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Multi-valued nonlinear perturbations of time fractional evolution equations in Banach spaces

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Abstract The paper is concerned with the fractional evolution inclusion ${}^{c}D_t^q u(t) \in Au(t) + F(t, u(t))$ in Banach spaces, where ${}^{c}D_t^q$, $0 < q < 1$, is the regularized Caputo fractional derivative of order q, A generates a compact semigroup, and F is a multi-valued function with convex, closed values. Constructing a suitable directionally L^p-integrable selection from F, we study the compactness and R_δ-structure of the set of trajectories on a closed domain. Moreover, we discuss the R_δ-structure of the set of trajectories to the control problem corresponding to the inclusion above. Finally, we apply our abstract theory to boundary value problems of fractional diffusion inclusions.

Keywords Fractional evolution inclusions · Topological structure of solution set · Invariance of reachability set

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1 Introduction

A great deal of studies on topological structure of solution sets (including R_δ, acyclicity, connectedness, compactness, and contractibility) to differential equations and inclusions have been made by many researchers. For references see, e.g., Andres et al. [1,2], Aronszajn [5], Bothe [9], Conti et al. [11], Gabor [17,18], Hu and Papageorgiou [22], and Zhu [45]; some of the more recent literatures are, e.g., Andres and Pavlačková [3], Bakowska and Gabor [6], Gabor and Grudzka [19], and one can find further references therein. In particular, in our previous work [10], we studied the topological structure of solution sets to the Cauchy problem of nonlinear delay differential inclusion

$$\begin{cases} u'(t) \in Au(t) + F(t, u(t), u_t), \\ u(t) = \phi(t), \quad t \in [-\tau, 0] \end{cases}$$

both on compact intervals and non-compact intervals, where $A : D(A) \subset X \to 2^X$ is an m-dissipative operator (possible multi-valued and/or nonlinear) and $F : \mathbb{R}^+ \times \overline{D(A)} \times C([-\tau, 0]; \overline{D(A)}) \to 2^X$ is a multi-valued function with convex, closed values. Moreover, we used the information of the structure on non-compact intervals to show the existence of global C^0-solutions for the evolution inclusion subject to nonlocal condition.

On the other hand, fractional evolution inclusions of the form $D_t^q u(t) \in Au(t) + F(t, u(t))$ in Banach spaces, where A is a closed linear operator and D_t^q is the fractional derivative of order q have been investigated to a large extent; For some recent contributions, we refer the reader to, for instance, Eidelman and Kochubei [15], Wang et al. [39,40], and Zhou [37,43,44] for the case of single-valued nonlinearities, and Ouahab [27] and Wang and Zhou [35] for the case of multi-valued nonlinearities. Fractional evolution inclusions are a kind of important differential inclusions describing the processes behaving in a much more complex way on time, which appear as a generalization of fractional evolution equations (such as (time-)fractional diffusion equations) through the application of multi-valued analysis. Comparing the fractional evolution equations, the researches on the theory of fractional differential inclusions are only on their initial stage of development. It is noted that El-Sayed and Ibrahim initialed the study of fractional differential inclusions in [16] and much interest has developed alone this line, see, e.g., [7,20]. A strong motivation for investigating this class of inclusions comes mainly from two compelling reasons: differential models with the fractional derivative providing an excellent instrument for the description of memory and hereditary properties have recently been proved valuable tools in the modeling of many physical phenomena (cf., [13,21,24,26]). As in [4,26], fractional diffusion equations describe anomalous diffusion on fractals (physical objects of fractional dimension, like some amorphous semiconductors or strongly porous materials). In normal diffusion described by, such as the heat equation, the mean square displacement of a diffusive particle behaves like const $\cdot\, t$ for $t \to \infty$. A typical behavior for anomalous diffusion is const $\cdot\, t^{\alpha}$ for some $0 < \alpha < 1$. Another of the reasons is that a lot of phenomena investigated in processes of controlled heat transfer, obstacle problems, and others can be described with the help of various differential inclusions, both linear and nonlinear. See, e.g., [12,23,32] for more comments and citations. One can find recent results in this direction from Paicu and Vrabie [28,29], Vrabie [33,34], Wang and Zhu [38], and the references therein.

However, as far as we know, there have been very few applicable results on the topological structure of solution sets for fractional evolution inclusions. This in fact is the main motivation of the present paper. In order to fill this gap, in this work we are interested in the problem of topological structure of set of trajectories (formed by mild solutions) to the Cauchy problem of evolution inclusion of the form

$$\begin{cases} {}^{\mathrm{c}}D_t^q u(t) \in Au(t) + F(t, u(t)), \quad t \in [0, b], \\ u(0) = u_0 \end{cases} \tag{1.1}$$

in the Banach space X, where ${}^{\mathrm{c}}D_t^q$, $0 < q < 1$, is the regularized Caputo fractional derivative of order q, A is the infinitesimal generator of a C_0-semigroup $\{T(t)\}_{t\geq 0}$ on X, and the forcing source $F : [0, b] \times X \to 2^X$ is a multi-valued function with convex, closed values for which $F(t, \cdot)$ is weakly upper semi-continuous for a.e. $t \in [0, b]$ and $F(\cdot, x)$ has a L^p-integrable selection for each $x \in X$.

Constructing a suitable directionally L^p-integrable ($p > 1$) selection from F and using an interplay between compactness arguments and multi-valued analysis techniques, we first consider the existence of mild solutions to Cauchy problem (1.1). Then, a new method, which can be considered as a modification of the method used in [10], is developed to discuss the R_δ-structure of the set of trajectories, which apparently cannot be obtained by the techniques of the previous work. All the results obtained are new in the case of single-valued nonlinearities. Moreover, we also deal with topological characterizations to the control problem corresponding to the inclusion above.

Another achievement of this paper, motivated by applications of the information about the structure to some problems in optimal control while certainly significant for its own sake, is establishing the invariance of a reachability set to the control problem mentioned above under single-valued nonlinear perturbations.

We remark that in the previous papers on topological structure such as [3,9,10], an elementary approach consisting in finding a strongly measurable selection f of the multi-valued function F was always available. However, in the present work, the concept of the mild solution involves a singular integral equation, which enables us to find that strongly measurable selections of multi-valued function F are not enough to obtain the desired results. Therefore, we have to impose the stronger assumption on F to get a L^p-integrable selection of multi-valued function F. Moreover, when dealing with the invariance of a reachability set under nonlinear perturbations, we note that the solution mapping corresponding to each control function is a multi-valued one, which prevents us from using the known tools as in [31] to show the desired results.

The remainder of this paper is organized as follows. Section 2 provides some preliminary material on differential inclusions and the notation to be used in what follows. Section 3 contains the main results of the paper. Finally, as a sample of applications, we present two examples to illustrate the feasibility of our results.

2 Preliminaries

As usual, for a Banach space W, 2^W is the set of all nonempty subsets of W, and $B_r(x_0)$ stands for the open ball in W with radius r and center $x_0 \in W$.

$C([0,b];X)$ is the Banach space of all continuous functions from $[0,b]$ to X equipped with the sup-norm and $L^p(0,b;X)$ $(1<p<+\infty)$ is the Banach space consisting of all Bochner integrable functions from $[0,b]$ to X satisfying $\int_0^b \|u(t)\|^p \mathrm{d}t < +\infty$, equipped with the norm

$$\|u\|_{L^p(0,b;X)} = \left(\int_0^b \|u(t)\|^p \mathrm{d}t\right)^{1/p}.$$

What followed is the criterion for weak compactness in $L^p(0,b;X)$ for $1<p<+\infty$ (see, e.g., [32, Corollary 1.3.1]).

Lemma 2.1 *Let X be reflexive and $1<p<+\infty$. A subset $K \subset L^p(0,b;X)$ is weakly relatively sequentially compact in $L^p(0,b;X)$ if and only if K is bounded in $L^p(0,b;X)$.*

Definition 2.1 A nonempty subset D of a metric space Y is said to be contractible if there exists a point $y_0 \in D$ and a continuous function $g:[0,1]\times D \to D$ such that $g(0,y)=y_0$ and $g(1,y)=y$ for every $y \in D$.

