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Noether symmetries for non-conservative Lagrange systems with time delay based on fractional model

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Abstract The fractional Noether symmetries and conserved quantities for non-conservative Lagrange systems with time delay are proposed and studied. Firstly, the fractional Hamilton variational principles for non-conservative Lagrange systems with time delay are established, and the fractional differential equations of motion with time delay are obtained. Secondly, based upon the invariance of the fractional Hamilton action with time delay under the group of infinitesimal transformations which depends on the generalized velocities, the generalized coordinates and the time, the fractional Noether symmetric transformations, the fractional Noether quasi-symmetric transformations and the fractional generalized Noether quasi-symmetric transformations with time delay are defined, and the criteria of the fractional symmetries are obtained. Finally, the relationship between the fractional symmetries and the fractional conserved quantities with time delay are studied, and the fractional Noether theories are established. At the end of the paper, two examples are given to illustrate the application of the results.

Keywords System with time delay · Fractional model · Fractional Hamilton action · Noether symmetry · Conserved quantity

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1 Introduction

Fractional calculus has played a significant role in many fields during the last several decades, such as, engineering, science, applied mathematics, astrophysics, etc [1–10]. The research of fractional variational problems can be traced back to Riewes's work [11,12]; he utilized the fractional calculus to develop a formalism which can be used for both conservative and non-conservative systems. Agrawal [13–15] continued the study of the fractional variational problems, for general fractional variational problems involving Riemann–Liouville, Caputo and Riesz fractional derivatives. The symmetric fractional derivative was introduced by Kilmek [16], and the Euler–Lagrange equations for models depending on sequential derivatives were obtained by using the minimal action principle. The fractional variational problems of the mechanical system within Riemann–Liouville and Caputo fractional derivatives were discussed by Mulish, Herzallah and Baleanu [17–20]. A new fractional variational problem was proposed by

El-Nabulsi [21,22], which was the fractional action-like variational problem. Frederico and Torres [23–26] studied the Noether's theorem for variational and optimal control problems based on the fractional models, and a new concept of fractional-conserved quantity, which was not constant in time, was given. Atanackovič [27] studied the fractional Noether theorem within the Riemann–Liouvill fractional derivatives based on the concept of classical conserved quantity. Later, Zhang [28–31] studied the differential equations of motion based on fractional models and the Noether symmetries and conserved quantities for variational problems based on the El-Nabulsi models.

The study of variational problems with time delay has a long history. For the variational problems with delay, argument was first introduced and discussed by El'sgol'c [32] in 1964. While the study of the fractional variational problems with time delay has only begun in recent years, Baleanu, Maaraba and Jarad [33–36] studied the fractional variational principles and optimal control problems with time delay within Riemann–Liouville and Caputo fractional derivatives and extended to the higher-order fractional variational and optimal control problems with time delay within Caputo fractional derivatives.

However, the study of the symmetries and conserved quantities with time delay has only just begun. In 2012, Fredrico and Torres [37] first discussed the Noether's theorem for variational and optimal control problems with time delay, while, in 2013, Zhang and Jin [38] studied the symmetries of dynamics for non-conservative system with time delay. The Noether symmetries for non-conservative system with time delay based on fractional models have not been investigated yet in the literature.

The main aim of the paper is to study the Noether symmetries and conserved quantities for the non-conservative system with time delay based on the fractional model. The structure of this paper is as follows: In Sect. 2, the definitions and properties of Riemann–Liouville fractional derivatives are given. The fractional Lagrange equations with time delay are presented in Sect. 3. In Sect. 4, the fractional Hamilton action with time delay of dynamics systems is discussed. In Sect. 5, the definitions and criteria of the fractional Noether symmetric transformations, the fractional Noether quasi-symmetric transformations and the fractional generalized Noether quasi-symmetric transformations with time delay are obtained. In Sect. 6, the inner relationship between the fractional Noether symmetries and the fractional-conserved quantities with time delay is studied. In Sect. 7, two examples are given to illustrate the application of the results.

2 Definitions and properties of Riemann–Liouville fractional derivative

In this section, we briefly review some basic definitions and properties of fractional derivatives used in the following sections. Detailed discussion and proof can be found in Refs. [6–8].

The left Riemann–Liouville fractional derivative is defined as

$$ {}_{t_1}D_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^n \int_{t_1}^{t} (t-\tau)^{n-\alpha-1} f(\tau)\mathrm{d}\tau \tag{1} $$

and the right Riemann–Liouville fractional derivative is defined as

$$ {}_tD_{t_2}^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \left(-\frac{\mathrm{d}}{\mathrm{d}t}\right)^n \int_{t}^{t_2} (\tau-t)^{n-\alpha-1} f(\tau)\mathrm{d}\tau \tag{2} $$

where $\Gamma(*)$ denotes the Euler Gamma function and α is the order of the derivative such that $n-1 \leq \alpha < n$. If α is an integer, the derivatives are defined in the usual sense, i.e.,

$$ \begin{aligned} {}_{t_1}D_t^\alpha f(t) &= \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^\alpha f(t), \\ {}_tD_{t_2}^\alpha f(t) &= \left(-\frac{\mathrm{d}}{\mathrm{d}t}\right)^\alpha f(t) \end{aligned} \tag{3} $$

If $f \in {}_{t_1}I_t^\alpha(L_p)$ and $g \in {}_tI_{t_2}^\alpha(L_p)$, then the formula of fractional integration by part is as follows: [34]

$$ \int_{t_1}^{r} g(t)\,{}_{t_1}D_t^\alpha f(t)\mathrm{d}t = \int_{t_1}^{r} f(t)\,{}_tD_r^\alpha g(t)\mathrm{d}t \tag{4} $$

and

$$ \int_{r}^{t_2} g(t)\,{}_{t_1}D_t^\alpha f(t)\mathrm{d}t = \int_{r}^{t_2} f(t)\,{}_tD_{t_2}^\alpha g(t)\mathrm{d}t $$

$$ -\frac{1}{\Gamma(\alpha)} \int_{t_1}^{r} ({}_{t_1}D_t^\alpha f(t)) $$

$$\times\left[\int_r^{t_2}({}_tD_{t_2}^{\alpha}g(z))(z-t)^{\alpha-1}\mathrm{d}z\right]\mathrm{d}t$$
$$=\int_r^{t_2} f(t)_tD_{t_2}^{\alpha}g(t)\mathrm{d}t-\frac{1}{\Gamma(\alpha)}\int_{t_1}^{r} f(t)_tD_r^{\alpha}$$
$$\times\left[\int_r^{t_2}({}_tD_{t_2}^{\alpha}g(z))(z-t)^{\alpha-1}\mathrm{d}z\right]\mathrm{d}t \quad (5)$$

where $r\in(t_1,t_2)$, ${}_{t_1}I_t^{\alpha}(L_p)$ and ${}_tI_{t_2}^{\alpha}(L_p)$ are the left and the right Riemann–Liouville fractional integrals, respectively.

The commutation relations of δ and ${}_{t_1}D_t^{\alpha}$ satisfy [27]

$$\delta_{t_1}D_t^{\alpha}f={}_{t_1}D_t^{\alpha}\delta f \quad (6)$$

3 Fractional equations of motion with time delay

Assume that the configuration of a mechanical system is determined by n generalized coordinates q_s $(s=1,\ldots,n)$. The Hamilton principle of a non-conservative system is

$$\int_{t_1}^{t_2}\left(\delta L+Q_s^{''}\delta q_s\right)\mathrm{d}t=0 \quad (7)$$

where the Lagrangian L is a C^2-function. And consider that the time delay exists in the system and the Lagrangian is

$$L=L\left(t,q_s(t),{}_{t_1}D_t^{\alpha}q_s(t),\dot{q}_s(t),q_s(t-\tau),\dot{q}_s(t-\tau)\right)$$
$$=L(t,q_s,{}_{t_1}D_t^{\alpha}q_s,\dot{q}_s,q_{s\tau},\dot{q}_{s\tau}) \quad (8)$$

and the generalized non-potential forces are

$$Q_s^{''}=Q_s^{''}(t,q_k,{}_{t_1}D_t^{\alpha}q_k,\dot{q}_k,q_{k\tau},\dot{q}_{k\tau}) \quad (9)$$

And subject the specified initial functions

$$q_s(t)=\Omega_s(t),\quad t_1-\tau\le t\le t_1, \quad (10)$$

and the terminal conditions

$$q_s(t)=q_s(t_2),\quad t=t_2,\quad (s=1,2,\ldots,n). \quad (11)$$

where $\Omega_s(t)$ is a given piecewise smooth function in $t_1-\tau\le t\le t_1$, τ is a given positive real number such that $\tau<t_2-t_1$ and the derivative order $0\le\alpha<1$, $q_s(t_2)$ are certain values. The principle (7) can be expressed as

$$\int_{t_1}^{t_2}\left[\frac{\partial L}{\partial q_s}(t)\delta q_s+\frac{\partial L}{\partial \dot{q}_s}(t)\delta\dot{q}_s+\frac{\partial L}{\partial_{t_1}D_t^{\alpha}q_s}(t)\delta_{t_1}D_t^{\alpha}q_s\right.$$
$$\left.+\frac{\partial L}{\partial q_{s\tau}}(t)\delta q_{s\tau}+\frac{\partial L}{\partial \dot{q}_{s\tau}}(t)\delta\dot{q}_{s\tau}+Q_s^{''}(t)\delta q_s\right]\mathrm{d}t$$
$$=0 \quad (12)$$

Making a linear change of variable $t=\theta+\tau$ and considering the initial functions (10), we have

$$\int_{t_1}^{t_2}\left[\frac{\partial L}{\partial q_{s\tau}}(t)\delta q_{s\tau}+\frac{\partial L}{\partial \dot{q}_{s\tau}}(t)\delta\dot{q}_{s\tau}\right]\mathrm{d}t$$
$$=\int_{t_1}^{t_2-\tau}\left[\frac{\partial L}{\partial q_{s\tau}}(\theta+\tau)\delta q_s+\frac{\partial L}{\partial \dot{q}_{s\tau}}(\theta+\tau)\delta\dot{q}_s\right]\mathrm{d}\theta \quad (13)$$

