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Non-linear fractional field equations: weak non-linearity at power-law non-locality

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Abstract Fractional non-relativistic field equations with the derivatives of non-integer order are considered. A connection of these equations with microscopic (lattice) models is discussed. The considered equations contain non-linear terms and fractional Laplacian in the Riesz form. Using the background field method and the mean field method, we obtain corrections to linear solution and equilibrium solution caused by the weak non-linearity.

Keywords Fractional field equations · Fractional derivative · Fractional dynamics · Background field method · Mean field method

1 Introduction

The theory of integration and differentiation of any arbitrary real (or complex) orders have a long history [1–4] and different fractional derivatives and integrals have been suggested by Riemann, Liouville, Riesz, Caputo, Grünwald, Letnikov, Marchaud, Weyl, Sonin, and others [4–7]. The fractional derivatives have a lot of unusual properties. For example, the fractional derivatives are noncommutative and nonassociative operators in general [5]. A violation of the usual Leibniz rule is a characteristic property for all types of fractional derivatives [8]. The fractional derivatives of products of two or more functions are represented as infinite series with derivatives and integrals of different non-integer orders [5]. The formula of fractional derivative of a composite function has a complex form (see Sect. 2.7.3 in [9]). The different fractional derivatives are related to each other. For example, the Grünwald–Letnikov derivatives coincide with the Marchaud derivatives for wide class of functions (see Sects. 20.2 and 20.3 in [4]) and the fact that the Marchaud derivatives coincide with Liouville derivatives (see Sects. 5.4 in [4]). In applications of fractional calculus, it is very important the non-commutativity and non-associativity actions of fractional derivatives and integrals, the violation of the Leibniz rule, and that fractional time evolution does not satisfy the semigroup property. These unusual properties of fractional integro-differentiation allow us to describe the unusual properties of complex systems, media, and processes with non-locality of power-law type, long-term memory, and fractality. Despite the difficulties, the fractional calculus has a wide application in mechanics and physics (for example see [10–19]). Moreover the theory of derivatives and integrals of non-integer orders [4–6] with respect to coordinates is a very powerful tool to describe the behavior of distributed systems that are characterized by non-locality of power-law type and fractality.

Various aspects of the fractional generalization of the field theory have been actively studied now (see for example [18,20–23]). In this paper, we consider non-relativistic field equations with the Riesz fractional

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derivatives of non-integer order. We demonstrate that this equation can be derived from microscopic (lattice) models with long-range interaction and non-linear external fields. Using the background field method and the mean field method, we obtain corrections to solutions of linear fractional equations and to the equilibrium solution, which are caused by the weak non-linearity.

2 Fractional differential equations for scalar fields

Let us consider a classical field model of distributed system, where states are described by scalar field $\varphi(\mathbf{x})$ in the n-dimensional space $\mathbb{R}^n$. For example, the field $\varphi(\mathbf{x})$ can describe the ordered field in the fluctuation theory of phase transitions [24,25], thermodynamic field in non-equilibrium thermodynamics [26], or the field functions in the continuum mechanics [27,28]. Note that nonlinear models in continuum mechanics has a wide applications [28–34].

Let us consider the nonlinear fractional differential equation

$$g((-\Delta)^{\alpha/2}\varphi)(\mathbf{x}) + \mu\varphi(\mathbf{x}) + \varepsilon N(\varphi(\mathbf{x})) = j(\mathbf{x}) \quad (\alpha > 0), \tag{1}$$

where $N(\varphi(\mathbf{x}))$ is the nonlinear function, g is the coupling constant, μ is the scale (or mass) parameter, $j(\mathbf{x})$ is the external field, ε is a small parameter of non-linearity. Here $(-\Delta)^{\alpha/2}$ is the fractional Laplacian in the Riesz form [5]. As a simple example of the nonlinear function we can consider

$$N(\varphi) = \varphi^3(\mathbf{x}). \tag{2}$$

Equation (1) with (2) is the fractional Ginzburg–Landau equation (see for example [35–37]). The synchronization effects for non-linear media, which is described by (1), with the power-law non-locality defined by long-range inter-particle interaction are considered in [38–41].

We note that Eq. (1) can be derived by continuous limit from the lattice models with long-range interactions [42,43]. In [42,43] we prove that the continuum equations with fractional Laplacian in the Riesz form [4,5] can be directly derived from lattice models with different types of long-range interactions (see also [38,39,44]).

