac{1}{\sqrt{(s - \widetilde{\varphi}_7^{\star})^2(s - \widetilde{\varphi}_{14})(s - \widetilde{\varphi}_{15})}} ds = \sqrt{-\frac{\alpha}{6}} \int_0^{\xi} ds. \quad (135)$$

From (134), (135) and noting that $u = \varphi(\xi)$ and $\xi = x - ct$, we get two periodic blow-up wave solutions $u_{23_\pm}(x,t)$ as (43).

(3) If $g = g_0$, we will consider two kinds of orbits.

(i) From the phase portrait, we see that there are two orbits $\widetilde{\Gamma}_8$ and $\widetilde{\Gamma}_8^*$, which have the same hamiltonian with the degenerate saddle point $(\varphi_+^*, 0)$. In (φ, ϕ)-plane the expressions of these two orbits are given as

$$\phi = \pm\sqrt{-\frac{\alpha}{6}}\sqrt{(\varphi - \varphi_+^*)^3(\varphi - \widetilde{\varphi}_{16})}, \quad (136)$$

where

$$\widetilde{\varphi}_{16} = -3\sqrt{\frac{\beta}{\alpha}}. \quad (137)$$

Substituting (136) into $\frac{d\varphi}{d\xi} = \phi$ and integrating them along these two orbits $\widetilde{\Gamma}_8$ and $\widetilde{\Gamma}_8^*$, it follows that

$$\pm\int_{\varphi}^{+\infty} \frac{1}{\sqrt{(s - \varphi_+^*)^3(s - \widetilde{\varphi}_{16})}} ds = \sqrt{-\frac{\alpha}{6}} \int_0^{\xi} ds, \quad (138)$$

$$\pm\int_{\widetilde{\varphi}_{16}}^{\varphi} \frac{1}{\sqrt{(s - \varphi_+^*)^3(s - \widetilde{\varphi}_{16})}} ds = \sqrt{-\frac{\alpha}{6}} \int_0^{\xi} ds. \quad (139)$$

From (138), (139) and noting that $u = \varphi(\xi)$ and $\xi = x - ct$, we get three blow-up solutions $u_{27}(x,t))$, $u_{28}(x,t)$ and $u_{29}(x,t)$ as (50), (51) and (52).

(ii) From the phase portrait, we see that there are two special orbits $\widetilde{\Gamma}_9$ and $\widetilde{\Gamma}_9^*$ passing the points $(\widetilde{\Gamma}_{17}, 0)$ and $(\widetilde{\Gamma}_{18}, 0)$. In (φ, ϕ)-plane, the expressions of the orbits are given as

$$\phi = \pm\sqrt{-\frac{\alpha}{6}}\sqrt{(\varphi - \widetilde{\varphi}_{18})(\varphi - \widetilde{\varphi}_{17})(\varphi - c_5)(\varphi - \bar{c}_5)}, \quad (140)$$

where $\widetilde{\varphi}_{17} < \widetilde{\varphi}_{16} < \widetilde{\varphi}_+^* < \widetilde{\varphi}_{18}$, c_5 and $\bar{c}_5$ are conjugate complex numbers.