Substituting formula (13) into formula (12), we have

$$\int_{t_1}^{t_2-\tau}\left\{\left[\frac{\partial L}{\partial q_s}(t)+\frac{\partial L}{\partial q_{s\tau}}(t+\tau)\right]\delta q_s\right.$$
$$+\frac{\partial L}{\partial_{t_1}D_t^{\alpha}q_s}(t)\delta_{t_1}D_t^{\alpha}q_s+\left(\frac{\partial L}{\partial \dot{q}_s}(t)+\frac{\partial L}{\partial \dot{q}_{s\tau}}(t+\tau)\right)\delta\dot{q}_s$$
$$\left.+Q_s^{''}(t)\delta q_s\right\}\mathrm{d}t+\int_{t_2-\tau}^{t_2}\left[\frac{\partial L}{\partial q_s}(t)\delta q_s+\frac{\partial L}{\partial \dot{q}_s}(t)\delta\dot{q}_s\right.$$
$$\left.+\frac{\partial L}{\partial_{t_1}D_t^{\alpha}q_s}(t)\delta_{t_1}D_t^{\alpha}q_s+Q_s^{''}(t)\delta q_s\right]\mathrm{d}t=0 \quad (14)$$

Considering the formulae (4), (5) and (6), we have

$$\int_{t_1}^{t_2-\tau}\frac{\partial L}{\partial_{t_1}D_t^{\alpha}q_s}(t)\delta_{t_1}D_t^{\alpha}q_s\mathrm{d}t$$
$$=\int_{t_1}^{t_2-\tau}\left({}_tD_{t_2-\tau}^{\alpha}\frac{\partial L}{\partial_{t_1}D_t^{\alpha}q_s}\right)(t)\delta q_s\mathrm{d}t \quad (15)$$

and

$$\int_{t_2-\tau}^{t_2}\frac{\partial L}{\partial_{t_1}D_t^{\alpha}q_s}(t)\delta_{t_1}D_t^{\alpha}q_s\mathrm{d}t$$
$$=\int_{t_2-\tau}^{t_2}\left({}_tD_{t_2}^{\alpha}\frac{\partial L}{\partial_{t_1}D_t^{\alpha}q_s}\right)(t)\delta q_s\mathrm{d}t$$

$$-\frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2-\tau}\delta q_s(t)\,{}_tD_{t_2-\tau}^{\alpha}\int_{t_2-\tau}^{t_2}\left({}_tD_{t_2}^{\alpha}\frac{\partial L}{\partial_{t_1}D_t^{\alpha}q_s}(z)\right)\times(z-t)^{\alpha-1}\mathrm{d}z\mathrm{d}t \tag{16}$$

By utilizing (15) and (16), integrating by parts, and taking use of initial functions (10) and terminal conditions (11), we have

$$\begin{aligned}&\int_{t_1}^{t_2-\tau}\left\{\left[\frac{\partial L}{\partial q_s}(t)+\frac{\partial L}{\partial q_{s\tau}}(t+\tau)\right]\delta q_s\right.\\&\quad+{}_tD_{t_2-\tau}^{\alpha}\frac{\partial L}{\partial_{t_1}D_t^{\alpha}q_s}(t)\delta q_s+Q_s^{''}(t)\delta q_s\\&\quad-\frac{1}{\Gamma(\alpha)}\delta q_s(t)\,{}_tD_{t_2-\tau}^{\alpha}\\&\quad\left.\times\left[\int_{t_2-\tau}^{t_2}\left({}_tD_{t_2-\tau}^{\alpha}\frac{\partial L}{\partial_{t_1}D_t^{\alpha}q_s}(z)\right)(z-t)^{\alpha-1}\mathrm{d}z\right]\right\}\mathrm{d}t\\&=-\left\{\delta q_s\int_t^{t_2-\tau}\left[\frac{\partial L}{\partial q_s}(\theta)+\frac{\partial L}{\partial q_{s\tau}}(\theta+\tau)\right.\right.\\&\quad+{}_\theta D_{t_2-\tau}^{\alpha}\frac{\partial L}{\partial_{t_1}D_\theta^{\alpha}q_s}(\theta)+Q_s^{''}(\theta)\\&\quad-\frac{{}_\theta D_{t_2-\tau}^{\alpha}}{\Gamma(\alpha)}\int_{t_2-\tau}^{t_2}\left({}_\theta D_{t_2}^{\alpha}\frac{\partial L}{\partial_{t_1}D_\theta^{\alpha}q_s}(z)\right)\\&\quad\left.\left.\times(z-\theta)^{\alpha-1}\mathrm{d}z\right]\mathrm{d}\theta\right\}_{t_1}^{t_2-\tau}\\&\quad+\int_{t_1}^{t_2-\tau}\delta\dot{q}_s\left\{\int_t^{t_2-\tau}\left[\frac{\partial L}{\partial q_s}(\theta)+\frac{\partial L}{\partial q_{s\tau}}(\theta+\tau)\right.\right.\\&\quad+{}_\theta D_{t_2-\tau}^{\alpha}\frac{\partial L}{\partial_{t_1}D_\theta^{\alpha}q_s}(\theta)+Q_s^{''}(\theta)\\&\quad-\frac{{}_\theta D_{t_2-\tau}^{\alpha}}{\Gamma(\alpha)}\int_{t_2-\tau}^{t_2}\left({}_\theta D_{t_2-\tau}^{\alpha}\frac{\partial L}{\partial_{t_1}D_\theta^{\alpha}q_s}(z)\right)\\&\quad\left.\left.\times(z-\theta)^{\alpha-1}\mathrm{d}z\right]\mathrm{d}\theta\right\}\mathrm{d}t\\&=\int_{t_1}^{t_2-\tau}\delta\dot{q}_s\left\{\int_t^{t_2-\tau}\left[\frac{\partial L}{\partial q_s}(\theta)+\frac{\partial L}{\partial q_{s\tau}}(\theta+\tau)\right.\right.\\&\quad+{}_\theta D_{t_2-\tau}^{\alpha}\frac{\partial L}{\partial_{t_1}D_\theta^{\alpha}q_s}(\theta)+Q_s^{''}(\theta)\\&\quad-\frac{{}_\theta D_{t_2-\tau}^{\alpha}}{\Gamma(\alpha)}\int_{t_2-\tau}^{t_2}\left({}_\theta D_{t_2}^{\alpha}\frac{\partial L}{\partial_{t_1}D_\theta^{\alpha}q_s}(z)\right)\\&\quad\left.\left.\times(z-\theta)^{\alpha-1}\mathrm{d}z\right]\mathrm{d}\theta\right\}\mathrm{d}t\end{aligned} \tag{17}$$

and

$$\begin{aligned}&\int_{t_2-\tau}^{t_2}\left[\frac{\partial L}{\partial q_s}(t)\delta q_s+{}_tD_{t_2}^{\alpha}\frac{\partial L}{\partial_{t_1}D_t^{\alpha}q_s}(t)\delta q_s+Q_s^{''}(t)\delta q_s\right]\mathrm{d}t\\&=\left\{\delta q_s\int_{t_2-\tau}^{t}\left[\frac{\partial L}{\partial q_s}(\theta)+{}_\theta D_{t_2}^{\alpha}\frac{\partial L}{\partial_{t_1}D_\theta^{\alpha}q_s}(\theta)\right.\right.\\&\quad\left.\left.+Q_s^{''}(\theta)\right]\mathrm{d}\theta\right\}_{t_2-\tau}^{t_2}\\&\quad-\int_{t_2-\tau}^{t2}\delta\dot{q}_s\left\{\int_{t_2-\tau}^{t}\left[\frac{\partial L}{\partial q_s}(\theta)+{}_\theta D_{t_2}^{\alpha}\frac{\partial L}{\partial_{t_1}D_\theta^{\alpha}q_s}(\theta)\right.\right.\\&\quad\left.\left.+Q_s^{''}(\theta)\right]\mathrm{d}\theta\right\}\mathrm{d}t\\&=-\int_{t_2-\tau}^{t_2}\delta\dot{q}_s\left\{\int_{t_2-\tau}^{t}\left[\frac{\partial L}{\partial q_s}(\theta)+{}_\theta D_{t_2}^{\alpha}\frac{\partial L}{\partial_{t_1}D_\theta^{\alpha}q_s}(\theta)\right.\right.\\&\quad\left.\left.+Q_s^{''}(\theta)\right]\mathrm{d}\theta\right\}\mathrm{d}t\end{aligned} \tag{18}$$

Substituting formulae (17) and (18) into formula (14), we obtain

$$\begin{aligned}&\int_{t_1}^{t_2-\tau}\delta\dot{q}_s\left\{\int_t^{t_2-\tau}\left[\frac{\partial L}{\partial q_s}(\theta)+\frac{\partial L}{\partial q_{s\tau}}(\theta+\tau)\right.\right.\\&\quad+{}_\theta D_{t_2-\tau}^{\alpha}\frac{\partial L}{\partial_{t_1}D_\theta^{\alpha}q_s}(\theta)+Q_s^{''}(\theta)\\&\quad-\frac{{}_\theta D_{t_2-\tau}^{\alpha}}{\Gamma(\alpha)}\int_{t_2-\tau}^{t_2}\left({}_\theta D_{t_2}^{\alpha}\frac{\partial L}{\partial_{t_1}D_\theta^{\alpha}q_s}(z)\right)\\&\quad\left.\times(z-\theta)^{\alpha-1}\mathrm{d}z\right]\mathrm{d}\theta\\&\quad\left.+\frac{\partial L}{\partial\dot{q}_s}(t)+\frac{\partial L}{\partial\dot{q}_{s\tau}}(t+\tau)\right\}\mathrm{d}t\end{aligned}$$

$$-\int_{t_2-\tau}^{t_2}\delta\dot{q}_s\left\{\int_{t_2-\tau}^{t}\left[\frac{\partial L}{\partial q_s}(\theta)+_{\theta}D_{t_2}^{\alpha}\frac{\partial L}{\partial_{t_1}D_{\theta}^{\alpha}q_s}(\theta)+Q_s''(\theta)\right]\mathrm{d}\theta+\frac{\partial L}{\partial\dot{q}_s}(t)\right\}\mathrm{d}t=0 \quad (19)$$

