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Finding coexisting attractors using amplitude control

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Abstract Nonlinear dynamical systems often have multiple stable states and thus can harbor coexisting and hidden attractors that may pose an inconvenience or even hazard in practical applications. Amplitude control provides one method to detect these coexisting attractors, and it explains the unpredictable and irreproducible behavior that sometimes occurs in carefully engineered systems. In this paper, two regimes of amplitude control are described to illustrate the method for detecting multistability and possible coexisting or hidden attractors.

Keywords Multiple stabilities · Amplitude control · Hidden attractors

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1 Introduction

For their potential application or hazards, chaotic systems and their synchronization have evoked great interest [1–6]. Especially, multistability with coexisting attractors in nonlinear dynamics and laser engineering [7–15] has attracted renewed attention because the presence of coexisting attractors may have serious technological implications and pose risks in applications and in amplitude control. Moreover, many dynamical systems have multiple coexisting attractors even without equilibria, where the attractors are hidden rather than self-excited [16,17] and whose basins of attraction do not contain neighborhoods of any equilibria.

The amplitude of oscillation in a dynamical system can often be controlled by changing the coefficient of one or more terms in the equations that describe the behavior without otherwise altering the characteristics of the oscillation such as its power spectral density and Lyapunov exponents [18–21]. In total amplitude control, all of the variables are simultaneously and proportionally controlled, whereas in partial amplitude control, only some are changed while the others are unaffected [22]. This method is often used in electrical circuit implementations to avoid saturating the amplifiers.

Amplitude control can be hindered by the existence of multistability, but it also provides a possible method to detect coexisting attractors, including hidden attractors. Even though Leonov et al. [16,17] used a special analytical–numerical algorithm to detect and localize the hidden attractors of Chua's circuit, it is still important to develop other effective methods to detect coexisting and hidden attractors. In this paper, we apply amplitude control and use a calculation of the Lyapunov exponents to give an indication of existence of multiple attractors. The method is also suitable for systems with no or stable equilibrium points, where all attractors are hidden. In Sect. 2, the basic method is described. Section 3 gives several examples of the method using both total and partial amplitude control. In Sect. 4, the advantages and limitations of the method are discussed along with future prospects.

2 Principle of amplitude control for detecting coexisting attractors

Amplitude control makes the attractor larger or smaller by changing the scale of some or all of the variables, and so it does not change the dynamical and topological properties of the attractor. However, if a system has multiple coexisting attractors, it is usually necessary to scale the initial conditions to remain within the desired basin of attraction. Otherwise, very different dynamical behaviors may occur for different settings of the amplitude control, which is an advantage if the goal is to identify coexisting attractors.

In general, these attractors may have very different dynamics including stable equilibria, periodicity, quasi-periodicity, chaos, and hyperchaos. A powerful method for identifying and quantifying the dynamic is the spectrum of Lyapunov exponents, whose number is equal to the number of dynamical variables and that are usually ordered from the largest (most positive) to the smallest (most negative). Consider a two-dimensional space of the largest two Lyapunov exponents λ_1 and λ_2 as shown in Fig. 1. From the definition, the exponents lie on or below the 45-degree line with stable equilibria (SE) in the lower left quadrant, limit cycles (LC) along the negative λ_2-axis, toruses (T) at the origin, chaotic attractors along the positive λ_1-axis, and hyperchaotic attractors in the upper right quadrant. If a dynamical system has coexisting attractors, they will usually have different values of one or both of their two largest Lyapunov exponents. Therefore, a scatter plot in the plane for different initial conditions will show clusters of points corresponding to the different coexisting attractors and will identify their types.

Similarly, for a given initial condition, different settings of the amplitude control will generally cause the system to visit the basins of most if not all of the attractors, especially if the basin boundaries are fractal [7–9] which is common in nonlinear dynamical systems that are multistable. However, for some dynamical systems, the attracting basin is simple or symmetrical according to some axis or original point, and so an appropriate initial condition must be chosen to increase the likelihood of visiting all the basins as the amplitude is adjusted. The amplitude control can be thought of as taking a particular straight-line path through the space of initial conditions.