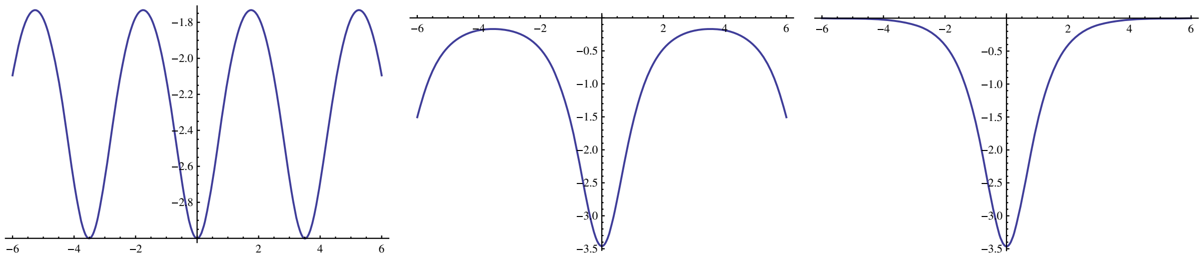

Fig. 3 The limiting process of u_1 tends to $u_{4\_}$ when φ_1 tends to φ_7

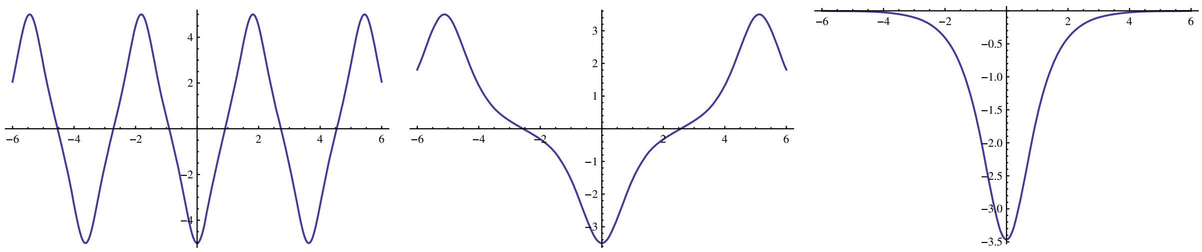

Fig. 4 The limiting process of $u_{3\_}$ tends to $u_{4\_}$ when φ_5 tends to φ_7

Substituting (140) into $\frac{\mathrm{d}\varphi}{\mathrm{d}\xi}=\phi$ and integrating them along $\widetilde{\Gamma}_9$ and $\widetilde{\Gamma}_9^*$, we have

$$\pm\int_{\widetilde{\varphi}_{18}}^{\varphi}\frac{1}{\sqrt{(s-\widetilde{\varphi}_{18})(s-\widetilde{\varphi}_{17})(s-c_5)(s-\bar{c}_5)}}\mathrm{d}s = \sqrt{-\frac{\alpha}{6}}\int_0^{\xi}\mathrm{d}s. \tag{141}$$

From (141) and noting that $u=\varphi(\xi)$ and $\xi=x-ct$, we get a periodic wave solutions $u_{30}(x,t)$ as (53).

Thus, we obtain the results given in Proposition 2.

3.4 The derivation of Proposition 3

In this section, we will give that the solitary wave solutions, periodic wave solutions, kink wave solutions, blow-up wave solutions and unbounded solutions can be obtained from the limits of the smooth periodic wave solutions or periodic blow-up solutions.

(1) Letting $\varphi_1\to\varphi_7$, it follows that $a_1\to-\frac{6\beta}{\alpha}$, $b_1\to\frac{6\beta}{\alpha}$, $c_1\to\sqrt{\frac{6\beta}{\alpha}}$, $d_1\to\sqrt{\frac{6\beta}{\alpha}}$, $\omega_1\to\frac{\sqrt{\beta}}{2}k_1\to 1$ and $\mathrm{sn}(\omega_1\xi,1)=\tanh(\omega_1\xi)$, and we have

$$\begin{aligned}\lim_{\varphi_1\to\varphi_7}u_1(x,t)&=\lim_{\varphi_1\to\varphi_7}\frac{a_1+b_1\mathrm{sn}^2(\omega_1\xi,k_1)}{c_1+d_1\mathrm{sn}^2(\omega_1\xi,k_1)}\\&=\frac{-\frac{6\beta}{\alpha}+\frac{6\beta}{\alpha}\tanh^2\left(\frac{\sqrt{\beta}}{2}\xi\right)}{\sqrt{\frac{6\beta}{\alpha}}+\sqrt{\frac{6\beta}{\alpha}}\tanh^2\left(\frac{\sqrt{\beta}}{2}\xi\right)}\\&=-\sqrt{\frac{6\beta}{\alpha}}\frac{1-\tanh^2\left(\frac{\sqrt{\beta}}{2}\xi\right)}{1+\tanh^2\left(\frac{\sqrt{\beta}}{2}\xi\right)}\\&=-\sqrt{\frac{6\beta}{\alpha}}\mathrm{sech}\left(\sqrt{\beta}\xi\right)=u_{4\_}(x,t).\end{aligned} \tag{142}$$

$$\begin{aligned}\lim_{\varphi_5\to\varphi_7}u_{3\_}(x,t)&=\lim_{\varphi_5\to\varphi_7}\varphi_5\mathrm{cn}\\&\quad\times\left(\sqrt{\frac{\alpha}{3}\varphi_5^2-\beta}\xi,-\varphi_5\sqrt{\frac{\alpha}{2\alpha\varphi_5^2-6\beta}}\right)\\&=-\sqrt{\frac{6\beta}{\alpha}}\mathrm{cn}\left(\sqrt{\beta}\xi,1\right)\\&=-\sqrt{\frac{6\beta}{\alpha}}\mathrm{sech}\left(\sqrt{\beta}\xi\right)=u_{4\_}(x,t).\end{aligned} \tag{143}$$

Therefore, the hyperbolic solitary wave solution $u_{4\_}(x,t)$ is the limit of the elliptic function periodic wave solutions $u_1(x,t)$ and $u_{3\_}(x,t)$. Their limiting process are in Figs. 3 and 4.

