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Successive lag synchronization on nonlinear dynamical networks via linear feedback control

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Abstract Successive lag synchronization (SLS) is defined as a new synchronization pattern, which means that lag synchronization appears between two successively numbered nodes in a dynamical network. Based on the topological structure of the considered network, linear feedback control and adaptive linear feedback control are proposed to achieve the SLS. By using Lyapunov function method and Barbalat Lemma, some sufficient conditions for the global stability of SLS are obtained. Moreover, the stability condition is independent on time delay. By using the proposed control method, successive lag consensus of a multi-agent system with second-order dynamics is also realized. By utilizing the Chua's circuit as the local nonlinear dynamics of all nodes in the network, several numerical examples are presented to verify the theoretical results.

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1 Introduction

During the past few decades, the synchronization of complex networks has been studied thoroughly and widely, due to its potential applications in various fields, such as secure communication, parameter identification of dynamical systems, seismology, parallel image processing, chemical reaction, animal behavior research, and so on (e.g., see [1–8]). Various patterns of synchronization have been defined and studied, including complete synchronization [9,10], cluster synchronization [11,12], phase synchronization [13], lag synchronization (LS) [14], outer synchronization [15,16], bubbling synchronization [17], projective synchronization [18], generalized synchronization [19], etc.

Lag synchronization appears as a coincidence of shifted-in-time states of two coupled systems, i.e., $y(t) \rightarrow x(t-\tau)$ with positive τ as $t \rightarrow +\infty$, which is investigated originally in [14,20]. LS has also been realized and studied in two coupled dynamical networks later [21,22]. Recently, generalized lag synchronization (GLS) of two coupled dynamical networks has become a research focus [23–26]. In these works, a common research problem is how to achieve LS between two coupled systems (or networks), where one is the drive system (or drive network), and the other is the response system (or response network). The main research method is to utilize nonlinear feedback control to achieve the LS. However, the LS in multi-coupled dynamical systems is still an open problem.

In fact, realization of LS in multi-agent systems [27–30] has important significance. For instance, let $x_i(t)$ be the position state of the ith agent at time t in a multi-agent system, where $i = 1, 2, \ldots, n$. In order to realize position synchronization (i.e., consensus) and avoid collision between agents, we need to obtain a coincidence of shifted-in-time states of these agents, i.e., $x_i(t-\tau) \to x_{i+1}(t)$ for a positive time delay τ as $t \to +\infty$. We name this new kind of lag synchronization in multi-coupled dynamical systems as the successive lag synchronization (SLS), which can be considered as a generalized pattern of the traditional LS in two coupled systems. That is to say, the realization of SLS means the state of the $(i+1)$-th node converges to the state of the i-th node with a constant time delay τ, which is achieved successively from the second node to the last one. In addition, compared with linear feedback control, nonlinear feedback control is harder to be implemented in the real dynamical systems. Therefore, it is natural to raise the following question: *just with linear feedback control, can we achieve the SLS in a multi-coupled dynamical system (or network)*?

In this paper, based on a general dynamical network model, we will give a positive answer to the above question. Depending on topological structure of the considered dynamical network, a linear feedback control is designed to realize its SLS. Moreover, in order to decrease the control strength, an adaptive linear feedback control is also proposed. By using Lyapunov function method and Barbalat Lemma, some sufficient conditions for the global stability of SLS are obtained. Finally, we give application of the proposed control in multi-agent systems with second-order dynamics. To verify these results, some numerical examples are presented.

The rest of this paper is organized as follows. In Sect. 2, we give some Lemmas and some preliminaries about the model formulation of a considered dynamical network under control. In Sect. 3, SLS global stabilities of the dynamical network under linear feedback control and adaptive linear feedback control are investigated, respectively. In Sect. 4, we study the application of the proposed control in a multi-agent system with second-order dynamics. In Sect. 5, some numerical verifications are performed. Finally, in Sect. 6, we conclude this paper.

2 Preliminaries

Let $\mathbb{R}^n$ denote the n-dimension Euclidean space. For $x \in \mathbb{R}^n$, its Euclidean norm $\|x\| = x^{\mathrm{T}}x$, where T denotes transposition. I_n represents the identity matrix in $\mathbb{R}^{n\times n}$. O_n represents the zero matrix in $\mathbb{R}^{n\times n}$. For two symmetric matrices $A, B \in \mathbb{R}^{n\times n}$, $A > B$ means that $A - B$ is positive definite. Let $\mathcal{C}([-r, 0], \mathbb{R})$ be the function space of all continuous functions from $[-r, 0]$ to $\mathbb{R}$, where $r > 0$.

The considered dynamical network under control is described by

$$\dot{x}_i(t) = f(x_i(t)) + c\sum_{j=1}^{n} a_{ij}x_j(t) + u_i(t), \tag{1}$$

where $i = 1, 2, \ldots, n$, $x_i(t) = (x_{i1}, x_{i2}, \ldots, x_{im})^{\mathrm{T}} \in \mathbb{R}^m$ is the state variable of the i-th node, and $u_i(t)$ denotes the feedback control input into the i-th node. The function $f(\cdot)$ is a continuously differentiable function which determines the local dynamical behavior of the nodes. The constant $c > 0$ denotes the coupling strength. The coupling matrix $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ with zero-sum rows shows the topological structure of the network. If nodes i and j are connected, then $a_{ij} = a_{ji} = 1$; otherwise $a_{ij} = a_{ji} = 0$. The diagonal elements of the coupling matrix A are

$$a_{ii} = -\sum_{j=1, j\neq i}^{n} a_{ij} = -k_i, \quad i = 1, 2, \ldots, n, \tag{2}$$

where k_i denotes the degree of node i. We suppose that the matrix A is irreducible, which means that the network is connected in the sense that there are no isolated clusters.