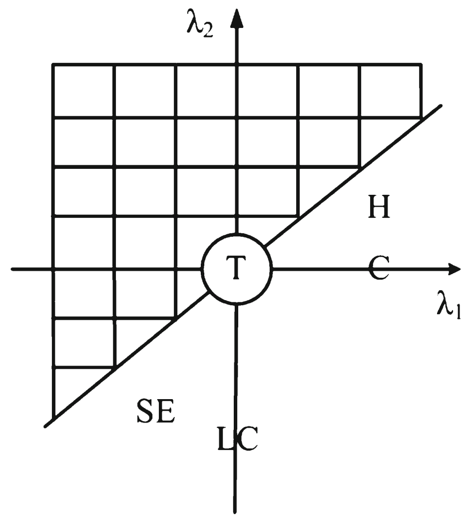

Fig. 1 Dynamical behaviors indicated by the two largest Lyapunov exponents

3 Finding coexisting attractors by amplitude control

3.1 Simple amplitude control

When we speak of an amplitude control we mean that the variables are controlled in proportion to another, and we do not consider cases in which variables are independently controlled. Amplitude control can be either partial (PAC) or total (TAC) depending on whether some or all of the dynamic variables are controlled. In an n-dimensional chaotic system, PAC means that anywhere from one to $n - 1$ of the state space variables are controlled, whereas TAC means all n of the variables are controlled.

Several chaotic systems based on absolute-value nonlinearities and with invariant Lyapunov exponents [18–20] have been studied, in which a constant (timeindependent) term in the equation determines the amplitude. In fact, such a constant is necessary in any system that contains only absolute-value nonlinearities since there would otherwise be nothing to determine the amplitude scale, and thus no attractor could exist. Therefore, the constant term is an amplitude parameter, which could realize TAC, and the amplitude parameter can be implemented electronically with an adjustable DC power supply.

Most model chaotic systems assume polynomial nonlinearities. Suppose there are four groups of state variable vectors, $\mathbf{X}, \mathbf{Y}, \mathbf{U}$ and $\mathbf{W}$. $\mathbf{X} = (x_1, x_2, \ldots, x_{l1})^{\mathrm{T}}$, $\mathbf{Y} = (y_1, y_2, \ldots, y_{l2})^{\mathrm{T}}$, $\mathbf{U} = (u_1, u_2, \ldots, u_{l1})^{\mathrm{T}}$, $\mathbf{W} = (w_1, w_2, \ldots, w_{l2})^{\mathrm{T}}$. There is no coupling or mixing of the linear terms between $\mathbf{X}$ and $\mathbf{Y}$. $\mathbf{D}$ is a constant vector, $\mathbf{D} = (d_1, d_2, \ldots, d_{l2})$, $\mathbf{p}$ and $\mathbf{q}$ are the index vectors associated with the state variable vector of the corresponding dimension, $\mathbf{p} = (p_1, p_2, \ldots, p_{l1})$, $\mathbf{q} = (q_1, q_2, \ldots, q_{l2})$, $p_i \geqq 0 (i = 1, 2, \ldots, l_1)$, $q_j \geqq 0 (j = 1, 2, \ldots, l_2)$, p_i and q_j are integers. $||\mathbf{p}||_1, ||\mathbf{q}||_1$ mean a vector of unit norm. $r \geqq 2, n \geqq 2, n \leqq r, r$ is the highest index of the nonlinear term, n is another positive integer representing the index of the nonlinear term of $\mathbf{X}$. Then the polynomial nonlinearity can be written as $g_{\mathbf{p},\mathbf{q}}(\mathbf{X}, \mathbf{Y}) = c^k_{[\mathbf{p},\mathbf{q}]} x_1^{p_1} x_2^{p_2} \cdots x_{l_1}^{p_{l_1}} y_1^{q_1} y_2^{q_2} \cdots y_{l_2}^{q_{l_2}}, k = 1, 2, \ldots, l_1, l_1 + 1, l_1 + 2, \ldots, l_1 + l_2$, where $c^k_{[\mathbf{p},\mathbf{q}]}$ is the coefficient of each term in the dimension k.