(2) Letting $\varphi_4\to\varphi_8$, it follows that $k_2\to 1$ and $\mathrm{sn}\left(\varphi_4\sqrt{\frac{\alpha}{6}}\xi,1\right)=\tanh(\sqrt{\beta}\xi)$, and we have

$$\lim_{\varphi_4\to\varphi_8}u_2(x,t) = \lim_{\varphi_4\to\varphi_8}\sqrt{\varphi_4^2-(2\varphi_4^2-\frac{6\beta}{\alpha})\mathrm{sn}^2\left(\varphi_4\sqrt{\frac{\alpha}{6}}\xi,k_2\right)}$$

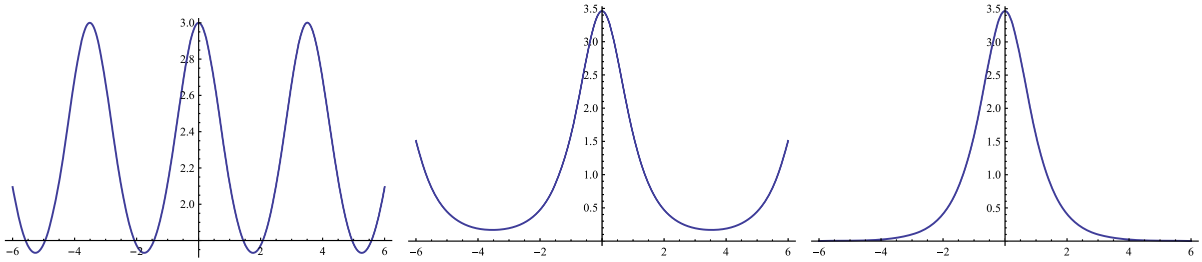

Fig. 5 The limiting process of u_2 tends to u_{4_+} when φ_4 tends to φ_8

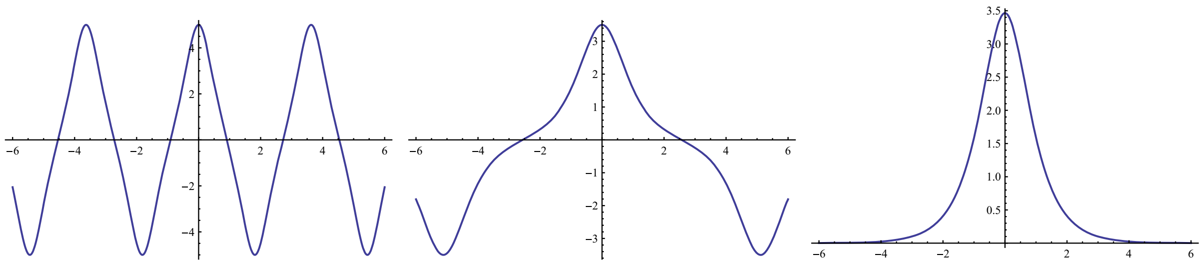

Fig. 6 The limiting process of u_{3_+} tends to u_{4_+} when φ_6 tends to φ_8

$$= \sqrt{\frac{6\beta}{\alpha} - \frac{6\beta}{\alpha}\tanh^2\left(\sqrt{\beta}\xi\right)} = \sqrt{\frac{6\beta}{\alpha}}\text{sech}\left(\sqrt{\beta}\xi\right)$$
$$= u_{4_+}(x,t). \tag{144}$$

$$\lim_{\varphi_6\to\varphi_8} u_{3_+}(x,t)$$
$$= \lim_{\varphi_6\to\varphi_8} \varphi_6\text{cn}\left(\sqrt{\frac{\alpha}{3}\varphi_6^2-\beta}\,\xi, \varphi_6\sqrt{\frac{\alpha}{2\alpha\varphi_6^2-6\beta}}\right)$$
$$= \sqrt{\frac{6\beta}{\alpha}}\text{cn}\left(\sqrt{\beta}\xi, 1\right) = \sqrt{\frac{6\beta}{\alpha}}\text{sech}\left(\sqrt{\beta}\xi\right)$$
$$= u_{4_+}(x,t). \tag{145}$$

Therefore, the hyperbolic solitary wave solution $u_{4_+}(x,t)$ is the limit of the elliptic function periodic wave solutions $u_2(x,t)$ and $u_{3_+}(x,t)$. Their limiting process are in Figs. 5 and 6.