Definition 2.2 A subset D of a metric space Y is called an R_δ-set if there exists a decreasing sequence $\{D_n\}$ of compact and contractible sets such that

$$D = \bigcap_{n=1}^{\infty} D_n.$$

Let Y be a metric space. Y is called an absolute retract (AR-space) if for any metric space H and any closed subset $D \subset H$, every continuous function $\varphi: D \to Y$ can be extended to a continuous function $\widetilde{\varphi}: H \to Y$.

Y is called an absolute neighborhood retract (ANR-space) if for any metric space H, closed subset $D \subset H$, and continuous function $\varphi: D \to Y$, there exists a neighborhood $U \supset D$ and a continuous extension $\widetilde{\varphi}: U \to Y$ of φ.

Obviously, if Y is an AR-space then it is an ANR-space. Furthermore, as in [14, Corollary 4.2], if D is a convex set in a locally convex linear space then it is an AR-space. This yields that each convex subset of a Fréchet space is an AR-space, since every Fréchet space is locally convex.

Any absolute retract is contractible.

The following hierarchy holds for nonempty subsets of a metric space:

$$\text{compact+convex} \subset \text{compact AR-space} \subset \text{compact} + \text{contractible} \subset R_\delta\text{-set},$$

and all the above inclusions are proper.

Below, let Y and Z be metric spaces. As usual, we denote $C(Y) = \{D \in 2^Y; D \text{ is closed}\}$, $C_v(Y) = \{D \in C(Y);\ D \text{ is convex}\}$, and $K(Y) = \{D \in C(Y);\ D \text{ is compact}\}$.

For the multi-valued mapping $\varphi: Y \to 2^Z$, we let $Gra(\varphi)$ stand for the graph of φ. If D is a subset of Z, then we denote by $\varphi^{-1}(D) = \{y \in Y; \varphi(y) \cap D \neq \emptyset\}$ the complete preimage of D under φ. φ is called closed if $Gra(\varphi)$ is closed in $Y \times Z$, quasi-compact if $\varphi(D)$ is relatively compact for each compact set $D \subset Y$, upper semi-continuous (shortly, u.s.c.) if $\varphi^{-1}(D)$ is closed for each closed set $D \subset Z$, and lower semi-continuous (shortly, l.s.c.) if $\varphi^{-1}(D)$ is open for each open set $D \subset Z$.

Definition 2.3 A multi-valued mapping $\varphi: Y \to 2^Z$ is an R_δ-mapping if φ is u.s.c. and $\varphi(y)$ is an R_δ-set for each $y \in Y$.

In the sequel, let W and V be Banach spaces.

Definition 2.4 Let $\varphi: D \subset W \to 2^V$ be a multi-valued mapping. Then,

(i) φ is called weakly upper semi-continuous (shortly, weakly u.s.c.) if $\varphi^{-1}(D')$ is closed in D for every weakly closed set $D' \subset V$.
(ii) φ is $\epsilon-\delta$ u.s.c. if for every $w_0 \in D$ and $\epsilon>0$ there exists $\delta>0$ such that $\varphi(y) \subset \varphi(w_0) + B_\epsilon(0)$ for all $y \in B_\delta(w_0) \cap D$.

Evidently u.s.c. is stronger than weakly u.s.c. and simple examples show that a weakly u.s.c. function with compact convex values may fail to be u.s.c.

We point out that u.s.c. is stronger than $\epsilon-\delta$ u.s.c., but for multi-valued mappings with compact values the two concepts coincide. Moreover, it was proved in [10, Lemma 2.2] that

Lemma 2.2 *Let D be a nonempty subset of W and $\varphi : D \to 2^V$ a multi-valued mapping with weakly compact values, then*

(i) *φ is weakly u.s.c. if φ is $\epsilon - \delta$ u.s.c.,*
(ii) *suppose further that φ has convex values and V is reflexive. Then φ is weakly u.s.c. if and only if for each sequence $\{(z_n, x_n)\} \subset D \times V$ such that $z_n \to z$ in W and $x_n \in \varphi(z_n)$, $n \geq 1$, it follows that there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and $x \in \varphi(z)$ such that $x_{n_k} \to x$ weakly in V.*

We set $I = (0, T)$ for some $T > 0$ and use the following notation for $q \geq 0$:

$$g_q(t) = \begin{cases} \frac{1}{\Gamma(q)} t^{q-1}, & t > 0, \\ 0, & t \leq 0, \end{cases}$$

and $g_0(t) = 0$, where $\Gamma(\cdot)$ is the Gamma function.

Definition 2.5 Let $f \in L^1(I; X)$ and $q \geq 0$. Then the express

$$J_t^q f(t) := (g_q * f)(t) = \frac{1}{\Gamma(q)} \int_0^t (t - s)^{q-1} f(s) \mathrm{d}s, \ t > 0, \ q > 0$$

with $J_t^0 f(t) = f(t)$, is called Riemann–Liouville integral of order q of f.

Definition 2.6 Let $f(t) \in C^{m-1}(I; X)$, $g_{m-q} * f \in W^{m,1}(I, X)$ $(m \in \mathbb{N}, 0 \leq m - 1 < q < m)$. The regularized Caputo fractional derivative of order q of f is defined by

$$_{\mathrm{c}}D_t^q f(t) = D_t^m J_t^{m-q} \left(f(t) - \sum_{i=0}^{m-1} f^{(i)}(0) g_{i+1}(t) \right), \tag{2.1}$$

where $D_t^m := \frac{d^m}{dt^m}$.

It is known that the Caputo definition for the fractional derivative incorporates the initial values of the function and of its integer derivatives of lower order and the relevant property that the derivative of a constant is zero is preserved.

Remark 2.1 Let us point out that in the treatment of abstract fractional evolution equations and inclusions in infinite dimensional spaces, one of the key points is to give reasonable concept of solutions according to the corresponding fractional derivative (see, e.g., [39,42]), which indicates that there is a strong influence on topological structures because of derivative differences.

Assume that $0 < q < 1$. We note that the setting determines the necessity to use the regularized fractional derivative (2.1). In particular, if, for example, one considers instead of (2.1) the Riemann–Liouville fractional derivative, but without subtracting $t^{-q}u(0)$, then the appropriate initial data will be the limit value, as $t \to 0$, of the fractional integral of a solution of the order $1 - q$, not the limit value of the solution itself. Also, we notice that for a smooth enough function $u(t)$, the Caputo fractional derivative $_{\mathrm{c}}D_t^q u$ can be written as

$$_{\mathrm{c}}D_t^q u(t) = \frac{1}{\Gamma(1-q)} \int_0^t (t - s)^{-q} u'(s) \mathrm{d}s.$$

In the physical literature, the expression on the right is used as the basic object for formulating fractional diffusion equations (cf., [15,39]).

Throughout this paper, A is a linear closed operator generating a C_0-semigroup $\{T(t)\}_{t \geq 0}$ on X and there exists a constant $M > 0$ such that

$$\sup\{\|T(t)\|, t \in \mathbb{R}^+\} \leq M.$$

Let us first introduce two families of operators on X:

$$Q(t) = \int_0^\infty \Psi_q(s) T(st^q) \mathrm{d}s, \quad t \geq 0,$$

$$P(t) = \int_0^\infty qs \Psi_q(s) T(st^q) \mathrm{d}s, \quad t \geq 0,$$

where Ψ_q is the function of Wright type:

$$\Psi_q(s) = \frac{1}{\pi q} \sum_{n=1}^\infty (-s)^{n-1} \frac{\Gamma(1 + qn)}{n!} \sin(n\pi q), \ s \in (0, \infty).$$

Then $Q(t)$ and $P(t)$ are linear and bounded operators on X, for which the following estimates hold:

$$\|Q(t)x\| \leq M\|x\|, \qquad \|P(t)x\| \leq \frac{qM\|x\|}{\Gamma(1+q)}, \quad t \geq 0, x \in X.$$

Also, $Q(t)$ and $P(t)$ are strongly continuous on X. Moreover, we have the following results.

Lemma 2.3 *Assume that $T(t)$ is compact for $t > 0$. Then*

(i) *$Q(t)$ and $P(t)$ are compact for every $t > 0$, and*

(ii) $Q(t)$ *and* $P(t)$ *are continuous in the uniform operator topology for* $t > 0$.

For a detailed account and bibliographic references for such families of operators see, e.g., the survey by Wang et al. [36,40] and Zhou and Jiao [42].