Theorem 1 *Suppose a differential equation of a chaotic system without a constant term can be expressed as*

$$\dot{\mathbf{X}} = \sum_{||\mathbf{p}||_1=1} g_{\mathbf{p}}(\mathbf{X}) + \sum_{||\mathbf{p}||_1=r} g_{\mathbf{p}}(\mathbf{X}). \tag{1}$$

Then the system (2) can realize TAC by a unified parameter introduction in all of the nonlinear coefficients, and all variables in the vector $\mathbf{X}$ *can be controlled to be* $f^{\frac{-1}{r-1}}$ *of the original scale.*

$$\dot{\mathbf{U}} = \sum_{||\mathbf{p}||_1=1} g_{\mathbf{p}}(\mathbf{U}) + f \sum_{||\mathbf{p}||_1=r} g_{\mathbf{p}}(\mathbf{U}), \tag{2}$$

where $\mathbf{U} = f^{\frac{-1}{r-1}}\mathbf{X}$, $f \neq 0$, *and the new introduced coefficient parameter* f *is an amplitude parameter for TAC.*

Proof 1 Substitute $\mathbf{U} = f^{\frac{-1}{r-1}}\mathbf{X}$ into Eq. (2) as follows,

$$f^{\frac{-1}{r-1}}\dot{\mathbf{X}} = \sum_{||\mathbf{p}||_1=1} f^{\frac{-1}{r-1}} g_{\mathbf{p}}(\mathbf{X}) + f \sum_{||\mathbf{p}||_1=r} (f^{\frac{-1}{r-1}})^r g_{\mathbf{p}}(\mathbf{X}) \tag{3}$$

After simplification, Eq. (3) turns into Eq. (1). This control mode is called TAC mode because all the variables in the chaotic system of with polynomial nonlinearity can be controlled by introducing a unified coefficient in each nonlinear term. □

Theorem 2 *Suppose a differential equation of a chaotic system can be expressed as*

$$\begin{cases} \dot{\mathbf{X}} = \sum\limits_{||\mathbf{p}||_1=1, ||\mathbf{q}||_1=0} g_{\mathbf{p},\mathbf{q}}(\mathbf{X}, \mathbf{Y}) \\ \quad + \sum\limits_{||\mathbf{p}||_1=1, ||\mathbf{p}||_1+||\mathbf{q}||_1=r} g_{\mathbf{p},\mathbf{q}}(\mathbf{X}, \mathbf{Y}), \\ \dot{\mathbf{Y}} = \sum\limits_{||\mathbf{p}||_1=0, ||\mathbf{q}||_1=1} g_{\mathbf{p},\mathbf{q}}(\mathbf{X}, \mathbf{Y}) \\ \quad + \sum\limits_{||\mathbf{p}||_1=n, ||\mathbf{p}||_1+||\mathbf{q}||_1=r} g_{\mathbf{p},\mathbf{q}}(\mathbf{X}, \mathbf{Y}) + \mathbf{D}. \end{cases} \tag{4}$$

Then the system (5) can realize PAC by a unified parameter introduction in some of the nonlinear coefficients, and the amplitude of the variable vector $\mathbf{X}$ *can be controlled to be* $1/\sqrt[n]{f}$ *of the original scale, while the amplitude of the variable vector* $\mathbf{Y}$ *remains constant.*

$$\begin{cases} \dot{\mathbf{U}} = \sum\limits_{||\mathbf{p}||_1=1, ||\mathbf{q}||_1=0} g_{\mathbf{p},\mathbf{q}}(\mathbf{U}, \mathbf{W}) \\ \quad + \sum\limits_{||\mathbf{p}||_1=1, ||\mathbf{p}||_1+||\mathbf{q}||_1=r} g_{\mathbf{p},\mathbf{q}}(\mathbf{U}, \mathbf{W}), \\ \dot{\mathbf{W}} = \sum\limits_{||\mathbf{p}||_1=0, ||\mathbf{q}||_1=1} g_{\mathbf{p},\mathbf{q}}(\mathbf{U}, \mathbf{W}) \\ \quad + f \sum\limits_{||\mathbf{p}||_1=n, ||\mathbf{p}||_1+||\mathbf{q}||_1=r} g_{\mathbf{p},\mathbf{q}}(\mathbf{U}, \mathbf{W}) + \mathbf{D}, \end{cases} \tag{5}$$

where $\mathbf{U} = \mathbf{X}/\sqrt[n]{f}$, $\mathbf{W} = \mathbf{Y}$, *and the new introduced coefficient parameter* $f (f > 0)$ *is an amplitude parameter for PAC.*