Due to the arbitrariness of integral interval, and considering the independence of $\delta\dot{q}_s$, we have

$$\begin{aligned}&\frac{\partial L}{\partial\dot{q}_s}(t)+\frac{\partial L}{\partial\dot{q}_{s\tau}}(t+\tau)+\int_{t}^{t_2-\tau}\Big[\frac{\partial L}{\partial q_s}(\theta)\\&+\frac{\partial L}{\partial q_{s\tau}}(\theta+\tau)+_{\theta}D_{t_2-\tau}^{\alpha}\frac{\partial L}{\partial_{t_1}D_{\theta}^{\alpha}q_s}(\theta)\\&+Q_s''(\theta)-\frac{_{\theta}D_{t_2-\tau}^{\alpha}}{\Gamma(\alpha)}\int_{t_2-\tau}^{t_2}\left(_{\theta}D_{t_2}^{\alpha}\frac{\partial L}{\partial_{t_1}D_{\theta}^{\alpha}q_s}(z)\right)\\&(z-\theta)^{\alpha-1}\mathrm{d}z\Big]\mathrm{d}\theta=0,\quad t\in[t_1,t_2-\tau]\\&-\int_{t_2-\tau}^{t}\left[\frac{\partial L}{\partial q_s}(\theta)+_{\theta}D_{t_2}^{\alpha}\frac{\partial L}{\partial_{t_1}D_{\theta}^{\alpha}q_s}(\theta)\right.\\&\left.+Q_s''(\theta)\right]\mathrm{d}\theta+\frac{\partial L}{\partial\dot{q}_s}(t)=0,\\&t\in(t_2-\tau,t_2]\end{aligned} \quad (20)$$

Taking derivative of equation (20) with respect to t, we have

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial\dot{q}_s}(t)+\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial\dot{q}_{s\tau}}(t+\tau)-_{t}D_{t_2-\tau}^{\alpha}\frac{\partial L}{\partial_{t_1}D_{t}^{\alpha}q_s}(t)\\&-\frac{\partial L}{\partial q_s}(t)-\frac{\partial L}{\partial q_{s\tau}}(t+\tau)\\&+\frac{_{t}D_{t_2-\tau}^{\alpha}}{\Gamma(\alpha)}\int_{t_2-\tau}^{t_2}\left(_{t}D_{t_2}^{\alpha}\frac{\partial L}{\partial_{t_1}D_{t}^{\alpha}q_s}(z)\right)(z-t)^{\alpha-1}\mathrm{d}z\\&=Q_s''(t),\quad t\in[t_1,t_2-\tau]\\&\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial\dot{q}_s}(t)-_{t}D_{t_2}^{\alpha}\frac{\partial L}{\partial_{t_1}D_{t}^{\alpha}q_s}(t)\\&-\frac{\partial L}{\partial q_s}(t)=Q_s''(t),\quad t\in(t_2-\tau,t_2]\end{aligned} \quad (21)$$

Equation (21) may be called the fractional Lagrange equations for non-conservative systems with time delay. If $Q_s''=0$, Eq. (21) can be expressed as

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial\dot{q}_s}(t)+\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial\dot{q}_{s\tau}}(t+\tau)-_{t}D_{t_2-\tau}^{\alpha}\frac{\partial L}{\partial_{t_1}D_{t}^{\alpha}q_s}(t)\\&-\frac{\partial L}{\partial q_s}(t)-\frac{\partial L}{\partial q_{s\tau}}(t+\tau)\\&+\frac{_{t}D_{t_2-\tau}^{\alpha}}{\Gamma(\alpha)}\int_{t_2-\tau}^{t_2}\left(_{t}D_{t_2}^{\alpha}\frac{\partial L}{\partial_{t_1}D_{t}^{\alpha}q_s}(z)\right)(z-t)^{\alpha-1}\mathrm{d}z\\&=0,\quad t\in[t_1,t_2-\tau]\\&\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial\dot{q}_s}(t)-_{t}D_{t_2}^{\alpha}\frac{\partial L}{\partial_{t_1}D_{t}^{\alpha}q_s}(t)\\&-\frac{\partial L}{\partial q_s}(t)=0,\quad t\in(t_2-\tau,t_2]\end{aligned} \quad (22)$$

Equation (22) may be called the fractional Euler–Lagrange equations with time delay.

4 Fractional Hamilton action with time delay

The fractional Hamilton action with time delay is

$$S(\gamma)=\int_{t_1}^{t_2}L(t,q_s(t),_{t_1}D_t^{\alpha}q_s(t),\dot{q}_s(t),q_s(t-\tau),\dot{q}_s(t-\tau))\mathrm{d}t \quad (23)$$

where γ is a curve. Introduce the infinitesimal transformations of r-parameter finite transformations group

$$\bar{t}=t+\triangle t,\quad \bar{q}_s(\bar{t})=q_s(t)+\triangle q_s,\quad (s=1,2,\ldots,n) \quad (24)$$

or their expansion formula is

$$\begin{aligned}&\bar{t}=t+\varepsilon_{\sigma}\xi_0^{\sigma}(t,q_k,_{t_1}D_t^{\alpha}q_k(t),\dot{q}_k),\ \bar{q}_s(\bar{t})\\&=q_s(t)+\varepsilon_{\sigma}\xi_s^{\sigma}(t,q_k,_{t_1}D_t^{\alpha}q_k(t),\dot{q}_k),\\&(s,k=1,2,\ldots,n)\end{aligned} \quad (25)$$

where $\varepsilon_{\sigma}(\sigma=1,2,\ldots,r)$ are infinitesimal parameters and $\xi_0^{\sigma},\xi_s^{\sigma}$ are the infinitesimal generators or generating functions of the infinitesimal transformations. Under the infinitesimal transformations (21), the curve γ will be transformed to a neighbor curve $\bar{\gamma}$, and the fractional Hamilton action (20) with time delay can be expressed as

$$S(\bar{\gamma})=\int_{\bar{t}_1}^{\bar{t}_2}L\left(\bar{t},\bar{q}_s(\bar{t}),_{\bar{t}_1}D_{\bar{t}}^{\alpha}\bar{q}_s(\bar{t}),\dot{\bar{q}}_s(\bar{t}),\bar{q}_s(\bar{t}-\tau),\dot{\bar{q}}_s(\bar{t}-\tau)\right)\mathrm{d}\bar{t} \quad (26)$$

The main linear part relative to ε in the difference $S(\bar{\gamma}) - S(\gamma)$ is

$$\Delta S = \int_{t_1}^{t_2} \left[\frac{\partial L}{\partial t}(t)\Delta t + \frac{\partial L}{\partial q_s}(t)\Delta q_s + \frac{\partial L}{\partial_{t_1} D_t^\alpha q_s}(t)\Delta_{t_1} D_t^\alpha q_s(t) + \frac{\partial L}{\partial q_{s\tau}}(t)\Delta q_{s\tau} + \frac{\partial L}{\partial \dot{q}_s}(t)\Delta \dot{q}_s + \frac{\partial L}{\partial \dot{q}_{s\tau}}(t)\Delta \dot{q}_{s\tau} + L\frac{\mathrm{d}}{\mathrm{d}t}(\Delta t)\right]\mathrm{d}t \tag{27}$$

By the linear change of variable $t = \theta + \tau$, considering the initial functions (10), we have

$$\int_{t_1}^{t_2} \left[\frac{\partial L}{\partial q_{s\tau}}(t)\Delta q_{s\tau} + \frac{\partial L}{\partial \dot{q}_{s\tau}}(t)\Delta \dot{q}_{s\tau}\right]\mathrm{d}t = \int_{t_1}^{t_2-\tau} \left[\frac{\partial L}{\partial q_{s\tau}}(\theta+\tau)\Delta q_s + \frac{\partial L}{\partial \dot{q}_{s\tau}}(\theta+\tau)\Delta \dot{q}_s\right]\mathrm{d}\theta \tag{28}$$

Substituting formula (28) into formula (27), we have

$$\Delta S = \int_{t_1}^{t_2-\tau} \left\{ \frac{\partial L}{\partial t}(t)\Delta t + \left[\frac{\partial L}{\partial q_s}(t) + \frac{\partial L}{\partial q_{s\tau}}(t+\tau)\right]\Delta q_s + \frac{\partial L}{\partial_{t_1} D_t^\alpha q_s}(t)\Delta_{t_1} D_t^\alpha q_s + \left[\frac{\partial L}{\partial \dot{q}_s}(t) + \frac{\partial L}{\partial \dot{q}_{s\tau}}(t+\tau)\right]\Delta \dot{q}_s + L\frac{\mathrm{d}}{\mathrm{d}t}(\Delta t)\right\}\mathrm{d}t + \int_{t_2-\tau}^{t_2} \left[\frac{\partial L}{\partial t}(t)\Delta t + \frac{\partial L}{\partial q_s}(t)\Delta q_s + \frac{\partial L}{\partial_{t_1} D_t^\alpha q_s}(t)\Delta_{t_1} D_t^\alpha q_s + \frac{\partial L}{\partial \dot{q}_s}(t)\Delta \dot{q}_s + L\frac{\mathrm{d}}{\mathrm{d}t}(\Delta t)\right]\mathrm{d}t \tag{29}$$

Taking notice that

$$\Delta_{t_1} D_t^\alpha q_s =_{t_1} D_t^\alpha \delta q_s + \frac{\mathrm{d}}{\mathrm{d}t}\left({}_{t_1} D_t^\alpha q_s\right)\Delta t, \quad \delta q_s = \Delta q_s - \dot{q}_s \Delta t \tag{30}$$