Consider the linear Cauchy problem in the form

$$\begin{cases} {}^{c}D_t^q u(t) = Au(t) + f(t), & t \in [0, b], \\ u(0) = u_0 \in X, \end{cases} \tag{2.2}$$

where $f \in L^p(0, b; X)$ with $p > 1$, $pq > 1$.

Definition 2.7 A function $u \in C([0, b]; X)$ is called a mild solution of Cauchy problem (2.2), if it satisfies the integral equation

$$u(t) = Q(t)u_0 + \int_0^t (t-s)^{q-1} P(t-s) f(s) \mathrm{d}s, \quad t \in [0, b].$$

From Wang and Yang [36] (see also Wang et al. [39] and Zhou and Jiao [42]), it follows that for each $u_0 \in X$ and $f \in L^p(0, b; X)$, there exists a unique mild solution to Cauchy problem (2.2) on $[0, b]$ which satisfies $u(0) = u_0$.

Let $u_0 \in X$ and $f \in L^p(0, b; X)$. Denote by $u(\cdot, u_0, f)$ the unique mild solution $u : [0, b] \to X$, of Cauchy problem (2.2) which verifies $u(0) = u_0$.

The following approximation result will be used later.

Lemma 2.4 *Assume that* B *is a bounded linear operator from the Banach space* V *to* X *and* $w \in L^p(0, b; V)$. *If the two sequences* $\{f_n\} \subset L^p(0, b; X)$ *and* $\{u_n\} \subset C([0, b]; X)$, *where* u_n *is a mild solution of the problem*

$$\begin{cases} {}^{c}D_t^q u_n(t) = Au_n(t) + f_n(t) + Bw(t), & t \in [0, b], \\ u_n(0) = u_0, \end{cases}$$

$\lim_{n\to\infty} f_n = f$ *weakly in* $L^p(0, b; X)$ *and* $\lim_{n\to\infty} u_n = u$ *in* $C([0, b]; X)$, *then* u *is a mild solution of the limit problem*

$$\begin{cases} {}^{c}D_t^q u(t) = Au(t) + f(t) + Bw(t), & t \in [0, b], \\ u(0) = u_0. \end{cases}$$

Proof An argument similar to that in the proof of [35, Lemma 2.4] shows that the assertion of lemma remains true. Here we omit the details. □

For the multi-valued nonlinearity $F : [0, b] \times X \to 2^X$ with convex, closed values, we have the following standing assumptions:

(H_1) $F(t, \cdot)$ is weakly u.s.c. for a.e. $t \in [0, b]$ and $F(\cdot, x)$ has a L^p-integrable selection for each $x \in X$,

(H_2) there exists $\alpha \in L^p(0, b; \mathbb{R}^+)$ such that

$$\|F(t, x)\| := \sup\{\|y\|;\, y \in F(t, x)\} \le \alpha(t)(1 + \|x\|)$$

for a.e. $t \in [0, b]$ and each $x \in X$,

where $p > 1$ and $pq > 1$.

We present an approximation lemma in the following, which is a slightly modified version of Lemma 3.3 in [10] (see also [17, Theorem 3.5], [12, Lemma 2.2]). We here omit the details for simplicity.

Lemma 2.5 *Assume that conditions* (H_1) *and* (H_2) *are satisfied. Then there exists a sequence of multi-valued functions* $\{F_n\}$ *with* $F_n : [0, b] \times X \to C_v(X)$ *such that*

(i) $F(t, x) \subset F_{n+1}(t, x) \subset F_n(t, x) \subset \overline{co}(F(t, B_{3^{1-n}}(x)))$ *for each* $n \ge 1$, $t \in [0, b]$ *and* $x \in X$;

(ii) $\|F_n(t, x)\| \le \alpha(t)(2 + \|x\|)$ *for each* $n \ge 1$, *a.e.* $t \in [0, b]$ *and* $x \in X$;

(iii) *there exists* $E \subset [0, b]$ *with* $mes(E) = 0$ *such that for each* $x^* \in X^*$, $\epsilon > 0$, *and* $(t, x) \in [0, b] \backslash E \times X$, *there exists a positive integer* $N > 0$ *such that for all* $n \ge N$,

$$x^*(F_n(t, x)) \subset x^*(F(t, x)) + (-\epsilon, \epsilon);$$

(iv) $F_n(t, \cdot) : X \to C_v(X)$ *is continuous for a.e.* $t \in [0, b]$ *with respect to Hausdorff metric for each* $n \ge 1$;

(v) *for each* $n \ge 1$, *there exists a selection* $g_n : [0, b] \times X \to X$ *of* F_n *such that* $g_n(\cdot, x)$ *is* L^p-*integrable* $(p > 1)$ *for each* $x \in X$ *and for any compact subset* $K \subset X$ *there exist constants* $C_V > 0$ *and* $\delta > 0$ *for which the estimate*

$$\|g_n(t, x_1) - g_n(t, x_2)\| \le C_V \alpha(t) \|x_1 - x_2\| \tag{2.3}$$

holds for a.e. $t \in [0, b]$ *and each* $x_1, x_2 \in V$ *with* $V := K + B_\delta(0)$;

(vi) *if* X *is reflexive, then* F_n *verifies* (H_1) *with* F *replaced by* F_n *for each* $n \ge 1$.

Remark 2.2 We note in particular that in the lemma above, it is assumed that $F(\cdot, x)$ has a L^p-integrable selection for each $x \in X$, which is different from those used in many previous papers such as [9,10]. But it is exactly what we need in the current situation.

3 Main results

In this section, we shall first study the R_δ-structure of set of solutions to Cauchy problem (1.1).

The following compactness characterizations of solution sets to Cauchy problem (2.2) will be useful.

Lemma 3.1 *Suppose that $T(t)$ is compact for $t > 0$. Let $D \subset X$ be relatively compact and $K \subset L^p(0,b;X)$ L^p-integrable bounded, that is,*

$$\|f(t)\| \le \chi(t) \text{ for all } f \in K \text{ and a.e. } t \in [0,b],$$

where $\chi \in L^p(0,b;\mathbb{R}^+)$. Then the set of mild solutions

$$\{u(\cdot,u_0,f);\ u_0 \in D,\ f \in K\}$$

is relatively compact in $C([0,b];X)$.

Proof Write

$$\Omega(D\times K) = \{u(\cdot,u_0,f);\ u_0 \in D,\ f \in K\}.$$

Let $t \in (0,b]$ be arbitrary and $\epsilon,\delta > 0$ small enough. Define the operator $J_{\epsilon,\delta} : \Omega(D\times K)(t) \to X$ by

$$J_{\epsilon,\delta}u(t) = Q(t)u_0 + T(\epsilon^q\delta)\int_0^{t-\epsilon}\int_\delta^\infty \times q\tau(t-s)^{q-1}\Psi_q(\tau)T((t-s)^q\tau - \epsilon^q\delta)f(s)\mathrm{d}\tau\mathrm{d}s$$

for $u(t) \in \Omega(D\times K)(t)$. Noticing $\int_0^\infty s^\gamma\Psi_q(s)\mathrm{d}\sigma = \frac{\Gamma(1+\gamma)}{\Gamma(1+q\gamma)}$, $\gamma \in [0,1]$, we have

$$\left\|\int_0^{t-\epsilon}\int_\delta^\infty q\tau(t-s)^{q-1}\Psi_q(\tau)T((t-s)^q\tau - \epsilon^q\delta)f(s)\mathrm{d}\tau\mathrm{d}s\right\| \le \frac{qM}{\Gamma(1+q)}\left((b^{\frac{pq-1}{p-1}} - \epsilon^{\frac{pq-1}{p-1}})\frac{p-1}{pq-1}\right)^{\frac{p-1}{p}}\|\chi\|_{L^p(0,b)},$$

which together with the compactness of $T(t)$ and $Q(t)$ for $t > 0$ enables us to conclude that the set $\{J_{\epsilon,\delta}u(t);\, u(t) \in \Omega(D\times K)(t)\}$ is relatively compact in X. Moreover, it follows that

$$\begin{aligned}
&\|J_{\epsilon,\delta}u(t) - u(t)\| \\
&\le qM\int_0^t (t-s)^{q-1}\chi(s)\mathrm{d}s\int_0^\delta \tau\Psi_q(\tau)\mathrm{d}\tau \\
&\quad + qM\int_{t-\epsilon}^t (t-s)^{q-1}\chi(s)\mathrm{d}s\int_\delta^\infty \tau\Psi_q(\tau)\mathrm{d}\tau \\
&\le qM\left(\frac{p-1}{pq-1}\right)^{\frac{p-1}{p}}\|\chi\|_{L^p(0,b)} \left(b^{q-\frac{1}{p}}\int_0^\delta \tau\Psi_q(\tau)\mathrm{d}\tau + \frac{\epsilon^{q-\frac{1}{p}}}{\Gamma(1+q)}\right) \\
&\to 0 \quad \text{as} \quad \epsilon \to 0,\delta \to 0.
\end{aligned}$$

Accordingly, we get

$$\|J_{\epsilon,\delta}u(t) - u(t)\| \to 0 \quad \text{as}\epsilon \to 0,\delta \to 0$$

uniformly for $t \in (0,b]$ and for $u(t) \in \Omega(D\times K)(t)$. This proves that the identity operator $I : \Omega(D\times K)(t) \to \Omega(D\times K)(t)$ is a compact operator, which yields that the set $\Omega(D\times K)(t)$ is relatively compact in X for each $t \in (0,b]$.