Proof 2 Substitute $\mathbf{U} = \mathbf{X}/\sqrt[n]{f}$, $\mathbf{W} = \mathbf{Y}$ into Eq. (5) as follows,

$$\begin{cases} \frac{1}{\sqrt[n]{f}}\dot{\mathbf{X}} = \sum\limits_{||\mathbf{p}||_1=1, ||\mathbf{q}||_1=0} \frac{1}{\sqrt[n]{f}} g_{\mathbf{p},\mathbf{q}}(\mathbf{X}, \mathbf{Y}) \\ \quad + \sum\limits_{||\mathbf{p}||_1=1, ||\mathbf{p}||_1+||\mathbf{q}||_1=r} \frac{1}{\sqrt[n]{f}} g_{\mathbf{p},\mathbf{q}}(\mathbf{X}, \mathbf{Y}), \\ \dot{\mathbf{Y}} = \sum\limits_{||\mathbf{p}||_1=0, ||\mathbf{q}||_1=1} g_{\mathbf{p},\mathbf{q}}(\mathbf{X}, \mathbf{Y}) \\ \quad + f \sum\limits_{||\mathbf{p}||_1=n, ||\mathbf{p}||_1+||\mathbf{q}||_1=r} (\frac{1}{\sqrt[n]{f}})^n g_{\mathbf{p},\mathbf{q}}(\mathbf{X}, \mathbf{Y}) + \mathbf{D} \end{cases} \tag{6}$$

After simplification, Eq. (6) becomes Eq. (4). This control mode is called PAC mode because the amplitude of the variables in the vector $\mathbf{X}$ is controlled by the introduced coefficients while the other variables in the vector $\mathbf{Y}$ remain unchanged. □

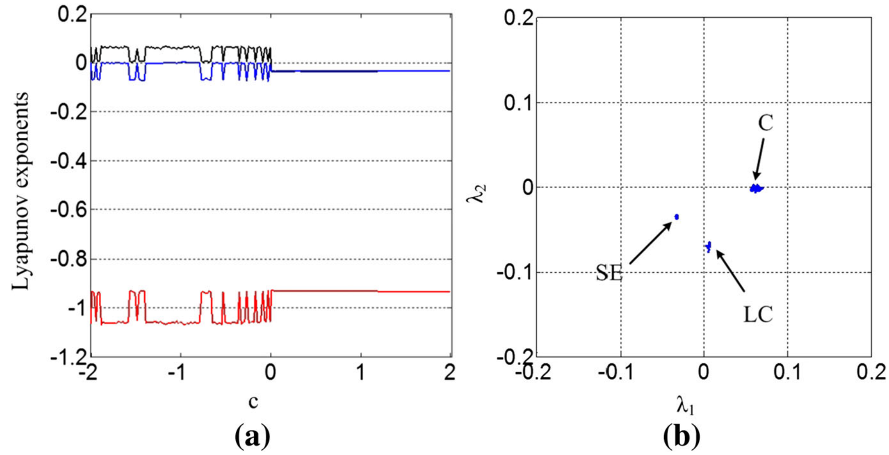

Fig. 2 Lyapunov exponent spectrum for system (8) and its distribution for initial conditions (0, 1, 0)