Definition 1 SLS of dynamical network (1) is said to be achieved if, for any initial condition $x_i(t) = \varphi_i(t) \in \mathcal{C}([-(i-1)\tau, 0], \mathbb{R})$ and every $i \in \{1, 2, \ldots, n-1\}$,

$$\lim_{t\to+\infty} \|x_i(t-\tau) - x_{i+1}(t)\| = 0, \tag{3}$$

where the time delay parameter $\tau > 0$.

First, for $i = 1, 2, \ldots, n-1$, some preliminary functions are defined by

$$v_i(t) = \sum_{k=1}^{n-1} cw_{ik}(t) + ca_{(i+1)1}x_1(t) - ca_{in}x_n(t-\tau)$$
$$-d(x_i(t-\tau) - x_{i+1}(t)),$$

where $d > 0$ denotes control strength, for $k \neq i$,

$$w_{ik}(t) = \begin{cases} 0, & \text{if } a_{ik} = a_{(i+1)(k+1)}, \\ -a_{ik}x_k(t-\tau), & \text{if } a_{ik} > a_{(i+1)(k+1)}, \\ a_{(i+1)(k+1)}x_{k+1}(t), & \text{if } a_{ik} < a_{(i+1)(k+1)}, \end{cases} \tag{4}$$

and

$$w_{ii}(t) = (a_{(i+1)(i+1)} - a_{ii})x_{i+1}(t). \tag{5}$$

Then, the control can be designed as

$$u_1(t) = 0,$$
$$u_i(t) = -v_{i-1}(t)$$
$$-v_{i-2}(t-\tau) - \cdots - v_1(t-(i-2)\tau),$$
$$i = 2, 3, \ldots, n. \tag{6}$$

By using this control scheme, we have

$$u_i(t-\tau) - u_{i+1}(t) = v_i(t), \quad i = 1, 2, \ldots, n-1. \tag{7}$$

One can intuitively understand the function of above preliminary functions $v_i(t)$ in the following way. From the error system (11) and the proof of Theorem 1 in Sect. 3, we can see that the term $-c\Big(\sum_{k=1}^{n-1} w_{ik}(t) + a_{(i+1)1}x_1(t) - a_{in}x_n(t-\tau)\Big)$ plays a suppressive role in the realization of SLS, while the term $-d(x_i(t-\tau) - x_{i+1}(t))$ is devoted to accelerate the realization.

Definition 2 ([31,32]) Function class QUAD(Δ, P, ω): Let a diagonal matrix $\Delta = \mathrm{diag}\{\delta_1, \delta_2, \ldots, \delta_m\}$ and a positive definite diagonal matrix $P = \mathrm{diag}\{p_1, p_2, \ldots, p_m\}$. QUAD($\Delta, P, \omega$) denotes a class of continuous functions $f(x,t) : \mathbb{R}^m \times [0,+\infty) \to \mathbb{R}^m$ satisfying

$$(x-y)^{\mathrm{T}} P\{[f(x,t) - f(y,t)] - \Delta[x-y]\}$$
$$\leq -\omega(x-y)^{\mathrm{T}}(x-y), \tag{8}$$

for some $\omega > 0$, all $x, y \in \mathbb{R}^m$ and all $t \geq 0$.

Lemma 1 (Barbalat Lemma [33]) *If $g(t) : \mathbb{R} \to \mathbb{R}^+$ is a uniformly continuous function for $t \geq 0$ and if the limit of the integral*

$$\lim_{t\to+\infty} \int_0^t g(s)ds \tag{9}$$

exists and is finite, then $\lim_{t\to+\infty} g(t) = 0$.

From this Lemma, we have the following result.

Lemma 2 *If $g(t) : \mathbb{R} \to \mathbb{R}^+$ is a uniformly continuous function for $t \geq 0$ and if the integral*

$$\int_0^t g(s)ds \tag{10}$$

is bounded on $[0,+\infty)$, then $\lim_{t\to+\infty} g(t) = 0$.

Proof Let $G(t) = \int_0^t g(s)ds$. Obviously, the function $G(t)$ is a monotone increasing function on $[0,+\infty)$. Since $G(t)$ is bounded, $\lim_{t\to+\infty} G(t)$ exists and is finite. According to Lemma 1, we can get this result. □

Lemma 3 ([34]) *For matrices A, B, C, and D with appropriate dimensions, the Kronecker product $\otimes$ satisfies*

(i) $(\phi A) \otimes B = A \otimes (\phi B)$,
(ii) $(A+B) \otimes C = A \otimes C + B \otimes C$,
(iii) $(A \otimes B)^{\mathrm{T}} = A^{\mathrm{T}} \otimes B^{\mathrm{T}}$,

where ϕ is a constant.