(3) Letting $\varphi_{11} \to \varphi_{13} - 0$, it follows that $c_1 \to \varphi_9^\star$, $\bar{c}_1 \to \varphi_9^\star$, $k_3 \to 0$, $\varphi_{10} \to \varphi_{12}+0$, $A_1 \to \varphi_{13} - \varphi_9^\star$ and $B_1 \to \varphi_{12} - \varphi_9^\star$, and we have

$$\lim_{\varphi_{11}\to\varphi_{13}} u_5(x,t) = \lim_{\varphi_{11}\to\varphi_{13}}$$
$$\frac{A_1\varphi_{10}+\varphi_{11}B_1+(A_1\varphi_{10}-\varphi_{11}B_1)\text{cn}\left(\sqrt{\frac{\alpha A_1B_1}{6}}\xi, k_3\right)}{A_1+B_1+(A_1-B_1)\text{cn}\left(\sqrt{\frac{\alpha A_1B_1}{6}}\xi, k_3\right)}$$

$$= \frac{-2\gamma_1+\eta_1\delta_1+\eta_1\sqrt{\mu_1}\cos\left(\sqrt{\frac{\alpha\gamma_1}{6}}\xi\right)}{\delta_1+\sqrt{\mu_1}\cos\left(\sqrt{\frac{\alpha\gamma_1}{6}}\xi\right)}$$
$$= u_6(x,t). \tag{146}$$

Therefore, the trigonometric function periodic wave solution $u_6(x,t)$ is the limit of the elliptic function periodic wave solution $u_5(x,t)$. The limiting process is in Fig. 7.

(6) Letting $\varphi_{22} \to \varphi_{26}$, it follows that $\varphi_{23} \to \varphi_+^*$, $c_3 \to \varphi_+^*$, $\bar{c}_3 \to \varphi_+^*$, $A_3 \to 0$, $B_3 \to 4\sqrt{\frac{\beta}{\alpha}}$ and $\text{cn}\left(\sqrt{\frac{\alpha A_3B_3}{6}}\xi, k_6\right) \to \text{cn}(0,k_6) = 1$, and we have

$$\lim_{\varphi_{22}\to\varphi_{26}} u_{12}(x,t) = \lim_{\varphi_{22}\to\varphi_{26}}$$
$$\frac{A_3\varphi_{22}+\varphi_{23}B_3+(A_3\varphi_{22}-\varphi_{23}B_3)\text{cn}\left(\sqrt{\frac{\alpha A_3B_3}{6}}\xi, k_6\right)}{A_3+B_3+(A_3-B_3)\text{cn}\left(\sqrt{\frac{\alpha A_3B_3}{6}}\xi, k_6\right)}$$
$$= \lim_{A_3\to 0}\frac{A_3\varphi_{22}+\varphi_{23}B_3+(A_3\varphi_{22}-\varphi_{23}B_3)\text{cn}\left(\sqrt{\frac{\alpha A_3B_3}{6}}\xi, k_6\right)}{A_3+B_3+(A_3-B_3)\text{cn}\left(\sqrt{\frac{\alpha A_3B_3}{6}}\xi, k_6\right)}$$
$$= \lim_{A_3\to 0}\frac{2\sqrt{6\alpha A_3B_3}\,(\varphi_{22}+\varphi_{22}\chi_1)-\alpha B_3(A_3\varphi_{22}-B_3\varphi_{23})\xi\chi_2\chi_3}{2\sqrt{6\alpha A_3B_3}\,(1+\chi_1)-\alpha B_3(A_3-B_3)\xi\chi_2\chi_3}$$
$$= \frac{\sqrt{\beta}(-9+2\beta\xi^2)}{\sqrt{\alpha}(3+2\beta\xi^2)} = u_{14}(x,t). \tag{147}$$