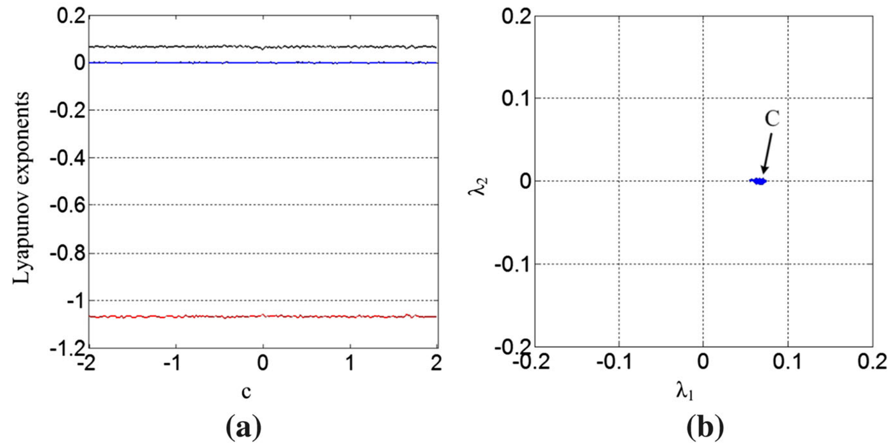

Fig. 3 Lyapunov exponent spectrum for system (8) and its distribution for initial conditions (0, 0, 0)

In short, for a specific chaotic system, we can use variable substitution to realize TAC or PAC, where the variables can be controlled in proportion at the same desired rate.

3.2 Examples of amplitude control with indication of coexisting attractors

3.2.1 Total amplitude control (TAC)

As a first example, we choose a system proposed in [7] given by

$$\begin{cases} \dot{x} = yz + a, \\ \dot{y} = x^2 - y, \\ \dot{z} = 1 - 4x, \end{cases} \tag{7}$$

that for $a = 0.01$ has three coexisting attractors: a stable equilibrium, a limit cycle, and a strange attractor. To achieve total amplitude control, we introduce a control parameter c in the constant and quadratic terms according to

$$\begin{cases} \dot{x} = \frac{1}{c} yz + ca, \\ \dot{y} = \frac{1}{c} x^2 - y, \\ \dot{z} = c - 4x, \end{cases} \tag{8}$$

Since a transformation $x = cu, y = cv, z = cw$ of Eq. (8) leads directly to Eq. (7), the parameter c proportionally controls the amplitude of variables x, y and z according to c.

With fixed initial conditions of (0, 1, 0), variation of c in the range $[-2, 2]$ causes the Lyapunov exponents to change as shown in Fig. 2 as the system transitions between the three different dynamics as confirmed by the corresponding clusters of points in the space of Lyapunov exponents.

To illustrate an inappropriate choice of initial conditions, consider the case (0, 0, 0) which gives the behavior shown in Fig. 3 for which the only observed attractor is the chaotic one. This selected initial condition does

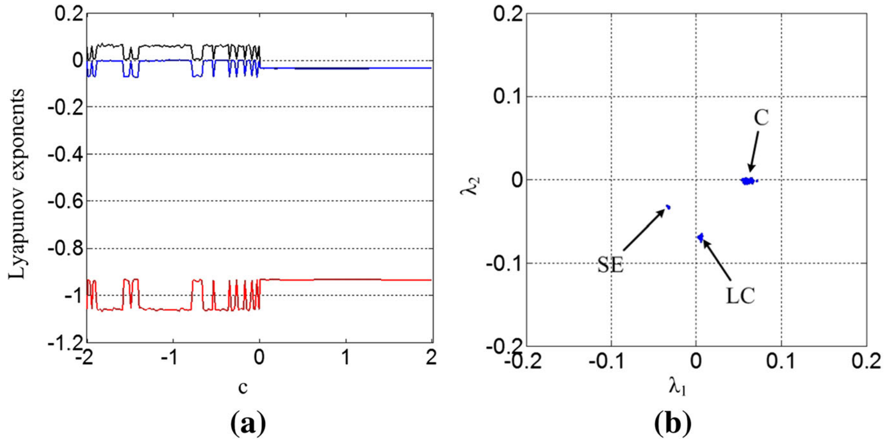

Fig. 4 Lyapunov exponent spectrum for system (9) and its distribution for initial conditions (0, 1, 0)