Remark 1 To realize SLS of dynamical network (1), the design of its control is not unique. As shown in control (6), the first node is not controlled, while the other nodes are all controlled. In fact, we can let $u_n(t) = 0$ and the other controllers $u_i(t) \neq 0$, which can be determined according to the corresponding SLS error systems as discussed in next section.

3 Global stability analysis of SLS

According to Definition 1, SLS errors of dynamical network (1) are defined as $e_i(t) = x_i(t-\tau) - x_{i+1}(t)$ for $i = 1, 2, \ldots, n-1$. Then, the error system can be described as

$$\dot{e}_i(t) = \dot{x}_i(t-\tau) - \dot{x}_{i+1}(t)$$
$$= f(x_i(t-\tau)) + c\sum_{j=1}^{n} a_{ij}x_j(t-\tau) + u_i(t-\tau)$$
$$- f(x_{i+1}(t)) - c\sum_{j=1}^{n} a_{(i+1)j}x_j(t) - u_{i+1}(t)$$
$$= f(x_i(t-\tau)) - f(x_{i+1}(t)) + c\sum_{j=1}^{n}(a_{ij}x_j(t-\tau)$$
$$-a_{(i+1)j}x_j(t)) + v_i(t)$$
$$= f(x_i(t-\tau)) - f(x_{i+1}(t)) + c\sum_{j=1}^{n-1} b_{ij}e_j(t)$$
$$-de_i(t), \tag{11}$$

where according to control (6) for $j \neq i$,

$$b_{ij} = \begin{cases} a_{ij}, & \text{if } a_{ij} = a_{(i+1)(j+1)}, \\ 0, & \text{if } a_{ij} \neq a_{(i+1)(j+1)}, \end{cases} \tag{12}$$

and

$$b_{ii} = a_{ii}. \tag{13}$$

It is easy to verify that B is also symmetric.

Theorem 1 *Let* $\Delta = \mathrm{diag}\{\delta_1, \delta_2, \ldots, \delta_m\}$ *be diagonal matrix and* $P = \mathrm{diag}\{p_1, p_2, \ldots, p_m\}$ *be positive definite diagonal matrix, such that* $f(x) \in \mathrm{QUAD}(\Delta, P)$. *If there exists* $d > 0$ *such that*

$$-\omega I_{(n-1)m} + I_{n-1} \otimes P\Delta + (cB - dI_{n-1}) \otimes P < O_{(n-1)m}, \tag{14}$$

then, the SLS of dynamical network (1) *under control* (6) *can be achieved for any initial conditions.*

Proof Define the following Lyapunov functional candidate

$$V(t) = \frac{1}{2}\sum_{i=1}^{n-1} e_i^{\mathrm{T}}(t) P e_i(t). \tag{15}$$

The derivative of $V(t)$ along trajectories of error system (11) can be obtained as follows:

$$\begin{aligned} \frac{\mathrm{d}V(t)}{\mathrm{d}t} &= \sum_{i=1}^{n-1} e_i^{\mathrm{T}}(t) P \Bigg(f(x_i(t-\tau)) - f(x_{i+1}(t)) \\ &\quad + c\sum_{j=1}^{n-1} (a_{ij}x_j(t-\tau) - a_{(i+1)j}x_j(t)) \\ &\quad + u_i(t-\tau) - u_{i+1}(t) \Bigg) \\ &= \sum_{i=1}^{n-1} e_i^{\mathrm{T}}(t) P \Bigg(f(x_i(t-\tau)) - f(x_{i+1}(t)) \\ &\quad + c\sum_{j=1}^{n-1} b_{ij}e_j(t) - de_i(t) \Bigg) \\ &\le \sum_{i=1}^{n-1} \Bigg(-\omega e_i^{\mathrm{T}} e_i + e_i^{\mathrm{T}} P\Delta e_i + ce_i^{\mathrm{T}} P \sum_{j=1}^{n-1} b_{ij}e_j \\ &\quad - de_i^{\mathrm{T}} P e_i \Bigg). \end{aligned} \tag{16}$$

Let $e(t) = (e_1^{\mathrm{T}}(t), e_2^{\mathrm{T}}(t), \ldots, e_{n-1}^{\mathrm{T}}(t))^{\mathrm{T}}$. From condition (31), there exists a constant $\varepsilon > 0$ such that $-\omega I_{(n-1)m} + I_{n-1} \otimes P\Delta + (cB - dI_{n-1}) \otimes P \le -\varepsilon I_{(n-1)m}$.