The fractional Laplacian $(-\Delta)^{\alpha/2}$ in the Riesz's form, which is used in Eq. (1), can be defined as the inverse Fourier's integral transform $\mathcal{F}^{-1}$ of $|\mathbf{k}|^\alpha$ by

$$((-\Delta)^{\alpha/2} f)(x) = \mathcal{F}^{-1}\Big(|k|^\alpha (\mathcal{F} f)(k)\Big), \tag{3}$$

where $\alpha > 0$ and $x \in \mathbb{R}^n$. For $\alpha > 0$, the fractional Laplacian in the Riesz's form usually is defined in the form of the hypersingular integral by

$$((-\Delta)^{\alpha/2} f)(x) = \frac{1}{d_n(m,\alpha)} \int_{\mathbb{R}^n} \frac{1}{|z|^{\alpha+n}} (\Delta_z^m f)(z) \mathrm{d}z,$$

where $m > \alpha$, and $(\Delta_z^m f)(z)$ is a finite difference of order m of a function $f(x)$ with a vector step $z \in \mathbb{R}^n$ and centered at the point $x \in \mathbb{R}^n$:

$$\left(\Delta_z^m f\right)(z) = \sum_{k=0}^{m} (-1)^k \frac{m!}{k!(m-k)!} f(x - kz).$$

The constant $d_n(m,\alpha)$ is defined by

$$d_n(m,\alpha) = \frac{\pi^{1+n/2} A_m(\alpha)}{2^\alpha \Gamma(1+\alpha/2)\Gamma(n/2+\alpha/2)\sin(\pi\alpha/2)},$$

where

$$A_m(\alpha) = \sum_{j=0}^{m} (-1)^{j-1} \frac{m!}{j!(m-j)!} j^\alpha.$$

Note that the hypersingular integral $((-\Delta)^{\alpha/2} f)(x)$ does not depend on the choice of $m > \alpha$. The Fourier transform $\mathcal{F}$ of the fractional Laplacian is given by $(\mathcal{F}(-\Delta)^{\alpha/2} f)(k) = |k|^\alpha (\mathcal{F} f)(k)$. This equation is valid for the Lizorkin space [4] and the space $C^\infty(\mathbb{R}^n)$ of infinitely differentiable functions on $\mathbb{R}^n$ with compact support.

3 Derivation of non-linear fractional field equation from the lattice model

In this section, we describe a connection of nonlinear Eq. (1) with microscopic (lattice) models. Let us consider a lattice model where all particles are displaced in one direction, and we assume that the displacement of particle from its equilibrium position is determined by a scalar field. The equations for one-dimensional lattice system of interacting particles have the form

$$-\frac{g_0}{M} \sum_{\substack{m=-\infty \\ m\neq n}}^{+\infty} K(n,m) \Big(\varphi_n(t) - \varphi_m(t)\Big) + \frac{\mu_0}{M}\varphi_n(t) + \frac{\varepsilon_0}{M} N(\varphi_n(t)) = j_n(t), \tag{4}$$

where $\varphi_n(t) = \varphi(n,t)$ is the displacement of n particle from its equilibrium position, g_0 is the coupling constant for inter-particle interactions in the lattice, the terms $\varepsilon_0 N$ characterize an non-linear interaction of the particles with the external on-site force, $\mu_0\varphi_n(t)$ is the linear external force, $j_n(t)$ is the external source. For simplicity, we assume that all particles have the same mass M. The elements $K(n,m)$ of Eq. (4) describe the inter-particle interaction in the lattice. For an unbounded homogeneous lattice, due to its homogeneity $K(n,m)$ has the form $K(n,m) = K(n-m)$. Equation (4) has the invariance with respect to its displacement of lattice as a whole in case of absence of external forces. It should be noted that the noninvariant terms lead to the divergences in the continuous limit [18].

In order to define the operation that transforms the lattice equations for $\varphi_n(t)$ into the continuum equation for a scalar field $\varphi(x,t)$, we use the methods suggested in [42,43]. We consider $\varphi_n(t)$ as Fourier series transform $\mathcal{F}_\Delta$ of some function $\hat{\varphi}(k,t)$ on $[-k_0/2, k_0/2]$, then we use the continuous limit (Lim) in the form $k_0 \to \infty$ to get $\tilde{\varphi}(k,t)$, and finally we apply the inverse Fourier integral transformation $\mathcal{F}^{-1}$ to obtain $\varphi(x,t)$. Diagrammatically this can be written in the following form:

$$\varphi_n(t) \xrightarrow{\mathcal{F}_\Delta} \hat{\varphi}(k,t) \xrightarrow{\text{Lim}} \tilde{\varphi}(k,t) \xrightarrow{\mathcal{F}^{-1}} \varphi(x,t). \tag{5}$$