We proceed to verify that the set $\Omega(D\times K)$ is equicontinuous on $(0,b]$. Taking $0 < t_1 < t_2 \le b$ and $\delta > 0$ small enough, we obtain that $u \in \Omega(D\times K)$,

$$\|u(t_1) - u(t_2)\| \le I_1 + I_2 + I_3 + I_4 + I_5, \quad u \in \Omega(D\times K),$$

where

$$\begin{aligned}
I_1 &= \|Q(t_1) - Q(t_2)\|\cdot\|u_0\|, \\
I_2 &= \frac{qM}{\Gamma(1+q)}(t_2-t_1)^{q-\frac{1}{p}}\left(\frac{p-1}{pq-1}\right)^{\frac{p-1}{p}}\|\chi\|_{L^p(0,b)}, \\
I_3 &= \sup_{s\in[0,t_1-\delta]}\|P(t_2-s) - P(t_1-s)\| \\
&\quad \times\left((t_1^{\frac{pq-1}{p-1}} - \delta^{\frac{pq-1}{p-1}})\frac{p-1}{pq-1}\right)^{\frac{p-1}{p}}\|\chi\|_{L^p(0,b)}, \\
I_4 &= \frac{2qM}{\Gamma(1+q)}(\delta)^{q-\frac{1}{p}}\left(\frac{p-1}{pq-1}\right)^{\frac{p-1}{p}}\|\chi\|_{L^p(0,b)}, \\
I_5 &= \frac{qM}{\Gamma(1+q)}\left((t_2-t_1)^{\frac{pq-1}{p-1}} + t_1^{\frac{pq-1}{p-1}} - t_2^{\frac{pq-1}{p-1}}\right)^{\frac{p-1}{p}} \\
&\quad \times\left(\frac{p-1}{pq-1}\right)^{\frac{p-1}{p}}\|\chi\|_{L^p(0,b)}.
\end{aligned}$$

Therefore, it is not difficult to see that I_i $(i = 2,4,5)$ tends to zero as $t_2 - t_1 \to 0, \delta \to 0$. Also, from Lemma

2.3(ii) it follows that $I_1, I_3 \to 0$ as $t_2 - t_1 \to 0, \delta \to 0$. Moreover, we see by the relative compactness of D that these limits remain true uniformly for $u \in \Omega(D \times K)$. That is to say that

$$\|u(t_1) - u(t_2)\| \to 0 \quad \text{as}\, t_2 - t_1 \to 0,$$

uniformly for $u \in \Omega(D \times K)$ and hence we get the desired result.

Thus, an application of Arzela–Ascoli's theorem justifies that $\Omega(D \times K)$ is relatively compact in $C([0, b]; X)$. This completes the proof. □

Let $c \in [0, b)$. Consider the singular integral equation of the form

$$u(t) = \varphi(t) + \int_c^t (t-s)^{q-1} P(t-s) g(s, u(s)) \mathrm{d}s, \quad t \in [c, b]. \tag{3.1}$$

We present the following existence result, which will be used to prove the contractibility of the solution set.

Lemma 3.2 *Let $p > 1$, $pq > 1$, and $T(t)$ be compact for $t > 0$. Assume that $g : [c, b] \times X \to X$ is a function such that $g(\cdot, x)$ is L^p-integrable for every $x \in X$. Suppose in addition that*

(1) *for any compact subset $K \subset X$ there exist $\delta > 0$ and $L_K \in L^p([c, b]; \mathbb{R}^+)$ such that*

$$\|g(t, x_1) - g(t, x_2)\| \le L_K(t)\|x_1 - x_2\|$$

for a.e. $t \in [c, b]$ and each $x_1, x_2 \in B_\delta(K)$;

(2) *there exists $\mu \in L^p([c, b]; \mathbb{R}^+)$ such that $\|g(t, x)\| \le \mu(t)(c' + \|x\|)$ for a.e. $t \in [c, b]$ and every $x \in X$, where c' is arbitrary, but fixed.*

Then Eq. (3.1) admits a unique solution for every $\varphi \in C([c, b]; X)$. Moreover, the solutions of equation (3.1) depend continuously on φ.

Proof Let $\varphi \in C([c, b]; X)$ be fixed. Write

$$B_\rho(\varphi, \xi) = \{u \in C([c, \xi]; X);\ \max_{t \in [c, \xi]} \|u(t) - \varphi(t)\| \le \rho\}$$

with

$$\frac{qM}{\Gamma(1+q)} (\xi - c)^{q - \frac{1}{p}} \left(\frac{p-1}{pq-1}\right)^{\frac{p-1}{p}} \times \|\mu\|_{L^p(c,\xi)} (c' + \rho + \max_{t \in [c, \xi]} \|\varphi(t)\|) \le \rho.$$

Let us define the operator W:

$$(Wu)(t) = \varphi(t) + \int_c^t (t-s)^{q-1} P(t-s) g(s, u(s)) \mathrm{d}s.$$

Then W mapping $B_\rho(\varphi, \xi)$ into itself is continuous due to conditions (1) and (2). Moreover, from the proof of Lemma 3.1 we see that W is a compact operator. Thus, there is a fixed point of W, denoted by u, which is a local solution to Eq. (3.1). In fact, this solution is unique due to condition (1). Also, making use of the known Gronwall type inequality, one can derive a universal bound for all mild solutions of Eq. (3.1) (if they exist).

In the sequel, the operator W is treated as a mapping from $C([c, b]; X)$ to $C([c, b]; X)$. Define the operator

$$\Psi : [c, b] \times C([c, b]; X) \to C([c, b]; X)$$

with

$$\Psi(t, v)(s) = \begin{cases} v(s), & s \in [c, t], \\ v(t), & s \in [t, b]. \end{cases}$$

Put

$$J = \{t \in [c, b];\ v^t \in C([c, b]; X),\ v^t = \Psi(t, W(v^t))\}.$$

Note that $u^\xi = \Psi(\xi, W(u^\xi))$ with $u^\xi = \Psi(\xi, u)$, which means $\xi \in J$, i.e., $J \ne \emptyset$. Moreover, it is easy to see that $[c, t] \subset J$ for all $t \in J$.

Let $\{t_n\} \subset J$ a monotonically increasing sequence such that t_n tends to $t_0 = \sup J$ as $n \to \infty$. Noticing that

$$\Psi(t_m, u^{t_m}) = \Psi(t_m, W(u^{t_m})),$$
$$\Psi(t_m, u^{t_n}) = \Psi(t_m, W(u^{t_n}))$$

on $[c, t_m]$ when $m \le n$, we obtain that $u^{t_m}(s) = u^{t_n}(s)$ for all $s \in [0, t_m]$. Also, note that

$$\|u^{t_m}(t_0) - u^{t_n}(t_0)\| = \|u^{t_m}(t_m) - u^{t_n}(t_n)\|.$$

Therefore, by the continuity of φ we conclude, using a similar argument with that in Lemma 3.1, that

$$\|u^{t_m}(t_0) - u^{t_n}(t_0)\| \to 0 \quad \text{as}\ n, m \to \infty.$$

Accordingly, the limit $\lim_{n\to\infty} u^{t_n}(t_0)$ exists.