and considering the formulae (4)–(6), we can express the formula (29) as

$$\begin{aligned}\Delta S = \int_{t_1}^{t_2-\tau} \varepsilon_\sigma \Bigg\{ \frac{\mathrm{d}}{\mathrm{d}t}\Bigg[L\xi_0^\sigma + \int_{t_1}^{t} \Bigg(\frac{\partial L}{\partial_{t_1} D_\theta^\alpha q_s}(\theta)_{t_1} D_\theta^\alpha \bar{\xi}_s^\sigma - {}_\theta D_{t_2-\tau}^\alpha \frac{\partial L}{\partial_{t_1} D_\theta^\alpha q_s}(\theta)\bar{\xi}_s^\sigma + \bar{\xi}_s^\sigma \frac{{}_\theta D_{t_2-\tau}^\alpha}{\Gamma(\alpha)} \\ \times \int_{t_2-\tau}^{t_2} \left({}_\theta D_{t_2}^\alpha \frac{\partial L}{\partial_{t_1} D_\theta^\alpha q_s}(z)\right)(z-\theta)^{\alpha-1}\mathrm{d}z\Bigg)\mathrm{d}\theta + \left(\frac{\partial L}{\partial \dot{q}_s}(t) + \frac{\partial L}{\partial \dot{q}_{s\tau}}(t+\tau)\right)\bar{\xi}_s^\sigma\Bigg] \\ + \bar{\xi}_s^\sigma \Bigg[\frac{\partial L}{\partial q_s}(t) + \frac{\partial L}{\partial q_{s\tau}}(t+\tau) +_t D_{t_2-\tau}^\alpha \frac{\partial L}{\partial_{t_1} D_t^\alpha q_s}(t) - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_{s\tau}}(t+\tau) - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_s}(t) \\ - \frac{{}_t D_{t_2-\tau}^\alpha}{\Gamma(\alpha)} \int_{t_2-\tau}^{t_2} \left({}_t D_{t_2}^\alpha \frac{\partial L}{\partial_{t_1} D_t^\alpha q_s}(z)\right)(z-t)^{\alpha-1}\mathrm{d}z\Bigg]\Bigg\}\mathrm{d}t \\ + \int_{t_2-\tau}^{t_2} \varepsilon_\sigma \Bigg[\frac{\mathrm{d}}{\mathrm{d}t}\Bigg(L\xi_0^\sigma + \int_{t_1}^{t}\Bigg(\frac{\partial L}{\partial_{t_1} D_\theta^\alpha q_s}(\theta)_{t_1} D_\theta^\alpha \bar{\xi}_s^\sigma - {}_\theta D_{t_2}^\alpha \frac{\partial L}{\partial_{t_1} D_\theta^\alpha q_s}(\theta)\bar{\xi}_s^\sigma\Bigg)\mathrm{d}\theta + \frac{\partial L}{\partial \dot{q}_s}(t)\bar{\xi}_s^\sigma\Bigg) \\ + \bar{\xi}_s^\sigma \left(\frac{\partial L}{\partial q_s}(t) +_t D_{t_2}^\alpha \frac{\partial L}{\partial_{t_1} D_t^\alpha q_s}(t) - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_s}(t)\right)\Bigg]\mathrm{d}t\end{aligned} \tag{31}$$

where

$$\bar{\xi}_s^\sigma = \xi_s^\sigma - \dot{q}_s \xi_0^\sigma \quad (s = 1, 2, \ldots, n) \tag{32}$$

Formulae (29) and (31) are the basic formulae for the fractional variation of Hamilton action with time delay.

5 Fractional Noether symmetries with time delay

In this section, we discuss the definitions and criteria of the fractional Noether symmetric transformations, the fractional Noether quasi-symmetric transformations and the fractional generalized Noether quasi-symmetric transformations with time delay.

Firstly, we give the definition and criteria of the fractional Noether symmetric transformations.

Definition 1 If the fractional Hamilton action (23) is an invariance of the group of infinitesimal transformations (24), the condition

$$\Delta S = 0 \tag{33}$$

is satisfied, then the infinitesimal transformations are called the fractional Noether symmetric transformations of the system with time delay.

From Definition 1 and formulae (29) and (31), we have the following criteria.

Criterion 1 For the group of infinitesimal transformations (24), the condition

$$\frac{\partial L}{\partial t}(t)\Delta t + \left[\frac{\partial L}{\partial q_s}(t) + \frac{\partial L}{\partial q_{s\tau}}(t+\tau)\right]\Delta q_s + \frac{\partial L}{\partial_{t_1} D_t^\alpha q_s}(t)\Delta_{t_1} D_t^\alpha q_s + \left[\frac{\partial L}{\partial \dot{q}_s}(t) + \frac{\partial L}{\partial \dot{q}_{s\tau}}(t+\tau)\right]\Delta \dot{q}_s + L\frac{\mathrm{d}}{\mathrm{d}t}(\Delta t) = 0 \tag{34}$$

is satisfied for $t_1 \leq t \leq t_2 - \tau$, the condition

$$\frac{\partial L}{\partial t}(t)\Delta t + \frac{\partial L}{\partial q_s}(t)\Delta q_s + \frac{\partial L}{\partial_{t_1} D_t^\alpha q_s}(t)\Delta_{t_1} D_t^\alpha q_s + \frac{\partial L}{\partial \dot{q}_s}(t)\Delta \dot{q}_s + L\frac{\mathrm{d}}{\mathrm{d}t}(\Delta t) = 0 \tag{35}$$

is satisfied for $t_2 - \tau < t \leq t_2$, then the infinitesimal transformations (24) are the fractional Noether symmetric transformations of the system with time delay.

The formulae (34) and (35) can be expressed as

$$\frac{\partial L}{\partial t}(t)\xi_0^\sigma + \left[\frac{\partial L}{\partial q_s}(t) + \frac{\partial L}{\partial q_{s\tau}}(t+\tau)\right]\xi_s^\sigma + \frac{\partial L}{\partial_{t_1} D_t^\alpha q_s}(t)\left({}_{t_1} D_t^\alpha \overline{\xi}_s^\sigma + \frac{\mathrm{d}}{\mathrm{d}t}\left({}_{t_1} D_t^\alpha q_s\right)\xi_0^\sigma\right) + \left[\frac{\partial L}{\partial \dot{q}_s}(t) + \frac{\partial L}{\partial \dot{q}_{s\tau}}(t+\tau)\right](\dot{\xi}_s^\sigma - \dot{q}_s\dot{\xi}_0^\sigma) + L\dot{\xi}_0^\sigma = 0 \tag{36}$$

for $t_1 \leq t \leq t_2 - \tau$, and

$$\frac{\partial L}{\partial t}(t)\xi_0^\sigma + \frac{\partial L}{\partial q_s}(t)\xi_s^\sigma + \frac{\partial L}{\partial_{t_1} D_t^\alpha q_s}(t) \times \left({}_{t_1} D_t^\alpha \overline{\xi}_s^\sigma + \frac{\mathrm{d}}{\mathrm{d}t}\left({}_{t_1} D_t^\alpha q_s\right)\xi_0^\sigma\right) + \frac{\partial L}{\partial \dot{q}_s}(t)(\dot{\xi}_s^\sigma - \dot{q}_s\dot{\xi}_0^\sigma) + L\dot{\xi}_0^\sigma = 0 \tag{37}$$

for $t_2 - \tau < t \leq t_2$. Where $\sigma = 1, 2, \ldots, r$. When $r = 1$, Eqs. (36) and (37) are the fractional Noether identities of the system with time delay.

Criterion 2 For the infinitesimal transformations of group (25), the r equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[L\xi_0^\sigma + \int_{t_1}^{t}\left(\frac{\partial L}{\partial_{t_1} D_\theta^\alpha q_s}(\theta)_{t_1} D_\theta^\alpha \bar{\xi}_s^\sigma - {}_\theta D_{t_2-\tau}^\alpha \frac{\partial L}{\partial_{t_1} D_\theta^\alpha q_s}(\theta)\bar{\xi}_s^\sigma + \bar{\xi}_s^\sigma \frac{{}_\theta D_{t_2-\tau}^\alpha}{\Gamma(\alpha)} \times \int_{t_2-\tau}^{t_2}\left({}_\theta D_{t_2}^\alpha \frac{\partial L}{\partial_{t_1} D_\theta^\alpha q_s}(z)\right) \times \left(z-\theta\right)^{\alpha-1}\mathrm{d}z\right)\mathrm{d}\theta + \left(\frac{\partial L}{\partial \dot{q}_s}(t) + \frac{\partial L}{\partial \dot{q}_{s\tau}}(t+\tau)\right)\bar{\xi}_s^\sigma\right] + \bar{\xi}_s^\sigma\left[\frac{\partial L}{\partial q_s}(t) + \frac{\partial L}{\partial q_{s\tau}}(t+\tau) + {}_t D_{t_2-\tau}^\alpha \frac{\partial L}{\partial_{t_1} D_t^\alpha q_s}(t) - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_{s\tau}}(t+\tau) - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_s}(t) - \frac{{}_t D_{t_2-\tau}^\alpha}{\Gamma(\alpha)} \times \int_{t_2-\tau}^{t_2}\left({}_t D_{t_2}^\alpha \frac{\partial L}{\partial_{t_1} D_t^\alpha q_s}(z)\right)(z-t)^{\alpha-1}\mathrm{d}z\right] = 0 \tag{38}$$

are satisfied for $t_1 \leq t \leq t_2 - \tau$ and the r equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[L\xi_0^\sigma + \int_{t_1}^{t}\left(\frac{\partial L}{\partial_{t_1} D_\theta^\alpha q_s}(\theta)_{t_1} D_\theta^\alpha \bar{\xi}_s^\sigma - {}_\theta D_{t_2}^\alpha \frac{\partial L}{\partial_{t_1} D_\theta^\alpha q_s}(\theta)\bar{\xi}_s^\sigma\right)\mathrm{d}\theta + \frac{\partial L}{\partial \dot{q}_s}(t)\bar{\xi}_s\right] + \bar{\xi}_s^\sigma\left[\frac{\partial L}{\partial q_s}(t) + {}_t D_{t_2}^\alpha \frac{\partial L}{\partial_{t_1} D_t^\alpha q_s}(t) - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_s}(t)\right] = 0, \tag{39}$$

are satisfied for $t_2-\tau < t \leq t_2$, where $\sigma = 1, 2, \ldots, r$. Then, the infinitesimal transformations (25) are the fractional Noether symmetric transformations of the system with time delay.

Secondly, we discuss the fractional Noether quasi-symmetric transformations of dynamics system with time delay within the left Riemann–Liouville fractional derivative.

Suppose that L_1 is another fractional Lagrangian with time delay, if the transformations (24) accurate to first-order infinitesimal satisfy the condition

$$\int_{t_1}^{t_2} L(t, q_s(t), {}_{t_1}D_t^\alpha q_s(t), \dot{q}_s(t), q_s(t-\tau), \dot{q}_s(t-\tau))\mathrm{d}t = \int_{\bar{t}_1}^{\bar{t}_2} L(\bar{t}, \bar{q}_s(\bar{t}), {}_{\bar{t}_1}D_{\bar{t}}^\alpha \bar{q}_s(\bar{t}), \dot{\bar{q}}_s(\bar{t}), \bar{q}_s(\bar{t}-\tau), \dot{\bar{q}}_s(\bar{t}-\tau))\mathrm{d}\bar{t} \quad (40)$$

then this invariance is called the quasi-invariance of the fractional Hamilton action (23) with time delay under the group of infinitesimal transformations (24). The Lagrangian L_1 and L determined by formula (40) satisfy the same differential equations. Then, the transformations are called the fractional Noether quasi-symmetric transformations. So, we have

Definition 2 If the fractional Hamilton action (23) is a quasi-invariance of the group of infinitesimal transformations (24), the condition

$$\Delta S = -\int_{t_1}^{t_2} \frac{\mathrm{d}}{\mathrm{d}t}(\Delta G)\mathrm{d}t \quad (41)$$

holds, where $G = G(t, q_s(t), {}_{t_1}D_t^\alpha q_s(t), \dot{q}_s(t), q_s(t-\tau), \dot{q}_s(t-\tau))$ is a gauge function, then the infinitesimal transformations (24) are the fractional Noether quasi-symmetric transformations of the system with time delay.