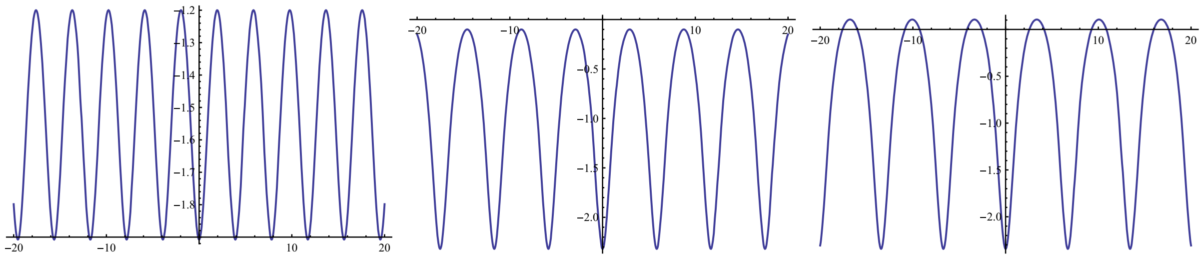

Fig. 7 The limiting process of u_5 tends to u_6 when φ_{11} tends to φ_{13}

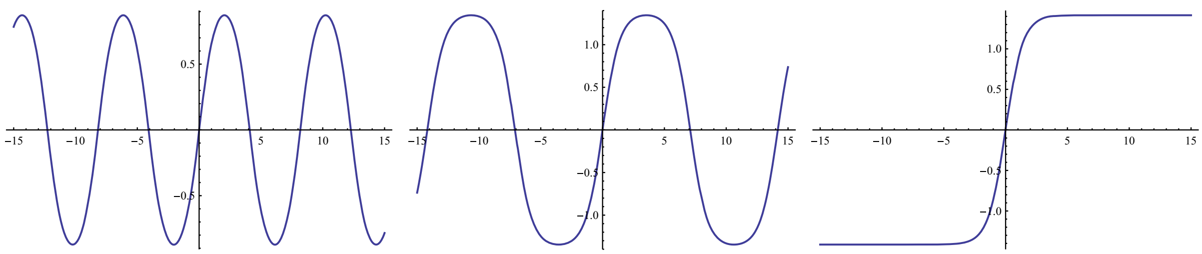

Fig. 8 The limiting process of u_{15_+} tends to u_{19_+} when $\widetilde{\varphi}_4$ tends to φ_+

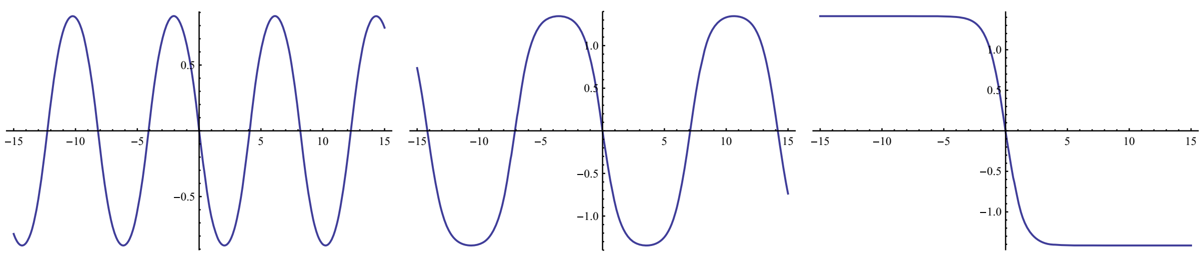

Fig. 9 The limiting process of u_{15_-} tends to u_{19_-} when $\widetilde{\varphi}_4$ tends to φ_+

where $\chi_1 = \mathrm{cn}\left(\sqrt{\frac{\alpha A_3 B_3}{6}}\xi, k_6\right)$, $\chi_2 = \mathrm{dn}\left(\sqrt{\frac{\alpha A_3 B_3}{6}}\xi, k_6\right)$, $\chi_3 = \mathrm{sn}\left(\sqrt{\frac{\alpha A_3 B_3}{6}}\xi, k_6\right)$.

Therefore, the fractional function solitary wave solution $u_{14}(x,t)$ is the limit of the elliptic function periodic wave solution $u_{12}(x,t)$. The limiting process is similar to that in Fig. 3.