From (16), we get

$$\begin{aligned} \frac{\mathrm{d}V(t)}{\mathrm{d}t} &\le -\omega e^{\mathrm{T}}(t) I_{(n-1)m} e(t) + e^{\mathrm{T}}(t)(I_{n-1} \otimes P\Delta)e(t) \\ &\quad + ce^{\mathrm{T}}(t)(B \otimes P)e(t) - de^{\mathrm{T}}(t)(I_{n-1} \otimes P)e(t) \\ &= e^{\mathrm{T}}(t)(-\omega I_{(n-1)m} + I_{n-1} \otimes P\Delta + cB \otimes P \\ &\quad - dI_{n-1} \otimes P)e(t) \le -\varepsilon e^{\mathrm{T}}(t)e(t) \\ &= -\varepsilon \sum_{i=1}^{n-1} e_i^{\mathrm{T}}(t)e_i(t). \end{aligned} \tag{17}$$

Integrating the above equation from 0 to t yields

$$\begin{aligned} \varepsilon \int_0^t \sum_{i=1}^{n-1} e_i^{\mathrm{T}}(s)e_i(s)ds &\le V(0) - V(t) \\ &\le V(0). \end{aligned} \tag{18}$$

So, the integral $\int_0^t \sum_{i=1}^{n-1} e_i^{\mathrm{T}}(s)e_i(s)ds$ is bounded. From Lemma 2, we obtain $\lim_{t\to+\infty} \|e_i(t)\| = 0$ for every $i \in \{1, 2, \ldots, n-1\}$, which in turn means that $\lim_{t\to+\infty} \|x_i(t-\tau) - x_{i+1}(t)\| = 0$. This completes the proof. □

From Theorem 1, we can always choose large enough control strength d such that inequality (31) is valid, then the SLS of dynamical network (1) under control (6) can be achieved. However, in practice, it is not allowed that the control strength is arbitrarily large. A feasible strategy is to adopt adaptive control method [31].

For $i = 1, 2, \ldots, n-1$, some preliminary functions are similarly defined by

$$\begin{aligned} v_i(t) &= \sum_{k=1}^{n-1} cw_{ik}(t) + ca_{(i+1)1}x_1(t) - ca_{in}x_n(t-\tau) \\ &\quad - d_i(t)e_i(t), \end{aligned}$$

where $d_i(t) \ge 0$ denotes the time-varying control strength, for $k \neq i$,

$$w_{ik}(t) = \begin{cases} 0, & \text{if } a_{ik} = a_{(i+1)(k+1)}, \\ -a_{ik}x_k(t-\tau), & \text{if } a_{ik} > a_{(i+1)(k+1)}, \\ a_{(i+1)(k+1)}x_{k+1}(t), & \text{if } a_{ik} < a_{(i+1)(k+1)}, \end{cases} \tag{19}$$

and

$$w_{ii}(t) = (a_{(i+1)(i+1)} - a_{ii})x_{i+1}(t). \tag{20}$$

Then, an adaptive control is designed as

$$u_1(t)=0,$$
$$u_i(t)=-v_{i-1}(t)-v_{i-2}(t-\tau)-\cdots-v_1(t-(i-2)\tau),$$
$$\dot{d}_j(t)=e_j^{\mathrm{T}}(t)Pe_j(t),\ i=2,\ldots,n,\quad j=1,\ldots,n-1. \tag{21}$$

Theorem 2 *Let* $\Delta=\mathrm{diag}\{\delta_1,\delta_2,\ldots,\delta_m\}$ *be diagonal matrix and* $P=\mathrm{diag}\{p_1,p_2,\ldots,p_m\}$ *be positive definite diagonal matrix, such that* $f(x)\in\mathrm{QUAD}(\Delta,P)$. *The SLS of dynamical network (1) can be achieved for any initial conditions under adaptive control (21).*

Proof Define the following Lyapunov functional candidate

$$V(t)=\frac{1}{2}\sum_{i=1}^{n-1}e_i^{\mathrm{T}}(t)Pe_i(t)+\sum_{i=1}^{n-1}\alpha(d_i(t)-d_i^*)^2, \tag{22}$$

where positive constants α and d_i^* will be decided later.

The derivative of $V(t)$ along trajectories of error system (11) can be obtained as follows:

$$\begin{aligned}\frac{\mathrm{d}V(t)}{\mathrm{d}t}&=\sum_{i=1}^{n-1}e_i^{\mathrm{T}}(t)P\Bigg(f(x_i(t-\tau))-f(x_{i+1}(t))\\&\quad+c\sum_{j=1}^{n-1}(a_{ij}x_j(t-\tau)-a_{(i+1)j}x_j(t))\\&\quad+u_i(t-\tau)-u_{i+1}(t)\Bigg)\\&\quad+2\alpha\sum_{i=1}^{n-1}(d_i(t)-d_i^*)e_i^{\mathrm{T}}(t)Pe_i(t)\\&=\sum_{i=1}^{n-1}e_i^{\mathrm{T}}(t)P\Bigg(f(x_i(t-\tau))-f(x_{i+1}(t))\\&\quad+c\sum_{j=1}^{n-1}b_{ij}e_j(t)-d_i(t)e_i(t)\Bigg)\\&\quad+2\alpha\sum_{i=1}^{n-1}(d_i(t)-d_i^*)e_i^{\mathrm{T}}(t)Pe_i(t).\end{aligned} \tag{23}$$