From Definition 2 and formulae (29) and (31), we have the following criteria.

Criterion 3 For the group of infinitesimal transformations (24), the condition

$$\frac{\partial L}{\partial t}(t)\Delta t + \left[\frac{\partial L}{\partial q_s}(t) + \frac{\partial L}{\partial q_{s\tau}}(t+\tau)\right]\Delta q_s + \frac{\partial L}{\partial {}_{t_1}D_t^\alpha q_s}(t)\Delta {}_{t_1}D_t^\alpha q_s + \left[\frac{\partial L}{\partial \dot{q}_s}(t) + \frac{\partial L}{\partial \dot{q}_{s\tau}}(t+\tau)\right]\Delta \dot{q}_s + L\frac{\mathrm{d}}{\mathrm{d}t}(\Delta t) = -\frac{\mathrm{d}}{\mathrm{d}t}(\Delta G) \quad (42)$$

is satisfied for $t_1 \le t \le t_2 - \tau$ and the condition

$$\frac{\partial L}{\partial t}(t)\Delta t + \frac{\partial L}{\partial q_s}(t)\Delta q_s + \frac{\partial L}{\partial {}_{t_1}D_t^\alpha q_s}(t)\Delta {}_{t_1}D_t^\alpha q_s + \frac{\partial L}{\partial \dot{q}_s}(t)\Delta \dot{q}_s + L\frac{\mathrm{d}}{\mathrm{d}t}(\Delta t) = -\frac{\mathrm{d}}{\mathrm{d}t}(\Delta G) \quad (43)$$

is satisfied for $t_2 - \tau < t \le t_2$. Then, the infinitesimal transformations (24) are the fractional Noether quasi-symmetric transformations of the system with time delay.

The formulae (42) and (43) can be expressed as

$$\frac{\partial L}{\partial t}(t)\xi_0^\sigma + \left[\frac{\partial L}{\partial q_s}(t) + \frac{\partial L}{\partial q_{s\tau}}(t+\tau)\right]\xi_s^\sigma + \frac{\partial L}{\partial {}_{t_1}D_t^\alpha q_s}(t)\left({}_{t_1}D_t^\alpha \bar{\xi}_s^\sigma + \frac{\mathrm{d}}{\mathrm{d}t}\left({}_{t_1}D_t^\alpha q_s\right)\xi_0^\sigma\right) + \left[\frac{\partial L}{\partial \dot{q}_s}(t) + \frac{\partial L}{\partial \dot{q}_{s\tau}}(t+\tau)\right](\dot{\xi}_s^\sigma - \dot{q}_s\dot{\xi}_0^\sigma) + L\dot{\xi}_0^\sigma = -\dot{G}^\sigma \quad (44)$$

for $t_1 \le t \le t_2 - \tau$ and

$$\frac{\partial L}{\partial t}(t)\xi_0^\sigma + \frac{\partial L}{\partial q_s}(t)\xi_s^\sigma + \frac{\partial L}{\partial {}_{t_1}D_t^\alpha q_s}(t)\left({}_{t_1}D_t^\alpha \bar{\xi}_s^\sigma + \frac{\mathrm{d}}{\mathrm{d}t}\left({}_{t_1}D_t^\alpha q_s\right)\xi_0^\sigma\right) + \frac{\partial L}{\partial \dot{q}_s}(t)(\dot{\xi}_s^\sigma - \dot{q}_s\dot{\xi}_0^\sigma) + L\dot{\xi}_0^\sigma = -\dot{G}^\sigma \quad (45)$$

for $t_2 - \tau < t \le t_2$, where $\sigma = 1, 2, \ldots, r$ and $\Delta G = \varepsilon_\sigma G^\sigma$. When $r = 1$, Eqs. (44) and (45) are the fractional Noether identities of the system with time delay.

Criterion 4 For the group of infinitesimal transformations (25), the r equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\Bigg[L\xi_0^\sigma + \int_{t_1}^{t}\Bigg(\frac{\partial L}{\partial {}_{t_1}D_\theta^\alpha q_s}(\theta)\,{}_{t_1}D_\theta^\alpha \bar{\xi}_s^\sigma - {}_\theta D_{t_2-\tau}^\alpha \frac{\partial L}{\partial {}_{t_1}D_\theta^\alpha q_s}(\theta)\bar{\xi}_s^\sigma + \bar{\xi}_s^\sigma \frac{{}_\theta D_{t_2-\tau}^\alpha}{\Gamma(\alpha)}\int_{t_2-\tau}^{t_2}\left({}_\theta D_{t_2}^\alpha \frac{\partial L}{\partial {}_{t_1}D_\theta^\alpha q_s}(z)\right)(z-\theta)^{\alpha-1}\mathrm{d}z\Bigg)\mathrm{d}\theta + \left(\frac{\partial L}{\partial \dot{q}_s}(t) + \frac{\partial L}{\partial \dot{q}_{s\tau}}(t+\tau)\right)\bar{\xi}_s^\sigma + G^\sigma\Bigg]$$

$$+\bar{\xi}_s^\sigma\left[\frac{\partial L}{\partial q_s}(t)+\frac{\partial L}{\partial q_{s\tau}}(t+\tau)+{}_tD_{t_2-\tau}^\alpha\frac{\partial L}{\partial_{t_1}D_t^\alpha q_s}(t)\right.$$
$$-\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_{s\tau}}(t+\tau)-\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_s}(t)$$
$$-\frac{{}_tD_{t_2-\tau}^\alpha}{\Gamma(\alpha)}\int_{t_2-\tau}^{t_2}\left({}_tD_{t_2}^\alpha\frac{\partial L}{\partial_{t_1}D_t^\alpha q_s}(z)\right)$$
$$\left.\times(z-t)^{\alpha-1}\mathrm{d}z\right]=0 \tag{46}$$

are satisfied for $t_1\le t\le t_2-\tau$ and the r equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[L\xi_0^\sigma+\int_{t_1}^{t}\left(\frac{\partial L}{\partial_{t_1}D_\theta^\alpha q_s}(\theta)_{t_1}D_\theta^\alpha\bar{\xi}_s^\sigma\right.\right.$$
$$\left.\left.-{}_\theta D_{t_2}^\alpha\frac{\partial L}{\partial_{t_1}D_\theta^\alpha q_s}(\theta)\bar{\xi}_s^\sigma\right)\mathrm{d}\theta+\frac{\partial L}{\partial \dot{q}_s}(t)\bar{\xi}_s^\sigma+G^\sigma\right]$$
$$+\bar{\xi}_s^\sigma\left[\frac{\partial L}{\partial q_s}(t)+{}_tD_{t_2}^\alpha\frac{\partial L}{\partial_{t_1}D_t^\alpha q_s}(t)-\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_s}(t)\right]=0 \tag{47}$$

are satisfied for $t_2-\tau<t\le t_2$, where $\sigma=1,2,\ldots,r$. Then, the infinitesimal transformations (25) are the fractional Noether quasi-symmetric transformations of the system with time delay.

By using Criterion 1–Criterion 4, or the fractional Noether identities (36) and (37), and (44) and (45) with time delay, we can verify the fractional Noether symmetry and fractional Noether quasi-symmetry of the system with time delay.

Finally, we study the fractional Noether generalized quasi-symmetric transformations of non-conservative system with time delay.

Assume the fractional dynamics system with time delay under the generalized non-potentials forces (9). If the transformations accurate to first-order infinitesimal satisfy the condition

$$\int_{t_1}^{t_2}L(t,q_s(t),{}_{t_1}D_t^\alpha q_s(t),$$
$$\dot{q}_s(t),q_s(t-\tau),\dot{q}_s(t-\tau))\mathrm{d}t$$
$$=\int_{\bar{t}_1}^{\bar{t}_2}L(\bar{t},\bar{q}_s(\bar{t}),{}_{\bar{t}_1}D_{\bar{t}}^\alpha\bar{q}_s(\bar{t}),\dot{\bar{q}}_s(\bar{t}),\bar{q}_s(\bar{t}-\tau),$$
$$\dot{\bar{q}}_s(\bar{t}-\tau))\mathrm{d}\bar{t}+\int_{t_1}^{t_2}Q_s''\delta q_s\mathrm{d}t \tag{48}$$

then this invariance is called the generalized quasi-invariance of the fractional Hamilton action (23) with time delay under the group of infinitesimal transformations (24) and the transformations are called the fractional Noether generalized quasi-symmetric transformations. So, we have

Definition 3 If the fractional Hamilton action (23) is a generalized quasi-invariance of the group of infinitesimal transformations (24), the condition

$$\Delta S=-\int_{t_1}^{t_2}\left[\frac{\mathrm{d}}{\mathrm{d}t}(\Delta G)+Q_s''\delta q_s\right]\mathrm{d}t \tag{49}$$

holds, then the infinitesimal transformations (24) are the fractional generalized Noether quasi-symmetric transformations of the system with time delay.

From Definition 3 and formulae (29) and (31), we have the following criteria.

Criterion 5 For the group of infinitesimal transformations (24), the condition

$$\frac{\partial L}{\partial t}(t)\Delta t+\left[\frac{\partial L}{\partial q_s}(t)+\frac{\partial L}{\partial q_{s\tau}}(t+\tau)\right]\Delta q_s$$
$$+\frac{\partial L}{\partial_{t_1}D_t^\alpha q_s}(t)\Delta_{t_1}D_t^\alpha q_s$$
$$+\left[\frac{\partial L}{\partial \dot{q}_s}(t)+\frac{\partial L}{\partial \dot{q}_{s\tau}}(t+\tau)\right]\Delta\dot{q}_s$$
$$+L\frac{\mathrm{d}}{\mathrm{d}t}(\Delta t)+Q_s''(\Delta q_s-\dot{q}_s\Delta t)$$
$$=-\frac{\mathrm{d}}{\mathrm{d}t}(\Delta G) \tag{50}$$

is satisfied for $t_1\le t\le t_2-\tau$, the condition

$$\frac{\partial L}{\partial t}(t)\Delta t+\frac{\partial L}{\partial q_s}(t)\Delta q_s$$
$$+\frac{\partial L}{\partial_{t_1}D_t^\alpha q_s}(t)\Delta_{t_1}D_t^\alpha q_s$$
$$+L\frac{\mathrm{d}}{\mathrm{d}t}(\Delta t)+\frac{\partial L}{\partial \dot{q}_s}(t)\Delta\dot{q}_s$$
$$+Q_s''(\Delta q_s-\dot{q}_s\Delta t)=-\frac{\mathrm{d}}{\mathrm{d}t}(\Delta G) \tag{51}$$

is satisfied for $t_2-\tau<t\le t_2$, then the infinitesimal transformations (24) are the fractional generalized Noether quasi-symmetric transformations of the system with time delay.