(7) Letting $\widetilde{\varphi}_4 \to \varphi_+ + 0$, it follows that $\sqrt{\frac{6\beta}{\alpha} - \widetilde{\varphi}_4^2} \to \sqrt{\frac{3\beta}{\alpha}}$, $\frac{1}{\widetilde{\varphi}_4}\sqrt{\frac{6\beta}{\alpha} - \widetilde{\varphi}_4^2} \to 1$ and $\mathrm{sn}\left(\sqrt{-\frac{\beta}{2}}\xi, 1\right) = \tanh\left(\sqrt{-\frac{\beta}{2}}\xi\right)$, and we have

$$\begin{aligned}
\lim_{\widetilde{\varphi}_4 \to \varphi_+} u_{15_\pm}(x,t) &= \lim_{\widetilde{\varphi}_4 \to \varphi_+} \pm\sqrt{\frac{6\beta}{\alpha} - \widetilde{\varphi}_4^2}\,\mathrm{sn}\left(\widetilde{\varphi}_4\sqrt{-\frac{\alpha}{6}}\xi, \frac{1}{\widetilde{\varphi}_4}\sqrt{\frac{6\beta}{\alpha} - \widetilde{\varphi}_4^2}\right) \\
&= \pm\sqrt{\frac{3\beta}{\alpha}}\tanh\left(\sqrt{-\frac{\beta}{2}}\xi\right) \\
&= u_{19_\pm}(x,t). \qquad (148)
\end{aligned}$$

Therefore, the kink wave solutions $u_{19_\pm}(x,t)$ are the limit of the elliptic function periodic wave solutions $u_{15_\pm}(x,t)$. Their limiting process are in Figs. 8 and 9.

(8) Letting $\widetilde{\varphi}_4 \to \varphi_+ + 0$, it follows that $\frac{1}{\widetilde{\varphi}_4}\sqrt{\frac{6\beta}{\alpha} - \widetilde{\varphi}_4^2} \to 1$ and $\mathrm{sn}\left(\sqrt{-\frac{\beta}{2}}\xi, 1\right) = \tanh\left(\sqrt{-\frac{\beta}{2}}\xi\right)$, and we have

$$\begin{aligned}
\lim_{\widetilde{\varphi}_4 \to \varphi_+} u_{16_\pm}(x,t) &= \lim_{\widetilde{\varphi}_4 \to \varphi_+} \pm\frac{\widetilde{\varphi}_4}{\mathrm{sn}\left(\widetilde{\varphi}_4\sqrt{-\frac{\alpha}{6}}\xi, \frac{1}{\widetilde{\varphi}_4}\sqrt{\frac{6\beta}{\alpha} - \widetilde{\varphi}_4^2}\right)} \\
&= \pm\sqrt{\frac{3\beta}{\alpha}}\coth\left(\sqrt{-\frac{\beta}{2}}\xi\right) = u_{20_\pm}(x,t). \qquad (149)
\end{aligned}$$

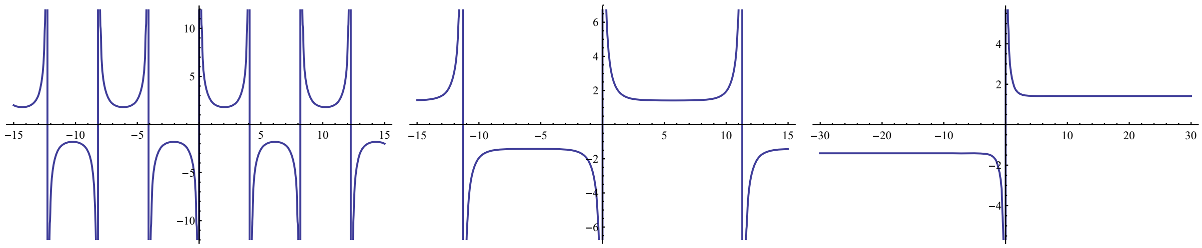

Fig. 10 The limiting process of u_{16_+} tends to u_{20_+} when $\widetilde{\varphi}_4$ tends to φ_+

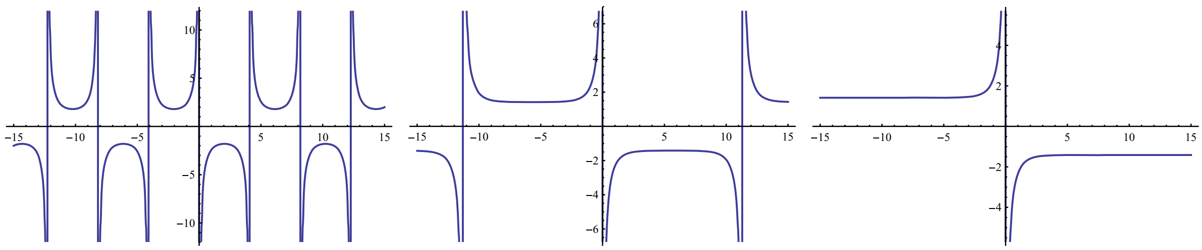

Fig. 11 The limiting process of u_{16_-} tends to u_{20_-} when $\widetilde{\varphi}_4$ tends to φ_+