Let $D(t)=\mathrm{diag}\{d_1(t),d_2(t),\ldots,d_{n-1}(t)\}$ and $D^*=\mathrm{diag}\{d_1^*,d_2^*,\ldots,d_{n-1}^*\}$, which are all positive definite. Similar to the proof of Theorem 1, from the condition of this theorem and (17), we get

$$\begin{aligned}\frac{\mathrm{d}V(t)}{\mathrm{d}t}&\le e^{\mathrm{T}}(t)(-\omega I_{(n-1)m}+I_{n-1}\otimes P\Delta+cB\otimes P\\&\quad-D(t)\otimes P)e(t)\\&\quad+2\alpha e^{\mathrm{T}}(t)(D(t)\otimes P-D^*\otimes P)e(t)\\&=e^{\mathrm{T}}(t)(-\omega I_{(n-1)m}+I_{n-1}\otimes P\Delta+cB\otimes P\\&\quad-2\alpha D^*\otimes P)e(t)\\&\quad-(1-2\alpha)e^{\mathrm{T}}(t)(D(t)\otimes P)e(t).\end{aligned} \tag{24}$$

First, we can choose appropriate α such that $1-2\alpha>0$. Then, we can further select appropriate D^* such that

$$\begin{aligned}&-\omega I_{(n-1)m}+I_{n-1}\otimes P\Delta+cB\otimes P-2\alpha D^*\otimes P\\&\quad\le-\tilde{\varepsilon}I_{(n-1)m},\end{aligned} \tag{25}$$

for a positive constant $\tilde{\varepsilon}$. Combining (24) and (25), we have

$$\frac{\mathrm{d}V(t)}{\mathrm{d}t}\le-\tilde{\varepsilon}\sum_{i=1}^{n-1}e_i^{\mathrm{T}}(t)e_i(t). \tag{26}$$

So, the integral $\int_0^t\sum_{i=1}^{n-1}e_i^{\mathrm{T}}(s)e_i(s)ds$ is bounded. From Lemma 2, we obtain $\lim_{t\to+\infty}\|e_i(t)\|=0$ for every $i\in\{1,2,\ldots,n-1\}$, which in turn means that $\lim_{t\to+\infty}\|x_i(t-\tau)-x_{i+1}(t)\|=0$. This completes the proof. □

Remark 2 In fact, the control methods (6) and (21) can be generalized to achieve the outer synchronization [15,16] of two or more coupled networks. For example, let U_i denotes the index set of all nodes in the i-th network, where $i=1,2,\ldots,N$. We can consider a new outer synchronization pattern, such that $\lim_{t\to+\infty}\|x_i(t)-x_j(t)\|=0,\forall\ i,j\in U_k$ and $\lim_{t\to+\infty}\|x_i(t-\tau)-x_j(t)\|=0,\forall\ i\in U_k,j\in U_{k+1}$. Since this consideration exceeds the work of this paper, we just present it for possible future researches.

4 Application in multi-agent systems with second-order dynamics

In this section, the proposed control method in above section will be generalized and applied to the control problem of successive lag consensus in multi-agent systems. A well-known multi-agent system with second-order dynamics [30,35] is described as follows:

$$\dot{x}_i(t) = y_i(t)$$
$$\dot{y}_i(t) = \alpha \sum_{j=1}^{n} a_{ij} x_j(t) - \beta \sum_{j=1}^{n} a_{ij} y_j(t),$$
$$i = 1, 2, \ldots, n, \qquad (27)$$

where $x_i \in \mathbb{R}^m$ and $y_i \in \mathbb{R}^m$ are the position and velocity state variables of the i-th agent, respectively. The parameters $\alpha > 0$ and $\beta > 0$ denote the coupling strengths, and coupling matrix $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ with zero-sum rows shows the topological structure of system (27). In this section, we consider the case $\alpha = \beta = c$, while the other cases have the similar analysis. By defining $z_i = (x_i^{\mathrm{T}}, y_i^{\mathrm{T}})^{\mathrm{T}}$, system (27) under control can be written as

$$\dot{z}_i(t) = f(z_i(t)) + c \sum_{j=1}^{n} a_{ij} \Gamma z_j(t) + u_i(t), i = 1, 2, \ldots, n, \qquad (28)$$

where $f(z_i(t)) = Hz_i(t)$, $H = \begin{pmatrix} O_m & I_m \\ O_m & O_m \end{pmatrix}$, and $\Gamma = \begin{pmatrix} O_m & O_m \\ I_m & I_m \end{pmatrix}$.

Definition 3 Successive lag consensus of system (28) is said to be achieved if, for any initial condition $z_i(t) = \varphi_i(t) \in \mathcal{C}([-(i-1)\tau, 0], \mathbb{R})$ and every $i \in \{1, 2, \ldots, n-1\}$,

$$\lim_{t\to+\infty} \|z_i(t-\tau) - z_{i+1}(t)\| = 0, \qquad (29)$$

where the time delay parameter $\tau > 0$.