The formulae (50) and (51) can be expressed as

$$\frac{\partial L}{\partial t}(t)\xi_0^\sigma + \left[\frac{\partial L}{\partial q_s}(t) + \frac{\partial L}{\partial q_{s\tau}}(t+\tau)\right]\xi_s^\sigma + \left[\frac{\partial L}{\partial \dot{q}_s}(t) + \frac{\partial L}{\partial \dot{q}_{s\tau}}(t+\tau)\right](\dot{\xi}_s^\sigma - \dot{q}_s\dot{\xi}_0^\sigma) + \frac{\partial L}{\partial_{t_1} D_t^\alpha q_s}(t)\left({}_{t_1}D_t^\alpha \bar{\xi}_s^\sigma + \frac{d}{dt}\left({}_{t_1}D_t^\alpha q_s\right)\xi_0^\sigma\right) + L\dot{\xi}_0^\sigma + Q_s^{''}\xi_s^\sigma = -\dot{G}^\sigma \tag{52}$$

for $t_1 \le t \le t_2 - \tau$ and

$$\frac{\partial L}{\partial t}(t)\xi_0^\sigma + \frac{\partial L}{\partial q_s}(t)\xi_s^\sigma + \frac{\partial L}{\partial_{t_1} D_t^\alpha q_s}(t)\left({}_{t_1}D_t^\alpha \bar{\xi}_s^\sigma + \frac{d}{dt}\left({}_{t_1}D_t^\alpha q_s\right)\xi_0^\sigma\right) + \frac{\partial L}{\partial \dot{q}_s}(t)(\dot{\xi}_s^\sigma - \dot{q}_s\dot{\xi}_0^\sigma) + L\dot{\xi}_0^\sigma + Q_s^{''}\xi_s^\sigma = -\dot{G}^\sigma \tag{53}$$

for $t_2 - \tau < t \le t_2$, where $\sigma = 1, 2, \ldots, r$. When $r = 1$, Eqs. (52) and (53) are the fractional Noether identities of the system with time delay.

Criterion 6 For the group of infinitesimal transformations (25), the r equations

$$\frac{d}{dt}\Bigg[L\xi_0^\sigma + \int_{t_1}^{t}\Bigg(\frac{\partial L}{\partial_{t_1} D_\theta^\alpha q_s}(\theta)\,{}_{t_1}D_\theta^\alpha \bar{\xi}_s^\sigma - {}_\theta D_{t_2-\tau}^\alpha \frac{\partial L}{\partial_{t_1} D_\theta^\alpha q_s}(\theta)\bar{\xi}_s^\sigma + \bar{\xi}_s^\sigma \frac{{}_\theta D_{t_2-\tau}^\alpha}{\Gamma(\alpha)}\int_{t_2-\tau}^{t_2}\left({}_\theta D_{t_2}^\alpha \frac{\partial L}{\partial_{t_1} D_\theta^\alpha q_s}(z)\right) \times (z-\theta)^{\alpha-1}dz\Bigg)d\theta + \Big(\frac{\partial L}{\partial \dot{q}_s}(t) + \frac{\partial L}{\partial \dot{q}_{s\tau}}(t+\tau)\Big)\bar{\xi}_s^\sigma + G^\sigma\Bigg] + \bar{\xi}_s^\sigma\Bigg[\frac{\partial L}{\partial q_s}(t) + \frac{\partial L}{\partial q_{s\tau}}(t+\tau) + {}_tD_{t_2-\tau}^\alpha \frac{\partial L}{\partial_{t_1} D_t^\alpha q_s}(t) - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_{s\tau}}(t+\tau) - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_s}(t) + Q_s^{''}(t) - \frac{{}_tD_{t_2-\tau}^\alpha}{\Gamma(\alpha)}\int_{t_2-\tau}^{t_2}\left({}_tD_{t_2}^\alpha \frac{\partial L}{\partial_{t_1} D_t^\alpha q_s}(z)\right)(z-t)^{\alpha-1}dz\Bigg] = 0 \tag{54}$$

are satisfied for $t_1 \le t \le t_2 - \tau$ and the r equations

$$\frac{d}{dt}\Bigg[L\xi_0^\sigma + \int_{t_1}^{t}\Bigg(\frac{\partial L}{\partial_{t_1} D_\theta^\alpha q_s}(\theta)\,{}_{t_1}D_\theta^\alpha \bar{\xi}_s^\sigma - {}_\theta D_{t_2}^\alpha \frac{\partial L}{\partial_{t_1} D_\theta^\alpha q_s}(\theta)\bar{\xi}_s^\sigma\Bigg)d\theta + \frac{\partial L}{\partial \dot{q}_s}(t)\bar{\xi}_s^\sigma + G^\sigma\Bigg] + \bar{\xi}_s^\sigma\Bigg[\frac{\partial L}{\partial q_s}(t) + {}_tD_{t_2}^\alpha \frac{\partial L}{\partial_{t_1} D_t^\alpha q_s}(t) - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_s}(t) + Q_s^{''}(t)\Bigg] = 0 \tag{55}$$

are satisfied for $t_2 - \tau < t \le t_2$, where $\sigma = 1, 2, \ldots, r$. Then the infinitesimal transformations (25) are the fractional generalized Noether quasi-symmetric transformations of the system with time delay.

By using Criterion 5 and Criterion 6, or the fractional Noether identities (52) and (53) with time delay, we can verify the fractional generalized Noether quasi-symmetry for non-conservative system with time delay.

6 Fractional Noether theorem with time delay

In this section, we discuss the fractional Noether theorems of dynamics system with time delay. Firstly, we give the definition of the fractional conserved quantity of dynamics system with time delay.

Definition 4 A function $I(t, t+\tau, q_s, {}_{t_1}D_t^\alpha q_s, \dot{q}_s, q_{s\tau}, \dot{q}_{s\tau}, q_s(t+\tau), \dot{q}_s, \dot{q}_s(t+\tau), {}_{t_1}D_t^\alpha q_s(t+\tau))$ is said to be a fractional conserved quantity of dynamics system (18) under study if, and only if

$$\frac{d}{dt}I(t, t+\tau, q_s, {}_{t_1}D_t^\alpha q_s, \dot{q}_s, q_{s\tau}, \dot{q}_{s\tau}, q_s(t+\tau), \dot{q}_s, \dot{q}_s(t+\tau), {}_{t_1}D_t^\alpha q_s(t+\tau)) = 0 \tag{56}$$

holds, along all the solution curves of the fractional differential equations of motion (21) with time delay.

For the fractional Lagrange system (22) with time delay that we discussed, if we can find a fractional Noether symmetric transformation or the fractional Noether quasi-symmetric transformations with time delay, then we can get a fractional conserved quantities corresponding to these symmetries by using the following Noether theorems.

Theorem 1 *For the fractional Lagrange system (22), if the group of infinitesimal transformations (24) is the fractional Noether symmetric transformations under Definition 1, then there exists a system of r linear independent fractional conserved quantities with time delay as follows, there are*

$$I^\sigma = L\xi_0^\sigma + \left[\frac{\partial L}{\partial \dot{q}_s}(t) + \frac{\partial L}{\partial \dot{q}_{s\tau}}(t+\tau)\right]\bar{\xi}_s^\sigma + \int_{t_1}^{t}\Bigg[\frac{\partial L}{\partial_{t_1} D_\theta^\alpha q_s}(\theta)_{t_1} D_\theta^\alpha \bar{\xi}_s^\sigma - {}_\theta D_{t_2-\tau}^\alpha \frac{\partial L}{\partial_{t_1} D_\theta^\alpha q_s}(\theta)\bar{\xi}_s^\sigma + \bar{\xi}_s^\sigma \frac{{}_\theta D_{t_2-\tau}^\alpha}{\Gamma(\alpha)} \int_{t_2-\tau}^{t_2}\left({}_\theta D_{t_2}^\alpha \frac{\partial L}{\partial_{t_1} D_\theta^\alpha q_s}(z)\right) \times (z-\theta)^{\alpha-1} dz\Bigg] d\theta = const. \quad (57)$$

for $t_1 \le t \le t_2-\tau$ and there are

$$I^\sigma = L\xi_0^\sigma + \int_{t_1}^{t}\left(\frac{\partial L}{\partial_{t_1} D_\theta^\alpha q_s}(\theta)_{t_1} D_\theta^\alpha \bar{\xi}_s^\sigma - {}_\theta D_{t_2}^\alpha \frac{\partial L}{\partial_{t_1} D_\theta^\alpha q_s}(\theta)\bar{\xi}_s^\sigma\right) d\theta + \frac{\partial L}{\partial \dot{q}_s}(t)\bar{\xi}_s^\sigma = const. \quad (58)$$

for $t_2-\tau < t \le t_2$, where $\sigma = 1, 2, \ldots, r$.