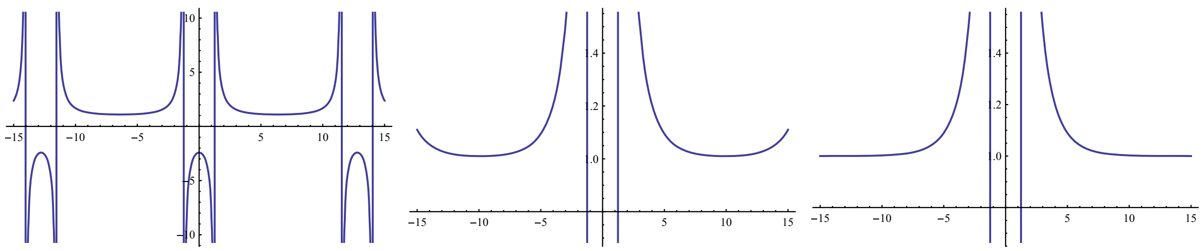

Fig. 12 The limiting process of u_{21} tends to u_{25} when $\widetilde{\varphi}_4$ tends to $\widetilde{\varphi}_7$

Therefore, the unbounded wave solutions $u_{20_\pm}(x,t)$ are the limit of the elliptic function periodic wave solutions $u_{16_\pm}(x,t)$. Their limiting process are in Figs. 10 and 11.

(9) Letting $\widetilde{\varphi}_{11} \to \widetilde{\varphi}_7 - 0$, it follows that $\widetilde{\varphi}_8 \to \widetilde{\varphi}_{12} - 0$, $\widetilde{\varphi}_9 \to \widetilde{\varphi}_{13} + 0$, $\widetilde{\varphi}_{10} \to \widetilde{\varphi}_7 + 0$, $k_8 \to 1$, $\omega_3 \to \frac{\sqrt{\beta - \alpha\widetilde{\varphi}_7^2}}{2}$ and $\mathrm{sn}(\omega_3\xi, 1) \to \tanh(\omega_3\xi)$, and we have

$$\begin{aligned}
&\lim_{\widetilde{\varphi}_{11}\to\widetilde{\varphi}_7} u_{21}(x,t)\\
&= \lim_{\widetilde{\varphi}_{11}\to\widetilde{\varphi}_7} \frac{(-\widetilde{\varphi}_{11}+\widetilde{\varphi}_9)\widetilde{\varphi}_8 + (\widetilde{\varphi}_{11}-\widetilde{\varphi}_8)\widetilde{\varphi}_9 \mathrm{sn}^2(\omega_3\xi, k_8)}{-\widetilde{\varphi}_{11}+\widetilde{\varphi}_9 + (\widetilde{\varphi}_{11}-\widetilde{\varphi}_8)\mathrm{sn}^2(\omega_3\xi, k_8)}\\
&= \frac{(-\widetilde{\varphi}_7+\widetilde{\varphi}_{13})\widetilde{\varphi}_{12} + (\widetilde{\varphi}_7-\widetilde{\varphi}_{12})\widetilde{\varphi}_{13}\tanh^2(\omega_3\xi)}{-\widetilde{\varphi}_7+\widetilde{\varphi}_{13} + (\widetilde{\varphi}_7-\widetilde{\varphi}_{12})\tanh^2(\omega_3\xi)}\\
&= \widetilde{\varphi}_7 + \frac{6\beta - 6\alpha\widetilde{\varphi}_7^2}{2\alpha\widetilde{\varphi}_7 + \sqrt{6\alpha\beta - 2\alpha^2\widetilde{\varphi}_7^2}\cosh\left(\sqrt{\beta - \alpha\widetilde{\varphi}_7^2}\,\xi\right)}\\
&= u_{25}(x,t).
\end{aligned} \tag{150}$$

Therefore, the blow-up wave solution $u_{25}(x,t)$ is the limit of the periodic blow-up wave solution $u_{21}(x,t)$. The limiting process is in Fig. 12.