For $i = 1, 2, \ldots, n-1$, define preliminary functions:

$$v_i(t) = \sum_{k=1}^{n-1} cw_{ik}(t) + ca_{(i+1)1}\Gamma z_1(t) - ca_{in}\Gamma z_n(t-\tau) - d(z_i(t-\tau) - z_{i+1}(t)),$$

where $d > 0$ denotes control strength, for $k \neq i$,

$$w_{ik}(t) = \begin{cases} 0, & \text{if } a_{ik} = a_{(i+1)(k+1)}, \\ -a_{ik}\Gamma z_k(t-\tau), & \text{if } a_{ik} > a_{(i+1)(k+1)}, \\ a_{(i+1)(k+1)}\Gamma z_{k+1}(t), & \text{if } a_{ik} < a_{(i+1)(k+1)}, \end{cases}$$

and

$$w_{ii}(t) = (a_{(i+1)(i+1)} - a_{ii})\Gamma z_{i+1}(t).$$

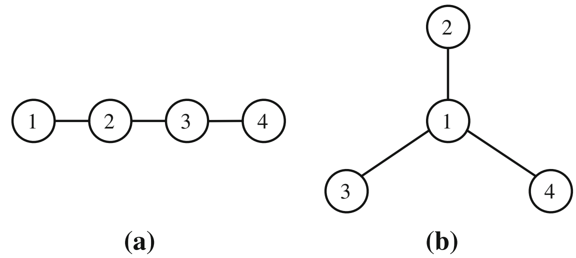

Fig. 1 Two kinds of topological structure for dynamical network (1) with size $n = 4$. **a** Chain-shaped network; **b** star-shaped network

Based on the proposed method in Sect. 2, the control of system (28) can be designed as

$$u_1(t) = 0,$$
$$u_i(t) = -v_{i-1}(t) - v_{i-2}(t-\tau) - \cdots - v_1(t-(i-2)\tau),$$
$$i = 2, 3, \ldots, n. \qquad (30)$$

Defining matrix B by (12)–(13), we have the following result.

Theorem 3 *If there exists $d > 0$ such that*

$$I_{n-1} \otimes H + cB \otimes \Gamma - dI_{(n-1)m} < O_{(n-1)m}, \qquad (31)$$

then the successive lag consensus of system (28) under control (30) can be achieved for any initial conditions.

Proof Define the following Lyapunov functional candidate

$$V(t) = \frac{1}{2}\sum_{i=1}^{n-1} e_i^{\mathrm{T}}(t)e_i(t), \qquad (32)$$

where $e_i(t) = z_i(t-\tau) - z_{i+1}(t)$. Then, by the similar analysis in Theorem 1, we can prove this theorem. For unnecessary repetition, we omit the detailed proof. □

5 Numerical experiments

To verify the effectiveness of our control method and the correctness of theoretical results, we will perform some numerical simulations in this section. Without loss of generality, the size of dynamical network (1) is set as $n = 4$, and its topological structure is shown in Fig. 1 with two cases. The local dynamics of all nodes in this network is characterized by a Chua's circuit which is described by

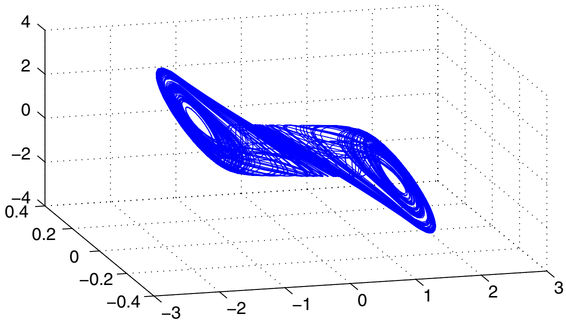

Fig. 2 The chaotic attractor of the Chua's circuit

$$\begin{cases} \dot{y}_1(t) = k[y_2(t) - h(y_1(t))], \\ \dot{y}_2(t) = y_1(t) - y_2(t) + y_3(t), \\ \dot{y}_3(t) = -l y_2(t), \end{cases} \tag{33}$$

where $k = 9$, $l = 100/7$, and $h(z) = (2/7)z - (3/14)[|z+1| - |z-1|]$. The chaotic attractor is shown in Fig. 2. For this system [32], we have $m = 3$ and can choose $P_1 = I_3$, $\Delta_1 = 10 I_3$ and $\omega = 0.6218$ to achieve the inequality (8). To realize SLS of dynamical network (1), the linear feedback control (6) and adaptive linear feedback control (21) are used in Sect. 5.1 and 5.2, respectively.

5.1 Linear feedback control

The topological structure of dynamical network is shown in Fig. 1a with network size $n = 4$ and chain-shaped topology. The star-shaped topology as shown in Fig. 1b will be considered in Sect. 5.2. The coupling matrix is

$$A = \begin{pmatrix} -1 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix}. \tag{34}$$

By (12) and (13), we get

$$B = \begin{pmatrix} -1 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -2 \end{pmatrix}. \tag{35}$$

According to control (6), we have

$$\begin{aligned} u_1(t) &= 0, \\ u_2(t) &= -cx_1(t) + cx_2(t) + d(x_1(t-\tau) - x_2(t)), \\ u_3(t) &= -cx_1(t-\tau) + cx_2(t-\tau) \\ &\quad + d(x_1(t-2\tau) - x_3(t)), \\ u_4(t) &= -cx_1(t-2\tau) + cx_2(t-2\tau) \\ &\quad - cx_4(t) + cx_4(t-\tau) \\ &\quad + d(x_1(t-3\tau) - x_4(t)). \end{aligned}$$