Proof According to Definition 1 and Criterion 2, and substituting Eq. (22) into (38) and (39), we have

$$\frac{d}{dt}\Bigg\{L\xi_0^\sigma + \left(\frac{\partial L}{\partial \dot{q}_s}(t) + \frac{\partial L}{\partial \dot{q}_{s\tau}}(t+\tau)\right)\bar{\xi}_s^\sigma + \int_{t_1}^{t}\Bigg[\frac{\partial L}{\partial_{t_1} D_\theta^\alpha q_s}(\theta)_{t_1} D_\theta^\alpha \bar{\xi}_s^\sigma - {}_\theta D_{t_2-\tau}^\alpha \frac{\partial L}{\partial_{t_1} D_\theta^\alpha q_s}(\theta)\bar{\xi}_s^\sigma + \bar{\xi}_s^\sigma \frac{{}_\theta D_{t_2-\tau}^\alpha}{\Gamma(\alpha)} \int_{t_2-\tau}^{t_2}\left({}_\theta D_{t_2}^\alpha \frac{\partial L}{\partial_{t_1} D_\theta^\alpha q_s}(z)\right) \times (z-\theta)^{\alpha-1} dz\Bigg] d\theta\Bigg\} = 0 \quad (59)$$

for $t_1 \le t \le t_2-\tau$ and

$$\frac{d}{dt}\left[L\xi_0^\sigma + \int_{t_1}^{t}\left(\frac{\partial L}{\partial_{t_1} D_\theta^\alpha q_s}(\theta)_{t_1} D_\theta^\alpha \bar{\xi}_s^\sigma - {}_\theta D_{t_2}^\alpha \frac{\partial L}{\partial_{t_1} D_\theta^\alpha q_s}(\theta)\bar{\xi}_s^\sigma\right) d\theta + \frac{\partial L}{\partial \dot{q}_s}(t)\bar{\xi}_s^\sigma\right] = 0 \quad (60)$$

for $t_2-\tau < t \le t_2$, where $\sigma = 1, 2, \ldots, r$. Integrating formulae (59) and (60), we obtain the results. □

Theorem 2 *For the fractional Lagrange system (22), if the group of infinitesimal transformations (24) is the fractional Noether quasi-symmetric transformations under Definition 2, then there exists a system of r linear independent conserved quantities with time delay as follows, there are*

$$I^\sigma = L\xi_0^\sigma + \left[\frac{\partial L}{\partial \dot{q}_s}(t) + \frac{\partial L}{\partial \dot{q}_{s\tau}}(t+\tau)\right]\bar{\xi}_s^\sigma + \int_{t_1}^{t}\Bigg[\frac{\partial L}{\partial_{t_1} D_\theta^\alpha q_s}(\theta)_{t_1} D_\theta^\alpha \bar{\xi}_s^\sigma - {}_\theta D_{t_2-\tau}^\alpha \frac{\partial L}{\partial_{t_1} D_\theta^\alpha q_s}(\theta)\bar{\xi}_s^\sigma + \bar{\xi}_s^\sigma \frac{{}_\theta D_{t_2-\tau}^\alpha}{\Gamma(\alpha)} \int_{t_2-\tau}^{t_2}\left({}_\theta D_{t_2}^\alpha \frac{\partial L}{\partial_{t_1} D_\theta^\alpha q_s}(z)\right) \times (z-\theta)^{\alpha-1} dz\Bigg] d\theta + G^\sigma = const. \quad (61)$$

for $t_1 \le t \le t_2-\tau$ and there are

$$I^\sigma = L\xi_0^\sigma + \int_{t_1}^{t}\left(\frac{\partial L}{\partial_{t_1} D_\theta^\alpha q_s}(\theta)_{t_1} D_\theta^\alpha \bar{\xi}_s^\sigma - {}_\theta D_{t_2}^\alpha \frac{\partial L}{\partial_{t_1} D_\theta^\alpha q_s}(\theta)\bar{\xi}_s^\sigma\right) d\theta + \frac{\partial L}{\partial \dot{q}_s}(t)\bar{\xi}_s^\sigma + G^\sigma = const. \quad (62)$$

for $t_2-\tau < t \le t_2$, where $\sigma = 1, 2, \ldots, r$.

Proof According to Definition 2 and Criterion 4, and substituting Eqs.(22) into (46) and (47), we have

$$\frac{d}{dt}\Bigg[L\xi_0^\sigma + \left(\frac{\partial L}{\partial \dot{q}_s}(t) + \frac{\partial L}{\partial \dot{q}_{s\tau}}(t+\tau)\right)\bar{\xi}_s^\sigma + \int_{t_1}^{t}\Bigg(\frac{\partial L}{\partial_{t_1} D_\theta^\alpha q_s}(\theta)_{t_1} D_\theta^\alpha \bar{\xi}_s^\sigma - {}_\theta D_{t_2-\tau}^\alpha \frac{\partial L}{\partial_{t_1} D_\theta^\alpha q_s}(\theta)\bar{\xi}_s^\sigma$$

$$+\bar{\xi}_s^\sigma \frac{{}_\theta D_{t_2-\tau}^\alpha}{\Gamma(\alpha)} \int_{t_2-\tau}^{t_2} \left({}_\theta D_{t_2}^\alpha \frac{\partial L}{\partial_{t_1} D_\theta^\alpha q_s}(z) \right) \times (z-\theta)^{\alpha-1} \mathrm{d}z \Bigg) \mathrm{d}\theta + G^\sigma \Bigg] = 0 \tag{63}$$

for $t_1 \le t \le t_2 - \tau$ and

$$\frac{\mathrm{d}}{\mathrm{d}t}[L\xi_0^\sigma + \int_{t_1}^{t} \left(\frac{\partial L}{\partial_{t_1} D_\theta^\alpha q_s}(\theta)_{t_1} D_\theta^\alpha \bar{\xi}_s^\sigma - {}_\theta D_{t_2}^\alpha \frac{\partial L}{\partial_{t_1} D_\theta^\alpha q_s}(\theta) \bar{\xi}_s^\sigma \right) \mathrm{d}\theta + \frac{\partial L}{\partial \dot{q}_s}(t) \bar{\xi}_s^\sigma + G^\sigma] = 0 \tag{64}$$

for $t_2 - \tau < t \le t_2$, where $\sigma = 1, 2, \ldots, r$. Integrating formulae (63) and (64), we obtain the results. □

Theorem 1 and Theorem 2 are called the fractional Noether theorems for the Lagrange system with time delay. According to the fractional Noether theorems, if we can find a fractional Noether symmetry with time delay, then we can find a fractional conserved quantity with time delay.

Finally, we discuss the fractional Noether theorem for non-conservative system (21) with time delay.

Theorem 3 *For the fractional non-conservative system (21), if the infinitesimal transformations of group (24) are the fractional generalized Noether quasi-symmetric transformations under Definition 3, then there exists a system of r linear independent conserved quantities with time delay as (61) and (62).*

Proof According to Definition 3 and Criterion 6, and substituting Eqs.(21) into (54) and (55), we have formulae (63) and (64). Integrating formulae (63) and (64), we obtain the results. □

Theorem 3 is called the fractional Noether theorems for the non-conservative systems with time delay. Especially, if the items with the fractional derivatives and with time delay vanish, Theorem 3 is reduced to the standard Noether theorem for non-conservative systems [39].

7 Examples

Example 1 Let us study a mechanical system with time delay whose Lagrangian is

$$L = \frac{1}{2}\left[\left({}_{t_1} D_t^\alpha q(t) \right)^2 + \dot{q}^2(t) + \dot{q}^2(t-\tau) \right] - \frac{\omega^2}{2} \left(q^2(t) + q^2(t-\tau) \right) \tag{65}$$

where $t \in [t_1, t_2]$ and $\tau < t_2 - t_1$ is a given real number. The following conditions are satisfied: when $t \in [t_1 - \tau, t_1]$, $q(t) = \Omega(t)$, where $\Omega(t)$ is a given piecewise smooth function in $[t_1 - \tau, t_1]$; when $t = t_2$, $q(t) = q(t_2)$, where $q(t_2)$ is a certain value [39].

The fractional Euler–Lagrange equation for the system is

$$\begin{aligned} &\ddot{q}(t) + \ddot{q}_\tau(t+\tau) - {}_t D_{t_2-\tau}^\alpha {}_{t_1} D_t^\alpha q(t) \\ &+\omega^2 (q(t) + q_\tau(t+\tau)) \\ &+\frac{{}_t D_{t_2-\tau}^\alpha}{\Gamma(\alpha)} \int_{t_2-\tau}^{t_2} \left[{}_t D_{t_2}^\alpha ({}_{t_1} D_t^\alpha q_s(z)) \right] \\ &(z-t)^{\alpha-1} \mathrm{d}z = 0, \quad t \in [t_1, t_2 - \tau] \\ &\ddot{q}(t) - {}_t D_b^\alpha \left({}_{t_1} D_t^\alpha q(t) \right) \\ &+\omega^2 q(t) = 0, \quad t \in (t_2 - \tau, t_2] \end{aligned} \tag{66}$$

Form the fractional Noether identity (44) and (45) with time delay, we have

$$\begin{aligned} &-\omega^2 (q(t) + q_\tau(t+\tau)) \xi_1 \\ &+ \left({}_{t_1} D_t^\alpha q(t) \right) \left[\left({}_{t_1} D_t^\alpha \bar{\xi}_1 \right) + \frac{\mathrm{d}}{\mathrm{d}t} \left({}_{t_1} D_t^\alpha q \right) \xi_0 \right] \\ &+ [\dot{q}(t) + \dot{q}_\tau(t+\tau)] (\dot{\xi}_1 - \dot{q}\dot{\xi}_0) \\ &+ L\dot{\xi}_0 = -\dot{G}, \quad t \in [t_1, t_2 - \tau] \\ &-\omega^2 q(t) \xi_1 + \left({}_{t_1} D_t^\alpha q(t) \right) \Bigg[\left({}_{t_1} D_t^\alpha \bar{\xi}_1 \right) \\ &+ \frac{\mathrm{d}}{\mathrm{d}t} \left({}_{t_1} D_t^\alpha q \right) \xi_0 \Bigg] + \dot{q}(t)(\dot{\xi}_1 - \dot{q}\dot{\xi}_0) \\ &+ L\dot{\xi}_0 = -\dot{G}, \quad t \in (t_2 - \tau, t_2]. \end{aligned} \tag{67}$$

Equation (67) has a solution

$$\xi_0^1 = 1, \xi_1^1 = 0, G^1 = 0, \quad t \in [t_1, t_2]. \tag{68}$$

$$\begin{aligned} &\xi_0^2 = 0, \xi_1^2 = \dot{q}(t), G^2 = \frac{\omega^2}{2}(q^2(t) + q_\tau^2(t+\tau)) \\ &-\frac{1}{2}\left[\left({}_{t_1} D_t^\alpha q(t) \right)^2 + \dot{q}_\tau^2(t+\tau) + \dot{q}^2(t) \right], \\ &t \in [t_1, t_2 - \tau] \\ &\xi_0^2 = 0, \xi_1^2 = \dot{q}(t), G^2 = \frac{\omega^2}{2} q^2(t) \\ &-\frac{1}{2}\left[\left({}_{t_1} D_t^\alpha q(t) \right)^2 + \dot{q}^2(t) \right], \quad t \in (t_2 - \tau, t_2] \end{aligned} \tag{69}$$