Similarly, we can derive the others cases. This has proved Proposition 3.

Remark 1 One may find that we only consider the case when $g \leq 0$ in Proposition 1(when $g \geq 0$ in Proposition 2). In fact, we may get exactly the same solutions in the opposite case.

Remark 2 By comparing with the solutions of Refs. [4–7], most of my results are new. After checking over those solutions carefully, when $a = 1$, we find that my results (15), (37) and (38), exactly the same as those results (5.19), (5.16), (5.17), (5.20), (5.21) given in Ref. [7]. When $a = 1$ and $c = \frac{\alpha}{1-2\alpha^2}$, we find that my results (39) and (40), exactly the same as those results

(5.6) given in Ref. [7]. To our knowledge, we believe that many other solutions are new.

4 Conclusions

In this paper, I have obtained many traveling wave solutions for the MBBM equation (2) by employing the bifurcation method and qualitative theory of dynamical systems. The traveling wave solutions have been given in Propositions 1 and 2. On the other hand, in Proposition 3, we prove that the solitary wave solutions, periodic wave solutions, kink wave solutions, blow-up wave solutions and unbounded solutions can be obtained from the limits of the smooth periodic wave solutions or periodic blow-up solutions. The method can be applied to many other nonlinear evolution equations, and we believe that many new results wait for further discovery by this method.

References

1. Benjamin, T.B., Bona, J.L., Mahony, J.J.: Model equations for long waves in nonlinear dispersive systems. Philos. Trans. R. Soc. Lond. Ser. A **272**(1220), 47–78 (1972)
2. Saut, J.C., Tzvetkov, N.: Global well-posedness for the KP-BBM equations. Appl. Math. Res. Express **2004**(1), 1–16 (2004)
3. Varlamov, V., Liu, Y.: Cauchy problem for the Ostrovsky equation. Discret. Contin. Dyn. Syst. **10**(3), 731–753 (2004)
4. Yusufoglu, E.: New solitonary solutions for the MBBM equations using Exp-function method. Phys. Lett. A **372**(4), 442–446 (2008)
5. Daghan, D., Donmez, O., Tuna, A.: Explicit solutions of the nonlinear partial differential equations. Nonlinear Anal.: Real World Appl. **11**(3), 2152–2163 (2010)
6. Abbasbandy, S., Shirzadi, A.: The first integral method for modified Benjamin–Bona–Mahony equation. Commun. Nonlinear Sci. Numer. Simul. **15**(7), 1759–1764 (2010)
7. Yusufoglu, E., Bekir, A.: The tanh and the sine–cosine methods for exact solutions of the MBBM and the Vakhnenko equations. Chaos Solitons Fract. **38**(4), 1126–1133 (2008)
8. Noor, M.A., Noor, K.I., Waheed, A., Al-Said, E.A.: Some new solitonary solutions of the modified Benjamin–Bona–Mahony equation. Comput. Math. Appl. **62**(4), 2126–2131 (2011)
9. Li, J.B., Liu, Z.R.: Smooth and non-smooth traveling waves in a nonlinearly dispersive equation. Appl. Math. Comput. **25**(1), 41–56 (2000)
10. Wang, Q.D., Tang, M.Y.: New exact solutions for two nonlinear equations. Phys. Lett. A **372**(17), 2995–3000 (2008)
11. Tang, S.Q., Huang, W.T.: Bifurcations of travelling wave solutions for the generalized double sinh-Gordon equation. Appl. Math. Comput. **189**(2), 1774–1781 (2007)
12. Liu, Z.R., Tang, H.: Explicit periodic wave solutions and their bifurcations for generalized Camassa–Holm equation. Int. J. Bifurc. Chaos **20**(8), 2507–2519 (2010)
13. Song, M., Liu, Z.R.: Periodic wave solutions and their limits for the ZK-BBM equation. Appl. Math. Comput. **232**, 9–26 (2014)
14. Song, M., Liu, Z.R.: Qualitative analysis and explicit traveling wave solutions for the Davey–Stewartson equation. Math. Methods Appl. Sci. **37**, 393–401 (2014)
15. Wen, Z.S.: Several new types of bounded wave solutions for the generalized two-component Camassa–Holm equation. Nonlinear Dyn. **77**(3), 849–857 (2014)
16. Wen, Z.S.: Bifurcation of solitons, peakons, and periodic cusp waves for θ-equation. Nonlinear Dyn. **77**(1–2), 247–253 (2014)