In order to achieve (31), we choose $c = 0.1$ and search the lower bound of the control strength d by solving the following Linear Matrix Inequality:

$$cB \otimes I_3 + (-\omega + 10 - d) I_9 \leq 0. \tag{36}$$

By the Matlab LMI and control toolboxes, we get the lower bound of the control strength $d_l = 9.36$. So, if $d > d_l$, then the conditions of Theorem 1 are satisfied, which means that the SLS of dynamical network (1) under control (6) can be achieved for any initial conditions. Without loss of generality, the initial condition is set as constant variable $(\varphi_1, \varphi_2, \varphi_3, \varphi_4) \in [0, 1]^{12}$. Figure 3 gives a realization under control strength $d = 11$ and time delay $\tau = 0.1$. In this simulation, we observe that there appears coincidence of shifted-in-time states in the network as $t \to +\infty$. So, the SLS is achieved under linear feedback control (6). To explore the influence of time delay on SLS, another realization is performed in Fig. 4 under control strength $d = 11$ and time delay $\tau = 0.5$. We can still achieve the SLS, which means that the stability condition in Theorem 1 is independent on time delay.

5.2 Adaptive linear feedback control

The topological structure of dynamical network is shown in Fig. 1b with network size $n = 4$ and star-shaped topological structure. By the definition in Sect. 2, the coupling matrix of this network is

$$A = \begin{pmatrix} -3 & 1 & 1 & 1 \\ 1 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix}. \tag{37}$$

According to the adaptive linear feedback control (21), we have

$$\begin{aligned} u_1(t) &= 0, \\ u_2(t) &= -cx_1(t) - 2cx_2(t) + cx_2(t-\tau) + cx_3(t-\tau) \\ &\quad + cx_4(t-\tau) + d_1(t)e_1(t), \\ u_3(t) &= -cx_1(t) - 2cx_2(t-\tau) \\ &\quad + cx_2(t-2\tau) + cx_3(t-2\tau) \\ &\quad + cx_4(t-2\tau) + d_1(t-\tau)e_1(t-\tau) \\ &\quad + d_2(t)e_2(t), \end{aligned}$$

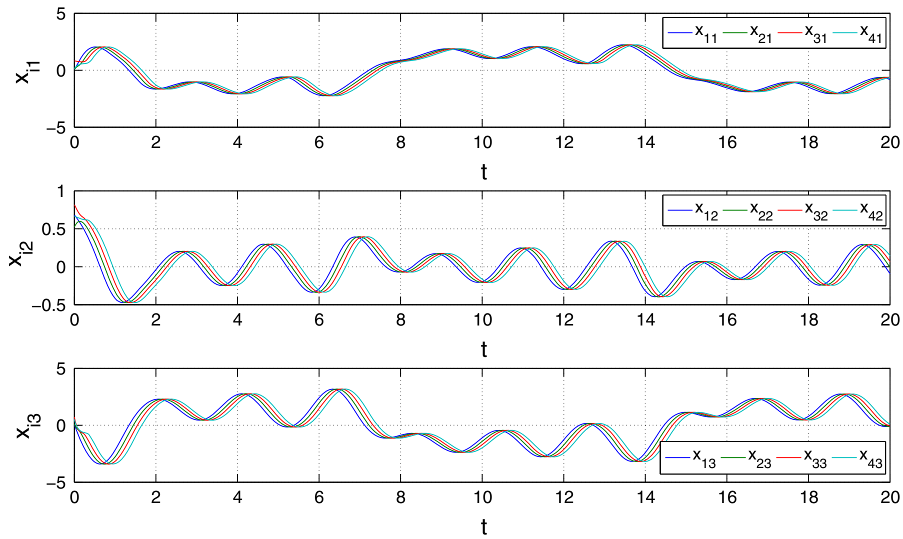

Fig. 3 The trajectories of all state variables x_{ij} of dynamical network (1) under control (6), where $i = 1, 2, 3, 4$, $j = 1, 2, 3$, the coupling strength $c = 0.1$, control strength $d = 11$, and time delay $\tau = 0.1$

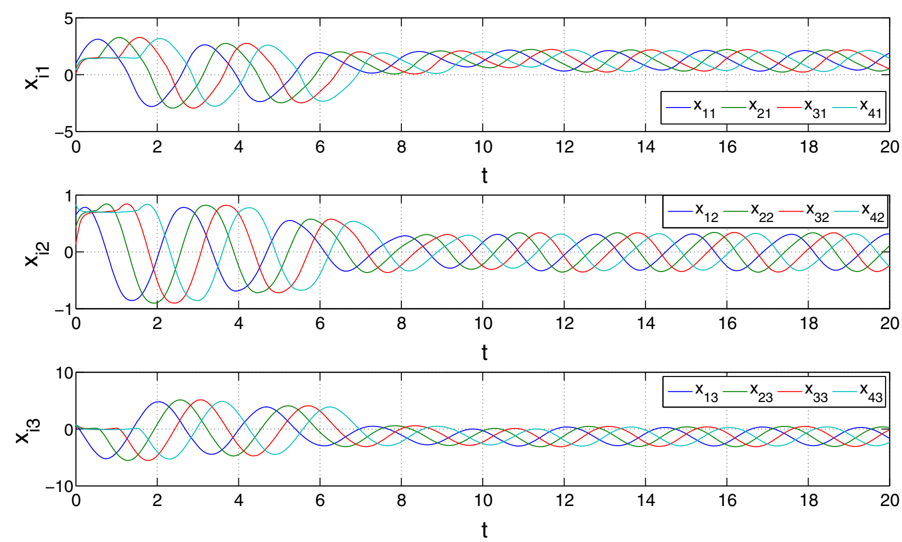

Fig. 4 The trajectories of all state variables x_{ij} of dynamical network (1) under control (6), where $i = 1, 2, 3, 4$, $j = 1, 2, 3$, the coupling strength $c = 0.1$, control strength $d = 11$, and time delay $\tau = 0.5$