Consider the function

$$u^{t_0}(s) = \begin{cases} u^{t_n}(s), & s \in [c, t_n], \\ \lim_{n\to\infty} u^{t_n}(t_0), & s \in [t_0, b], \end{cases}$$

where $n \to \infty$, which maps $[c, b]$ into X. It follows from the equicontinuity of the family $\{u^{t_n}\}$ that u^{t_0} is continuous. Also, note that $u^{t_0}(t) = W(u^{t_0})(t)$ for all $t \in [c, t_0)$. Moreover, it is easy to verify, by Lebesgue's Dominated Convergence Theorem, that

$$u^{t_0}(t_0) = \lim_{n\to\infty}\left(\varphi(t_n) + \int_c^{t_n} (t_n - s)^{q-1} P(t_n - s) g(s, u^{t_n}(s)) \mathrm{d}s\right) = \varphi(t_0) + \int_c^{t_0} (t_0 - s)^{q-1} P(t_0 - s) g(s, u^{t_0}(s)) \mathrm{d}s.$$

Thus, we find that

$u^{t_0} = \Psi(t_0, W(u^{t_0}))$,

which yields that $t_0 \in J$.

Next, we show that $t_0 = b$. If this is not the case, then $t_0 < b$. Put

$$\widetilde{\varphi}(t) = \varphi(t) + \int_c^{t_0} (t - s)^{q-1} P(t - s) g(s, u^{t_0}(s)) \mathrm{d}s$$

with $\widetilde{\varphi} \in C([t_0, b]; X)$. As we can see there exists $\xi' > 0$ such that the following integral equation:

$$u(t) = \widetilde{\varphi}(t) + \int_{t_0}^{t} (t - s)^{q-1} P(t - s) g(s, u(s)) \mathrm{d}s.$$

has a solution $w \in C([t_0, t_0 + \xi']; X)$.

Denote

$$u^{t_0+\xi'}(s) = \begin{cases} u^{t_0}(s), & s \in [c, t_0], \\ w(s), & s \in [t_0, t_0 + \xi'], \\ w(t_0 + \xi'), & s \in [t_0 + \xi', b]. \end{cases}$$

Then it is clear that $u^{t_0+\xi'} \in C([c, b]; X)$. Moreover, one finds that

$$u^{t_0+\xi'}(t) = \varphi(t) + \int_c^{t} (t-s)^{q-1} P(t-s) g(s, u^{t_0+\xi'}(s)) \mathrm{d}s$$

for $t \in [c, t_0 + \xi']$, which implies that

$u^{t_0+\xi'} = \Psi(t_0 + \xi', W(u^{t_0+\xi'}))$.

This yields that $t_0 + \xi' \in J$, a contradiction.

Finally, Let $\varphi_n \to \varphi_0$ in $C([c, b]; X)$ as $n \to \infty$ and u_n the solution of Eq. (3.1) with the perturbation φ_n, i.e.,

$$u_n(t) = \varphi_n(t) + \int_c^{t} (t - s)^{q-1} P(t - s) g(s, u_n(s)) \mathrm{d}s, \quad t \in [c, b], n \geq 1. \tag{3.2}$$

Then from condition (2) and the compactness of $T(t)$ for $t > 0$ it follows that the set

$$\left\{ \int_c^{t} (t - s)^{q-1} P(t - s) g(s, u_n(s)) \mathrm{d}s; n \geq 1 \right\}$$

is relatively compact in $C([c, b]; X)$. This gives that the family $\{u_n\}$ is relatively compact in $C([c, b]; X)$. We assume, by passing to a subsequence if necessary, that $u_n \to \widetilde{u}$ in $C([c, b]; X)$ as $n \to \infty$. Therefore, taking the limit in (3.2) as $n \to \infty$, one finds, again by Lebesgue's Dominated Convergence Theorem, that $\widetilde{u}$ is the solution of Eq. (3.1) with the perturbation φ_0. This completes the proof. □

With the help of conditions (H_1)–(H_2), it can be demonstrated that the superposition function $F(t, u(t))$ admits a L^p-integrable selection for each $u \in C([0, b]; X)$. Consider the selection set of F: for each $u \in C([0, b]; X)$,

$$\mathrm{Sel}_F(u) = \{f \in L^p(0, b; X); f(t) \in F(t, u(t)) \text{ for } a.e.\ t \in [0, b]\}.$$

It will provide some useful properties of Sel_F in the following lemma.

Lemma 3.3 *Let conditions (H_1) and (H_2) be satisfied. Suppose in addition that X is reflexive. Then $\mathrm{Sel}_F : C([0, b]; X) \to 2^{L^p(0,b;X)}$ is weakly u.s.c. with nonempty, convex and weakly compact values.*

Proof Let us first show that $\mathrm{Sel}_F(u)$ is nonempty for each $u \in C([0, b]; X)$. To this aim, let $\{u_n\}$ be a sequence of step functions such that $u_n \to u$ in $C([0, b]; X)$. Therefore, from (H_1) it follows that $F(\cdot, u_n(\cdot))$ has a selection $f_n(\cdot) \in L^p(0, b; X)$ for each $n \geq 1$. Moreover, in view of (H_2) we have $\{f_n\}$ is bounded in $L^p(0, b; X)$. Then, applying Lemma 2.1 yields that $\{f_n\}$ is weakly relatively compact in $L^p(0, b; X)$. We may assume, by passing to a subsequence if necessary, that $f_n \to f$ weakly in $L^p(0, b; X)$. An application of Mazur's theorem enables us to find that there exists a sequence $\{\tilde{f}_n\} \subset L^p(0, b; X)$ such that $\tilde{f}_n \in co\{f_k; k \geq n\}$ for each $n \geq 1$ and $\tilde{f}_n \to f$ in $L^p(0, b; X)$. Hence, $\tilde{f}_{n_k}(t) \to f(t)$ in X for a.e. $t \in [0, b]$ with some subsequence $\{\tilde{f}_{n_k}\}$ of $\{\tilde{f}_n\}$.

Denote by J the set of all $t \in [0, b]$ such that $\tilde{f}_{n_k}(t) \to f(t)$ in X and $f_n(t) \in F(t, u_n(t))$ for all

$n \geq 1$. Let $x^* \in X^*$, $\epsilon > 0$, and $t \in J$ be fixed. From (H_1), it follows immediately that $(x^* \circ F)(t, \cdot) : X \to 2^{\mathbb{R}}$ is u.s.c. with compact convex values, so $\epsilon - \delta$ u.s.c. with compact convex values. Accordingly, we have

$$x^*(\tilde{f}_{n_k}(t)) \in \text{co}\{x^*(f_k(t)); k \geq n\} \subset x^*(F(t, u_n(t)) \subset x^*(F(t, u(t)) + (-\epsilon, \epsilon)$$

with k large enough. Therefore, we obtain that $x^*(f(t)) \in x^*(F(t, u(t))$ for each $x^* \in X^*$ and $t \in J$. Since F has convex and closed values, we conclude that $f(t) \in F(t, u(t))$ for each $t \in J$, which implies that $f \in \text{Sel}_F(u)$. This proves the desired result.

Finally, the similar argument (with $\{u_n\} \subset C([0, b]; X)$ instead of the step functions) together with Lemma 2.2 (ii) shows that Sel_F is weakly u.s.c. with convex and weakly compact values, completing the proof. □

Remark 3.1 We will make use of the L^p-integrable selection established in Lemma 3.3 to ensure the existence of solutions to Cauchy problem (1.1) in the sequel. One wishes to point out that in [10], solutions of given problems were constructed as fixed points of a suitable transformation with contractible values, relying on a strongly measurable selection for weakly upper semi-continuous multi-valued functions.

Here $u \in C([0, b]; X)$ is called a mild solution of Cauchy problem (1.1) if u is the mild solution of Cauchy problem (2.2) with some $f \in \text{Sel}_F(u)$.

We denote by Sf the unique mild solution to Cauchy problem (2.2) corresponding to $f \in L^p(0, b; X)$ for simplicity.