The generators (68) and (69) correspond to the fractional Noether symmetry and the fractional Noether quasi-symmetry of the system, respectively. According to Theorems 1 and 2, we obtain

$$\begin{aligned}
I^1 &= \frac{1}{2}\left[({}_{t_1}D_t^\alpha q(t))^2 - \dot{q}^2(t) + \dot{q}^2(t-\tau) - \omega^2\left(q^2(t)+q^2(t-\tau)\right)\right] - \dot{q}_\tau(t+\tau)\dot{q}(t) \\
&\quad + \int_{t_1}^{t}\Big[-{}_{t_1}D_\theta^\alpha q(\theta)\,{}_{t_1}D_\theta^\alpha \dot{q}(\theta) + {}_\theta D_{t_2-\tau}^\alpha({}_{t_1}D_t^\alpha q(\theta))\dot{q}(\theta) \\
&\quad - \dot{q}(\theta)\frac{{}_\theta D_{t_2-\tau}^\alpha}{\Gamma(\alpha)}\int_{t_2-\tau}^{t_2}\left({}_\theta D_{t_2}^\alpha\left({}_{t_1}D_\theta^\alpha q_s\right)(z)\right) \times (z-\theta)^{\alpha-1}\mathrm{d}z\Big]\mathrm{d}\theta = \text{const.}, \quad t\in[t_1,t_2-\tau], \\
I^1 &= \frac{1}{2}\left[({}_{t_1}D_t^\alpha q(t))^2 - \dot{q}^2(t) + \dot{q}^2(t-\tau) - \omega^2\left(q^2(t)+q^2(t-\tau)\right)\right] \\
&\quad + \int_{t_1}^{t}\left[-{}_{t_1}D_\theta^\alpha q(\theta)\,{}_{t_1}D_\theta^\alpha \dot{q}(\theta) + {}_\theta D_{t_2}^\alpha({}_{t_1}D_\theta^\alpha q(\theta))\dot{q}(\theta)\right]\mathrm{d}\theta \\
&= \text{const.}, \quad t\in(t_2-\tau,t_2],
\end{aligned} \tag{70}$$

$$\begin{aligned}
I^2 &= \frac{1}{2}\left[-({}_{t_1}D_t^\alpha q(t))^2 + \omega^2\left(q^2(t)+q_\tau^2(t+\tau)\right) + \dot{q}^2(t) + \dot{q}_\tau^2(t+\tau)\right] \\
&\quad + \int_{t_1}^{t}\Big[{}_{t_1}D_\theta^\alpha q(\theta)\,{}_{t_1}D_\theta^\alpha \dot{q}(\theta) - {}_\theta D_{t_2-\tau}^\alpha({}_{t_1}D_t^\alpha q(\theta))\dot{q}(\theta) \\
&\quad + \dot{q}(\theta)\frac{{}_\theta D_{t_2-\tau}^\alpha}{\Gamma(\alpha)}\int_{t_2-\tau}^{t_2}\Big({}_\theta D_{t_2}^\alpha\left({}_{t_1}D_\theta^\alpha q_s\right) \times (z)\Big)(z-\theta)^{\alpha-1}\mathrm{d}z\Big]\mathrm{d}\theta = \text{const.}, \quad t\in[t_1,t_2-\tau], \\
I^2 &= \frac{1}{2}\left[-\left({}_{t_1}D_t^\alpha q(t)\right)^2 + \dot{q}^2(t) + \omega^2 q^2(t)\right] \\
&\quad + \int_{t_1}^{t}\left[{}_{t_1}D_\theta^\alpha q(\theta)\,{}_{t_1}D_\theta^\alpha \dot{q}(\theta) - {}_\theta D_{t_2}^\alpha({}_{t_1}D_\theta^\alpha q(\theta))\dot{q}(\theta)\right]\mathrm{d}\theta \\
&= \text{const.}, \quad t\in(t_2-\tau,t_2]
\end{aligned} \tag{71}$$

Formulae (70) and (71) are the fractional Noether conserved quantities corresponding to the fractional Noether symmetry (68) and the fractional Noether quasi-symmetry (69) for Lagrange system with time delay, respectively. If the item with fractional derivatives vanishes, Eq. (66) can be expressed as

$$\begin{aligned}
&\ddot{q}(t) + \ddot{q}_\tau(t+\tau) + \omega^2\left(q(t)+q_\tau(t+\tau)\right) = 0, \quad t\in[t_1,t_2-\tau] \\
&\ddot{q}(t) + \omega^2 q(t) = 0, \quad t\in(t_2-\tau,t_2]
\end{aligned} \tag{72}$$

Equation (72) is the differential equations of motion for the system with time delay. And the formulae (70) and (71) can be expressed as

$$\begin{aligned}
I^1 &= \frac{1}{2}\left[\dot{q}^2(t-\tau) - \dot{q}^2(t) - \omega^2\left(q^2(t)+q^2(t-\tau)\right)\right] - \dot{q}_\tau(t+\tau)\dot{q}(t) = \text{const.}, \quad t\in[t_1,t_2-\tau], \\
I^1 &= \frac{1}{2}\left[\dot{q}^2(t-\tau) - \dot{q}^2(t) - \omega^2\left(q^2(t)+q^2(t-\tau)\right)\right] = \text{const.}, \quad t\in(t_2-\tau,t_2].
\end{aligned} \tag{73}$$

$$\begin{aligned}
I^2 &= \frac{1}{2}\left[\omega^2\left(q^2(t)+q_\tau^2(t+\tau)\right) + \dot{q}^2(t) + \dot{q}_\tau^2(t+\tau)\right] = \text{const.}, \quad t\in[t_1,t_2-\tau], \\
I^2 &= \frac{\omega^2}{2}q^2(t) + \frac{1}{2}\dot{q}^2(t) = \text{const.}, \quad t\in(t_2-\tau,t_2]
\end{aligned} \tag{74}$$

The formulae (73) and (74) are corresponding Noether conserved quantities for the system with time delay. If the delay constant vanishes, Eq. (76) can be expressed as

$$\ddot{q}(t) + \omega^2 q(t) = 0, \tag{75}$$

Equation (75) is the differential equations of motion for classical system. The formulae (73) and (74) can be expressed as

$$I^1 = -\frac{1}{2}\left[\dot{q}^2(t) + \omega^2 q^2(t)\right] = \text{const.} \tag{76}$$

$$I^2 = \frac{\omega^2}{2}q^2(t) + \frac{1}{2}\dot{q}^2(t) = \text{const.} \tag{77}$$

The formulae (76) and (77) are corresponding Noether conserved quantities for classical system.

Example 2 Let us study a system whose Lagrangian and generalized non-potential force are

$$\begin{aligned}
L &= \frac{1}{2}m\left[\left({}_{t_1}D_t^\alpha q(t)\right)^2 + \dot{q}^2(t)\right] \\
Q'' &= -c\dot{q}(t-\tau)
\end{aligned} \tag{78}$$

where m, c and τ are real numbers.

From the fractional Lagrange equations (21), we have

$$m\ddot{q}(t) - m\,{}_tD_{t_2}^\alpha({}_{t_1}D_t^\alpha q(t)) = -c\dot{q}(t-\tau) \tag{79}$$

From the fractional Noether identities (52) and (53) with time delay, we have

$$\begin{aligned}
&m\,{}_{t_1}D_t^\alpha q(t)({}_{t_1}D_t^\alpha \bar{\xi}_1 + \frac{\mathrm{d}}{\mathrm{d}t}({}_{t_1}D_t^\alpha q(t))\xi_0) \\
&\quad + m\dot{q}(t)(\dot{\xi}_1 - \dot{q}\dot{\xi}_0) - c\dot{q}(t-\tau)(\xi_1 - \dot{q}\xi_0) \\
&\quad + \frac{1}{2}m\left({}_{t_1}D_t^\alpha q(t)\right)^2\dot{\xi}_0 \\
&= -\dot{G}
\end{aligned} \tag{80}$$

Equation (80) has solutions

$$\xi_0^1 = 1, \quad \xi_1^1 = \dot{q}(t), \quad G^1 = -\frac{m}{2}\left[\left({}_{t_1}D_t^{\alpha} q(t)\right)^2 + q^2(t)\right], \tag{81}$$

$$\xi_0^2 = 0, \quad \xi_1^2 = 1, \quad G^1 = cq(t-\tau) - m\int_{t_1}^{t} ({}_{t_1}D_t^{\alpha} q(t))({}_{t_1}D_t^{\alpha} 1)\mathrm{d}t \tag{82}$$

The generators (81) and (82) correspond to the fractional generalized Noether quasi-symmetry of the system. Substituting the generators (81) and (82) into the formulae (61) and (62), we have

$$I^1 = 0, \tag{83}$$

$$I^2 = cq(t-\tau) + m\dot{q}(t) - m\int_{t_1}^{t} [{}_tD_{t_2}^{\alpha}({}_{t_1}D_t^{\alpha} q(t))]\mathrm{d}t = \text{const.} \tag{84}$$

The conserved quantity (83) is trivial. If the item with fractional derivatives vanishes, Eq. (79) can be expressed as

$$m\ddot{q}(t) = -c\dot{q}(t-\tau) \tag{85}$$

Equation (85) is the differential equation of motion for the system with time delay. And formula (84) can be expressed as

$$I^2 = cq(t-\tau) + m\dot{q}(t) = \text{const.} \tag{86}$$

The formula (86) is the Noether conserved quantity for non-conservative system with time delay. If $\tau = 0$, Eq. (85) can be expressed as

$$m\ddot{q}(t) = -c\dot{q}(t) \tag{87}$$

Equation (87) is the differential equation of motion for classical system. And formula (86) can be expressed as

$$I^2 = cq(t) + m\dot{q}(t) = \text{const.} \tag{88}$$

The formula (88) is the Noether conserved quantity for classical non-conservative system.

8 Conclusion

The phenomenon of time delay is commonly found in nature and engineering; once we consider the influence of time delay, even a very simple issue, the dynamics behavior may become very complex. Therefore, it is more essential and more realistic description of mechanical system to consider a non-conservative mechanical system based on a fractional order model with time delay. In this paper, the fractional Noether symmetries and the conserved quantities for a non-conservative system with time delay within left Riemann–Liouville fractional derivatives are presented and discussed. The fractional differential equations of motion for the non-conservative system with time delay are obtained. The definitions and criteria of the fractional Noether symmetric transformations, the fractional Noether quasi-symmetric transformations and the fractional generalized Noether quasi-symmetric transformations of the system are given. And the relationship between the fractional Noether symmetries and the fractional conserved quantities with time delay is studied. If the fractional derivatives and the time delay vanish, the non-conservative Lagrange system with time delay based on fractional model is reduced to classical non-conservative Lagrange system [39]. The results of this paper are of universal significance. The approach of this paper can be further generalized to the fractional constrained mechanical systems with time delay, the fractional optimal control systems with time delay, the fractional Birkhoffian systems with time delay and so on.

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