$$
\begin{aligned}
u_4(t) = &-cx_1(t) - 2cx_2(t-2\tau) + cx_2(t-3\tau) \\
&+ cx_3(t-3\tau) + cx_4(t-3\tau) \\
&+ d_1(t-2\tau)e_1(t-2\tau) \\
&+ d_2(t-\tau)e_2(t-\tau) + d_3(t)e_3(t).
\end{aligned}
$$

Without the loss of generality, the initial condition is set as constant variables $\varphi_i \in [0,1]^3$, $i = 1, 2, 3, 4$ and $\varphi_5 = 0$, $\varphi_6 = 0$, $\varphi_7 = 0$ for control strength functions $d_1(t)$, $d_2(t)$, $d_3(t)$, respectively. Figure 5 presents a realization under adaptive linear feedback control (21) with coupling strength $c = 1$ and time delay $\tau = 3$. As shown in Fig. 5, a coincidence of shifted-in-time states in the network appears, as all control strengths converge to equilibrium states. So, under adaptive linear feedback control (21) the SLS of dynamical network (1) with star-shaped topology is achieved.

If we adopt linear feedback control (6), according to (31), from Theorem 1 the lower bound of the control strength d to realize the SLS can be obtained by solving the following Linear Matrix Inequality:

$$
cB \otimes I_3 + (-\omega + 10 - d)I_9 \leq 0, \tag{38}
$$

where $\omega = 0.6218$ and

$$
B = \begin{pmatrix} -3 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{39}
$$

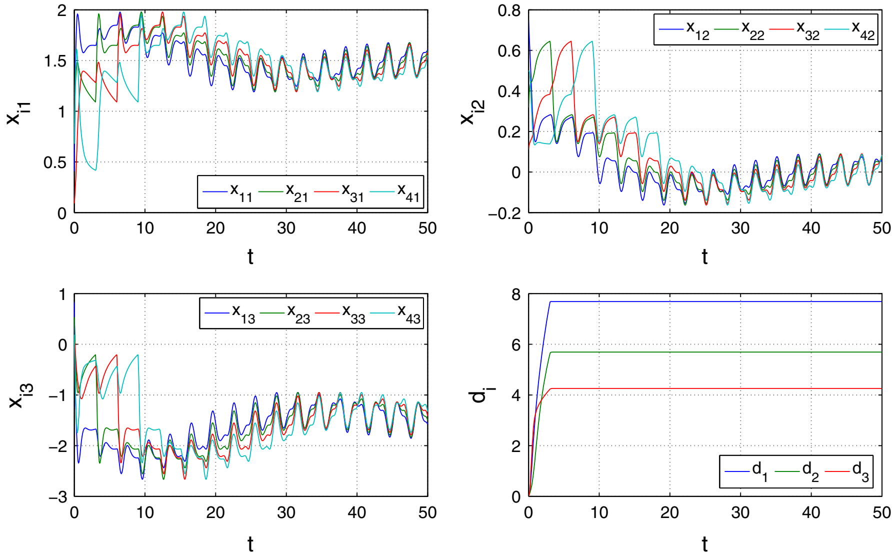

Fig. 5 The trajectories of all state variables x_{ij} of dynamical network (1) under control (21), where $i = 1, 2, 3, 4$, $j = 1, 2, 3$, the coupling strength $c = 1$, and time delay $\tau = 3$

By the Matlab LMI and control toolboxes, we get the lower bound of the control strength $d_l = 8.38$. From Fig. 5, we can see that the equilibrium points of all control strengths is smaller than d_l, which illuminates the adaptive linear feedback control can decrease the control strength.

6 Conclusions

Though many kinds of synchronization pattern on complex dynamical networks have been proposed and studies in the past decades, more realistic and reasonable synchronization patterns in real systems are still worth to be investigated in the future. From the viewpoint of real application, in this paper, we introduce and address a new synchronization pattern, i.e., the SLS, to realize coincidence of shifted-in-time states of a multi-coupled dynamical system. This kind of synchronization can be considered as a generalized pattern of traditional lag synchronization (LS) of two coupled dynamical networks.

As we know, linear control is superior to nonlinear control as the linear control is easier to be performed in real systems. In this paper, by using the linear feedback control depended on topological structure of the considered network, the SLS of the network has been realized. In order to reduce the control strength, we have designed an adaptive linear feedback control. By theoretical analysis, we have obtained some sufficient conditions for the global stability of SLS under the linear feedback control and adaptive linear feedback control, respectively. Finally, since the consensus and synchronization in dynamical systems have the similar principle, the proposed control method is generalized and applied to the control problem of successive lag consensus in a multi-agent system.

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