Now we are able to prove

Theorem 3.1 *Let conditions (H_1) and (H_2) be satisfied. Suppose in addition that X is reflexive and $T(t)$ is compact for $t > 0$. Then the solution set of Cauchy problem (1.1) for fixed $u_0 \in X$ is a nonempty compact subset of $C([0, b]; X)$. Moreover, it is a compact R_δ-set. In particular, it is connected.*

Proof We shall find a compact convex subset of $C([0, b]; X)$ which is invariant under

$$G := S \circ \text{Sel}_F.$$

It follows from Lemma 3.3 that Sel_F is weakly u.s.c. with convex and weakly compact values. Moreover, using Lemma 2.2(ii), Lemma 2.4 (with $B = 0$), and Lemma 3.1 an similar argument with that in [10, pp. 2052–2053] enables us to find that $G : C([0, b]; X) \to 2^{C([0,b];X)}$ is quasi-compact and closed. This yields that G is u.s.c due to [23, Theorem 1.1.12].

Let $\beta \in C[0, b]$ be the unique continuous solution of the integral equation

$$\beta(t) = a_1 + a_2 \int_0^t (t - s)^{q-1} \alpha(s)\beta(s)\text{d}s, \quad t \in [0, b],$$

in which a_1 and a_2 are defined as

$$a_1 = M\|u_0\| + \frac{qMb^{q-\frac{1}{p}}}{\Gamma(1+q)} \left(\frac{p-1}{pq-1}\right)^{\frac{p-1}{p}} \|\alpha\|_{L^p(0,b)}, \quad a_2 = \frac{qM}{\Gamma(1+q)}. \tag{3.3}$$

Therefore, if u is a mild solution of Cauchy problem (1.1), then it follows from (H_2) that for each $t \in [0, b]$,

$$\|u(t)\| \leq \|Q(t)u_0\| + \int_0^t (t-s)^{q-1} \|P(t-s) f(s)\| \text{d}s \leq \beta(t),$$

where $f \in \text{Sel}_F(u)$.

Write

$$D_0 = \{u \in C([0, b]; X); \|u(t)\| \leq \beta(t) \text{ for } t \in [0, b]\}, \quad \widetilde{D} = \overline{\text{conv}}(G(D_0)).$$

It is clear that D_0 is closed, bounded, and convex and $G(D_0) \subset D_0$. Also, $\widetilde{D}$ is invariant under G, i.e.,

$$G(\widetilde{D}) \subset \widetilde{D}.$$

Moreover, G has compact values and $\widetilde{D}$ is a compact set in $C([0.b]; X)$ since G is quasi-compact and closed.

Thus, we obtain, thanks to [10, Lemma 2.2], that the solution set of Cauchy problem (1.1) is nonempty if one can show that G has contractible values.

Given $u \in \widetilde{D}$. Fix $f^* \in \text{Sel}_F(u)$ and put $u^* = Sf^*$. Define a function $h : [0, 1] \times G(u) \to G(u)$ as

$$h(\lambda, v)(t) = \begin{cases} v(t), & t \in [0, \lambda b], \\ u(t; \lambda, v), & t \in (\lambda b, b] \end{cases}$$

for each $(\lambda, v) \in [0, 1] \times G(u)$, where

$$u(t; \lambda, v) = Q(t)u_0 + \int_0^{\lambda b} (t-s)^{q-1} P(t-s)\widetilde{f}(s)\mathrm{d}s + \int_{\lambda b}^{t} (t-s)^{q-1} P(t-s) f^*(s)\mathrm{d}s,$$

where $\widetilde{f} \in \mathrm{Sel}_F(u)$ such that $v = S\widetilde{f}$. It is easy to see that h is well defined and it is continuous. Moreover, note that

$$h(0, v) = u^* \text{ and } h(1, v) = v \quad \text{for every } v \in G(u).$$

Thus, it follows that G has contractible values.

Now, we let $M(u_0)$ denote the set of all mild solutions of Cauchy problem (1.1). One finds readily that $M(u_0)$ is a compact subset of $C([0.b]; X)$. Next, we are going to show that it is also an R_δ-set.

To this aim, let us consider the Cauchy problem of evolution inclusion of the form

$$\begin{cases} {}^{\mathrm{c}}D_t^q u(t) \in Au(t) + F_n(t, u(t)), & t \in [0, b], \\ u(0) = u_0 \in X, \end{cases} \tag{3.4}$$

where $n \geq 1$ and the sequence of multi-valued functions $\{F_n\}$ with $F_n : [0, b] \times X \to C_v(X)$ is established in Lemma 2.5.

From Lemma 2.5(ii) and (vi), it follows that F_n verifies the conditions (H_1) and (H_2) for each $n \geq 1$. Then from Lemma 3.3 one finds that Sel_{F_n} is weakly u.s.c. with convex and weakly compact values. Moreover, one can see from the above arguments that the solution set of Cauchy problem (3.4) is nonempty and compact in $C([0, b]; X)$ for each $n \geq 1$.

Let $M_n(u_0)$ denote the set of all mild solutions of Cauchy problem (3.4). We show that $M_n(u_0)$ is contractible for each $n \geq 1$. To do this, let $u \in M_n(u_0)$ and f_n be the selection of F_n, $n \geq 1$. We deal with the existence and uniqueness of solutions to the integral equation

$$v(t) = Q(t)u_0 + \int_0^{\lambda b} (t-s)^{q-1} P(t-s) f^u(s)\mathrm{d}s + \int_{\lambda b}^{t} (t-s)^{q-1} P(t-s) f_n(s, v(s))\mathrm{d}s \tag{3.5}$$

where $f^u \in \mathrm{Sel}_{F_n}(u)$. From Lemma 2.5(iv) and (v), we know $f_n(t, \cdot)$ is continuous for a.e. $t \in [0, b]$ and $f_n(\cdot, x)$ is L^p-integrable for each $x \in X$. Moreover, it follows from Lemma 2.5(ii) that

$$\|f_n(t, x)\| \leq \alpha(t)(2 + \|x\|)$$

for a.e. $t \in [0, b]$ and each $x \in X$. Therefore, noticing Lemma 2.5(v) one finds, thanks to Lemma 3.2, that Eq. (3.5) has a unique solution on $[\lambda b, b]$, denote it by $v(\cdot, \lambda, u)$.

We define a function

$$\Phi : [0, 1] \times M_n(u_0) \to M_n(u_0)$$

by

$$\Phi(\lambda, u)(t) = \begin{cases} u(t), & t \in [0, \lambda b], \\ v(t, \lambda, u), & t \in (\lambda b, b] \end{cases}$$

for each $(\lambda, u) \in [0, 1] \times M_n(u_0)$. It is easy to see that Φ is well defined. Also, it is clear that

$$\Phi(0, u) = v(\cdot, 0, u), \quad \Phi(1, u) = u \quad \text{on } M_n(u_0).$$

Moreover, it follows readily that Φ is continuous. Thus, we have proved that $M_n(u_0)$ is contractible for each $n \geq 1$.

Finally, it is easy to verify that

$$M(u_0) \subset \cdots \subset M_n(u_0) \subset \cdots \subset M_2(u_0) \subset M_1(u_0)$$

in view of Lemma 2.5(i), and

$$M(u_0) = \bigcap_{n=1}^{+\infty} M_n(u_0)$$

in view of Lemma 2.5(ii) and (iii), Lemmas 2.1 and 2.4 (cf., [10, Theorem 3.2]). Consequently, we conclude that $M(u_0)$ is an R_δ-set, completing this proof. □

Remark 3.2 It is noted that the R_δ-structure of the set of trajectories to Cauchy problem (1.1) under the weaker condition that the semigroup generated by A is only equicontinuous remains still an unsolved problem.

As an interesting application of Theorem 3.1, we shall deal with mild solutions of the Cauchy problem with nonlocal initial condition in the form

$$\begin{cases} {}^{\mathrm{c}}D_t^q u(t) = Au(t), & t \in [0, b], \\ u(0) = g(u), \end{cases} \tag{3.6}$$

where A is defined as that in the problem (1.1) with $M \leq 1$ (it is key) and

$$g : C([0, b]; X) \to X$$

is continuous. Assume also that

(H_g) for some $r > 0$, $\|g(u)\| \le r$ for all $u \in \Omega_r := \{u \in C([0,b]; X); \|u(t)\| \le r$ for all $t \in [0,b]\}$, and

(H_g') for each $\mathcal{Q} \subset \Omega_r$ which restricted to $[\eta, b]$ is relatively compact in $C([\eta, b]; X)$ for each $\eta \in (0,b)$, $g(\mathcal{Q})$ is relatively compact in X.