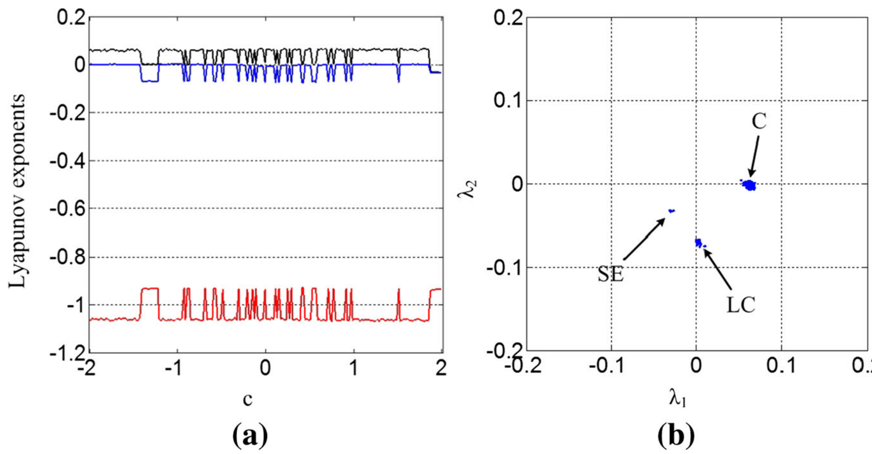

Fig. 5 Lyapunov exponent spectrum for system (10) and its distribution for initial conditions (1, 1, 0)

not pass through all the basins of attraction, and thus it is a blind spot for coexisting attractors. For systems with symmetries, it is necessary to avoid initial condition along the axis (or axes) of symmetry.

3.2.2 Partial amplitude control (PAC)

For a three-dimensional chaotic system, partial amplitude control can be one-dimensional or two-dimensional. If $x = u, y = cv, z = w$, the resulting one-dimensional control system from Eq. (9) is identical to (7), which indicates that the variable y is controlled according to the parameter c, while the amplitude of the variables x and z remains unchanged. The resulting system is

$$\begin{cases} \dot{x} = \frac{1}{c}yz + a, \\ \dot{y} = cx^2 - y, \\ \dot{z} = 1 - 4x, \end{cases} \tag{9}$$

and the result of varying c over the range $[-2, 2]$ for initial conditions (0, 1, 0) is shown in Fig. 4.

For an example of two-dimensional partial amplitude control, let $x = cu$, $y = cv$, $z = w$. The resulting control system from Eq. (10) is identical to (7), and thus the parameter c controls the amplitude of variables x and y according to c, while the amplitude of z remains unchanged. The resulting system is

$$\begin{cases} \dot{x} = yz + ca, \\ \dot{y} = \frac{x^2}{c} - y, \\ \dot{z} = 1 - 4\frac{x}{c}, \end{cases} \tag{10}$$

and the result of varying c over the range $[-2, 2]$ for initial conditions (1, 1, 0) is shown in Fig. 5. Both cases of partial amplitude control pass through all three basins of attraction and thus correctly identify all dynamics of the system.

4 Discussion and conclusion

Amplitude control with fixed initial conditions provides a tool for identifying coexisting attractors in a dynamical system that may be more convenient in a practical application than exploring all possible initial conditions. It also provides an explanation for why a system may exhibit different dynamical behaviors for different settings of an amplitude control adjustment even when the system is started with the same initial conditions.

However, this method has some disadvantages. The Lyapunov exponent calculation typically converges more slowly than other measures such as the location and size of the attractor, especially the second largest Lyapunov exponent. Furthermore, it cannot easily distinguish two symmetrical attractors with the same Lyapunov exponent spectra, which is common in systems with symmetries. In such a case, the method can be used with whatever measure is deemed most appropriate for the system under consideration, or a combination of measures could be used.

Even though this phenomenon has practical consequences in that it might render the prediction of a system's behavior difficult, it is still an easy way to find multiple stabilities in dynamical systems. In general, when the amplitude parameter varies in a range to control the size of the attractors, dynamical systems will often, but not always, pass though the different attracting basins. Thus, the method of amplitude control opens up interesting possibilities in the identification and study of multistability with coexisting attractors, including coexisting hidden attractors.

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