Let $M_0(u_0)$ denote the set of all mild solutions of Cauchy problem (3.6) with $g(u) = u_0$. Then from Theorem 3.1 it follows that for fixed $u_0 \in X$, $M_0(u_0)$ is a nonempty compact subset of $C([0,b]; X)$. Moreover, it is a compact R_δ-set, which implies, for given $r_0 > 0$, that the multi-valued map $M_0 : \Omega^0_{r_0} := \{x \in X; \|x\| \le r_0\} \to 2^{C([0,b];X)}$ is an R_δ-mapping.

Next, letting $u_0 \in \Omega^0_{r_0}$ and $u \in M_0(u_0)$ we have

$$\|u(t)\| = \|Q(t)u_0\| \le r_0.$$

Accordingly, we obtain $M_0(\Omega^0_{r_0}) \subset \Omega_{r_0}$.

Let

$$\begin{aligned}\hat{\Omega}^0_{r_0} &= \overline{\mathrm{conv}}(g(\Omega_{r_0})),\\ \tilde{\Omega}_{r_0} &= \overline{\mathrm{conv}}(M_0(\hat{\Omega}^0_{r_0})),\\ \tilde{\Omega}^0_{r_0} &= \overline{\mathrm{conv}}(g(\tilde{\Omega}_{r_0})).\end{aligned}$$

Note that $\tilde{\Omega}_{r_0}$ and $\tilde{\Omega}^0_{r_0}$ are AR-spaces and $M_0(\tilde{\Omega}^0_{r_0})$ is compact due to (H_g').

Now, by (H_g) it is not difficult to verify that the multi-valued mapping

$$M_0 \circ g : \tilde{\Omega}_{r_0} \to \tilde{\Omega}_{r_0}$$

is well defined. Therefore, from the result $M_0(g(\tilde{\Omega}_{r_0})) \subset M_0(\tilde{\Omega}^0_{r_0})$ and [10, Theorem 2.1] we infer that $M_0 \circ g$ admits a fixed point in $\tilde{\Omega}_{r_0}$, which in fact is a mild solution of Cauchy problem (3.6).

Remark 3.3 We mention that condition (H_g') is satisfied when condition (H_g) above and the following condition hold:

(H_g'') there exists $\delta' \in (0,b]$ such that for every $u, w \in \Omega_r$ satisfying $u(t) = w(t)$ ($t \in [\delta', b]$), $g(u) = g(w)$.

Let us note that condition (H_g'') is the case when the values of the solution $u(t)$ for t near zero do not affect $g(u)$, which was used in some situations of previous research such as [36,40].

In what follows, we treat the topological structure of control problem for fractional evolution inclusion of the form

$$\begin{cases} {}^cD_t^q u(t) \in Au(t) + F(t, u(t)) + Bw(t), t \in [0,b],\\ u(0) = u_0,\end{cases} \tag{3.7}$$

where $u_0 \in X$ is given, the control function w takes values in the Banach space V, B is a bounded linear operator from V to X, and A and F are defined the same as those in the problem (1.1).

We first consider the linear control problem

$$\begin{cases} {}^cD_t^q u(t) = Au(t) + f(t) + Bw(t), \quad t \in [0,b],\\ u(0) = u_0,\end{cases} \tag{3.8}$$

where $f \in L^p(0,b;X)$.

Denote by $u(\cdot, f, w)$ the unique mild solution of linear control problem (3.8).

We present the following compactness characterization of solution set to linear control problem (3.8), whose proof is very closely to that of Lemma 3.1.

Lemma 3.4 *Suppose that $T(t)$ is compact for $t > 0$. Let $p > 1, pq > 1$, $K \subset L^p(0,b;X)$ be L^p-integrable bounded and $\widetilde{K} \subset L^p(0,b;V)$ be bounded. Then the set of mild solutions*

$$\{u(\cdot, f, w); f \in K, w \in \widetilde{K}\}$$

is relatively compact in $C([0,b]; X)$.

The topological characterization to control problem (3.7) is given in the following theorem.

Theorem 3.2 *Let conditions (H_1), (H_2) be satisfied. Suppose in addition that X is reflexive and $T(t)$ is compact for $t > 0$. Then the solution set of control problem (3.7) for fixed $w \in L^p(0,b;V)$ is a compact R_δ-set.*

Proof Let $\hat{\beta} \in C[0,b]$ be the unique continuous solution of the integral equation

$$\hat{\beta}(t) = a_1' + a_2 \int_0^t (t-s)^{q-1}\alpha(s)\hat{\beta}(s)\mathrm{d}s, \quad t \in [0,b],$$

in which a_2 is the constant appearing in (3.3) and a_1' is defined as

$$a_1' = M\|u_0\| + \frac{qMb^{q-\frac{1}{p}}}{\Gamma(1+q)}\left(\frac{p-1}{pq-1}\right)^{\frac{p-1}{p}} \times \left(\|\alpha\|_{L^p(0,b)} + \|Bw\|_{L^p(0,b;X)}\right).$$

Denote by $S'f$ the unique mild solution to linear control problem (3.8) corresponding to $f \in L^p(0,b;X)$. Just the same with the proof of Theorem 3.1, we can find a compact convex subset $\widetilde{E}$ of $C([0,b];X)$ which is invariant under

$$G' := S' \circ \mathrm{Sel}_F,$$

where

$$\widetilde{E} = \overline{\mathrm{conv}}(G'(E_0)),$$
$$E_0 = \{u \in C([0,b];X);\ \|u(t)\| \le \hat{\beta}(t) \text{ for } t \in [0,b]\}.$$

With the above preparation, the rest of the proof follows from an argument similar to the latter part in the proof of Theorem 3.1 (with Lemma 3.4 instead of Lemma 3.1). We here omit the proof for simplicity. The proof is completed. □

Remark 3.4 The extra condition: X is reflexive in Theorem 3.2 can be dropped the case of F being a single-valued function.

As an application of Theorem 3.2, we can prove the invariance of a reachability set of control problem (3.7) under single-valued nonlinear perturbations.

Define the operator $T : L^p(0,b;X) \to C([0,b];X)$ by

$$(Tf)(t) = \int_0^t (t-s)^{q-1} P(t-s) f(s)\mathrm{d}s,$$
$$f \in L^p(0,b;X),$$

where $p > 1$ and $pq > 1$. Evidently T is a linear bounded operator.

Let us assume that

- (H_3) the single-valued function $F : [0,b] \times X \to X$ is continuous and $\|F(t,x)\| \le \alpha(t)(1+\|x\|)$ for each $t \in [0,b]$, $x \in X$, where α is the function appearing in (H_2),
- (H_4) for each $f \in L^p(0,b;X)$, there exists $w \in L^p(0,b;V)$ such that $(T(Bw))(b) = (Tf)(b)$.

Remark 3.5 (H_4) first introduced in Seidman [31] is fulfilled if B is surjective.

The set

$$K_F = \{u(b,F,w);\ w \in L^p(0,b;V)\}$$

is called the reachability set of control problem (3.7). By K_0 we denote the reachability set for the corresponding linear problem ($F \equiv 0$).

We refer the reader to [8] for the basic notions and facts of control problems.

Theorem 3.3 *Let $p > 1$, $pq > 1$ and conditions (H_3), (H_4) be satisfied. Suppose in addition that $T(t)$ is compact for $t > 0$. Then there exists $r_0 > 0$ such that the reachability set of control problem (3.7) is invariant under nonlinear perturbations, i.e., $K_F = K_0$ if $\|B\|_{V\to X} < r_0$.*

Proof Let $N(w)$ for fixed $w \in L^p(0,b;V)$ denote the set of all mild solutions of control problem (3.7). Then by Theorem 3.2 we see that $N(w)$ is an R_δ-set for every $w \in L^p(0,b;V)$. Furthermore, it is easy to verify that the multi-valued mapping $N : L^p(0,b;V) \to 2^{C([0,b];X)}$ is an R_δ-mapping. Based on this, an argument similar to that in Seidman [31] enables us to find that the assertion of the theorem remains true. This completes the proof. □

4 Examples

In this section, we present examples showing how to apply our abstract results to specific problem.

Consider the following system of partial differential inclusion:

$$\begin{cases} {}^{c}D_t^q u(t,\xi) - u_{\xi\xi}(t,\xi) \in \\ F(t,\xi,u(t,\xi)), \quad (t,\xi) \in [0,b] \times [0,\pi], \\ u(t,0) = u(t,\pi) = 0, \quad t \in [0,b], \\ u(0,\xi) = u_0(\xi), \quad \xi \in [0,\pi], \end{cases} \tag{4.1}$$

where ${}^{c}D_t^q$, $\frac{1}{2} < q < 1$, is the regularized Caputo fractional derivative of order q, that is,

$${}^{c}D_t^q u(t,\xi) = \frac{1}{\Gamma(1-q)} \times \left(\frac{\partial}{\partial t} \int_0^t (t-s)^{-q} u(s,\xi)\mathrm{d}s - t^{-q} u(0,\xi) \right),$$

and

$$F(t,\xi,u) = [f_1(t,\xi,u), f_2(t,\xi,u)]$$

is a closed interval for each $(t,\xi,u) \in [0,b] \times [0,\pi] \times \mathbb{R}$.

We let the functions

$$f_i : [0,b] \times [0,\pi] \times \mathbb{R} \to \mathbb{R}, \quad i = 1,2$$

be such that

- (F_1) f_1 is l.s.c. and f_2 is u.s.c.,
- (F_2) $f_1(t,\xi,u) \le f_2(t,\xi,u)$ for each $(t,\xi,u) \in [0,b] \times [0,\pi] \times \mathbb{R}$,

(F_3) there exist $\alpha_1, \alpha_2 \in L^\infty(0, b; \mathbb{R}^+)$ such that

$$|f_i(t, \xi, u)| \le \alpha_1(t)|u| + \alpha_2(t), \quad i = 1, 2$$

$$\text{for each } (t, \xi, u) \in [0, b] \times [0, \pi] \times \mathbb{R},$$

Take $X = L^2(0, \pi)$ and denote its norm by $\|\cdot\|$ and inner product by $(\cdot, \cdot)$. Now we are able to prove

Theorem 4.1 *Let the conditions (F_1)–(F_3) be satisfied. Then the set of all mild solutions to system (4.1) is a compact R_δ-set. In particular, it is connected.*

Proof From our assumptions on f_1 and f_2, it follows readily that the multi-valued function

$$F(\cdot, \cdot, \cdot) : [0, b] \times [0, \pi] \times \mathbb{R} \to 2^{\mathbb{R}}$$

is u.s.c. with nonempty, convex, and compact values. Also, we see, for given $x \in X$, that $(t, \xi) \to f_i(t, \xi, x(\xi)), i = 1, 2$, are measurable on $[0, \pi]$. Moreover, it is easy to verify that

$$\max\{|f_1(t, \xi, x(\xi))|, |f_2(t, \xi, x(\xi))|\} \le \alpha_1(t)|x(\xi)| + \alpha_2(t)$$

a.e. for $(t, \xi) \in [0, b] \times [0, \pi]$, which yields that $\xi \mapsto \alpha_1(t)|x(\xi)| + \alpha_2(t)$ belongs to X a.e for $t \in [0, b]$. Therefore, we conclude that $(t, \xi) \mapsto f_i(t, \xi, x(\xi)), i = 1, 2$, belong to $L^2([0, b]; X)$. Thus, the multi-valued function $F : [0, b] \times X \to 2^X$ define as

$$F(t, x) = \{f \in X; f(\xi) \in [f_1(t, \xi, x(\xi)), f_2(t, \xi, x(\xi))] \text{a.e. in } [0, \pi]\}$$

for each $(t, x) \in [0, b] \times X$, has nonempty and convex values and $F(\cdot, x)$ has a L^2-integrable selection for each $x \in X$. Also, it follows that for each $f \in F(t, x)$,

$$|f(\xi)| \le \|\alpha_1\|_{L^\infty(0,b)}|x(\xi)| + \|\alpha_2\|_{L^\infty(0,b)}$$

a.e. for $\xi \in [0, \pi]$. This proves that $F(t, x)$ is bounded in X. Therefore, noticing that X is a Hilbert space, we conclude that $F(t, x)$ is weakly compact. Moreover, it is to see that $F(t, \cdot)$ is weakly u.s.c. for a.e. $t \in [0, b]$ (cf., e.g., [33, Lemma 5.1]) and a direct calculation shows that for each $f \in F(t, x)$,

$$\|f\| \le \alpha(t)(1 + \|x\|)$$

a.e. for each $t \in [0, b], x \in X$, where $\alpha(\cdot) = \sqrt{\pi} \max\{\alpha_1(\cdot), \alpha_2(\cdot)\}$.

The operator $A : D(A) \subset X \to X$ is defined as

$$Ax = x'', \quad x \in D(A), \quad D(A) = H^2(0, \pi) \cap H_0^1(0, \pi).$$

As in Pazy [30], A has a discrete spectrum and its eigenvalues are $-n^2, n \in \mathbb{N}^+$ with the corresponding normalized eigenvectors $x_n(\xi) = \sqrt{\frac{2}{\pi}} \sin(n\xi)$. Moreover, A generates a compact C_0-semigroup $\{T(t)\}_{t\ge0}$ on X such that $\|T(t)\| \le e^{-t}$ for all $t \ge 0$. Denote by $E_{q,l}$ the generalized Mittag–Leffler special function defined by

$$E_{q,l}(t) = \sum_{j=0}^{\infty} \frac{t^j}{\Gamma(qj + l)} \quad q, l > 0, t \in \mathbb{R}$$

(cf., e.g., [40]). Therefore, we have that for $x \in X$,

$$Q(t)x = \sum_{n=1}^{\infty} E_q(-n^2 t^q)(x, x_n)x_n,$$

$$\|Q(t)\| \le 1 \quad \text{for all } t \ge 0,$$

$$P(t)x = \sum_{n=1}^{\infty} e_q(-n^2 t^q)(x, x_n)x_n,$$

$$\|P(t)\| \le \frac{q}{\Gamma(1+q)} \quad \text{for all } t \ge 0,$$

where $E_q(t) := E_{q,1}(t)$ and $e_q(t) := E_{q,q}(t)$. According to the compactness of $T(t)$ for $t > 0$, we know that $Q(t)$ and $P(t)$ are compact operators for $t > 0$ (see [36,42]).

Then, system (4.1) can be rewritten as an abstract Cauchy problem of the form (1.1). Accordingly, Theorem 3.1 can apply to the situation, and hence we assert that the set of all mild solutions to system (4.1) is a compact R_δ-set. The proof is completed. □

We continue to use the setting as in the above example and consider the system (4.1) replacing the boundary condition with homogeneous Neumann boundary condition, i.e.,

$$u_\xi(t, 0) = u_\xi(t, \pi) = 0, \quad t \in [0, b].$$

In this case, let the operator $A : D(A) \subset X \to X$ is defined as

$$Ax = x'', \quad x \in D(A),$$
$$D(A) = \{x \in H^2(0, \pi) : x'(0) = x'(\pi) = 0\}.$$

Then the operator $-A$ has eigenvalues $0 = \lambda_1 < \lambda_2 < \cdots < \lambda_n \to \infty$ with the property that λ_j has finite multiplicity Γ_j (which equals to the dimension of the corresponding eigenspace). Also, there is a complete set $\{\Phi_{j,k}\}$ of eigenvectors of $-A$ and the operator A generates an analytic semigroup $\{T(t)\}_{t\ge0}$ on X defined by

$$T(t)x = E_1 x + \sum_{j=2}^{\infty} e^{-\lambda_j t} E_j x, \quad x \in X,$$

where the family $\{E_j\}$ is a family of complete orthonormal projections in X and

$$E_j x = \sum_{k=1}^{\Gamma_j} (x, \Phi_{j,k}) \Phi_{j,k}$$

Moreover, using the standard energy estimate method and Sobolev embedding theorems one finds that $\|T(t)\| \leq 1$ for all $t \geq 0$ and $T(t)$ is compact for each $t > 0$ (see, e.g., [25,41] for more details).

With the preparation above at hand, we conclude that the set of all mild solutions for the above system is a compact R_δ-set, which can be proved by the same kind of manipulations as in the proof of Theorem 4.1.

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