We performed the similar transformation for differential equations to map the lattice equation into the equation for the elastic continuum. We can represent the set of operation in the form of the following diagrams.

$$\begin{array}{ccc} \varphi_n(t) & \xrightarrow{\text{From Particle to Field}} & \varphi(x,t) \\ \mathcal{F}_\Delta \downarrow & & \uparrow \mathcal{F}^{-1} \\ \hat{\varphi}(k,t) & \xrightarrow[\text{Lim } \Delta x \to 0]{} & \tilde{\varphi}(k,t) \end{array} \tag{6}$$

Therefore the transform operation that map our lattice model into a continuum model is a sequence of the following three actions (for details see [42,43]):

1. The Fourier series transform $\mathcal{F}_\Delta : \varphi_n(t) \to \mathcal{F}_\Delta\{\varphi_n(t)\} = \hat{\varphi}(k,t)$ that is defined by

$$\hat{\varphi}(k,t) = \sum_{n=-\infty}^{+\infty} \varphi_n(t)\, e^{-ikx_n} = \mathcal{F}_\Delta\{\varphi_n(t)\}, \tag{7}$$

$$\varphi_n(t) = \frac{1}{k_0} \int_{-k_0/2}^{+k_0/2} dk\, \hat{\varphi}(k,t)\, e^{ikx_n} = \mathcal{F}_\Delta^{-1}\{\hat{\varphi}(k,t)\}, \tag{8}$$

where $x_n = n\Delta x$ and $\Delta x = 2\pi/k_0$ is the inter-particle distance. To simplify our consideration we assume that all lattice particles have the same inter-particle distance Δx.

2. The passage to the limit $\Delta x \to 0$ $(k_0 \to \infty)$ denoted by $\text{Lim} : \hat{\varphi}(k,t) \to \text{Lim}\{\hat{\varphi}(k,t)\} = \tilde{\varphi}(k,t)$. The function $\tilde{\varphi}(k,t)$ can be derived from $\hat{\varphi}(k,t)$ in the limit $\Delta x \to 0$. Note that $\tilde{\varphi}(k,t)$ is a Fourier integral transform of the field $\varphi(x,t)$, and $\hat{\varphi}(k,t)$ is a Fourier series transform of $\varphi_n(t)$, where we use

$$\varphi_n(t) = \frac{2\pi}{k_0}\varphi(x_n,t)$$

considering $x_n = n\Delta x = 2\pi n/k_0 \to x$.

3. The inverse Fourier's integral transform $\mathcal{F}^{-1} : \tilde{\varphi}(k,t) \to \mathcal{F}^{-1}\{\tilde{\varphi}(k,t)\} = \varphi(x,t)$ that is defined by

$$\tilde{\varphi}(k,t) = \int_{-\infty}^{+\infty} \mathrm{d}x\, e^{-ikx}\varphi(x,t) = \mathcal{F}\{\varphi(x,t)\}, \tag{9}$$

$$\varphi(x,t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \mathrm{d}k\, e^{ikx}\tilde{\varphi}(k,t) = \mathcal{F}^{-1}\{\tilde{\varphi}(k,t)\}. \tag{10}$$

The combination of these three actions $\mathcal{F}^{-1}$ Lim $\mathcal{F}_\Delta$ allows us to realize the transformation of lattice models into continuum models [42,43].

In the continuous limit the equations for lattice with interaction of power-law type [18,42,43] gives the fractional field equation. Note that Eqs. (7) and (8) in the limit $\Delta x \to 0$ $(k_0 \to \infty)$ gives the Fourier integral transform Eqs. (9) and (10), where the sum is changed by integral.

In the continuous limit $\Delta x \to 0$, the lattice Eq. (4) with the long-range interaction of power-law type gives (for details see [18]) the fractional field equation

$$g\,((-\Delta)^{\alpha/2}u)(x,t) + \mu\varphi(x,t) + \varepsilon\, N(\varphi(x,t)) = j(x,t), \tag{11}$$

with the fractional Laplacian $(-\Delta)^{\alpha/2}$ of order α. Here the variables x and Δx are dimensionless,

$$g = \frac{g_0 \, |\Delta x|^{\alpha}}{M}, \quad \mu = \frac{\mu_0}{M}, \quad \varepsilon = \frac{\varepsilon_0 (\Delta x)^2}{M} \tag{12}$$

are the finite parameters.

4 Particular solution of linear fractional equation

Let us derive a particular solution of Eq. (1) with $N(\varphi) = 0$. To solve the linear fractional differential equation

$$g \, ((-\Delta)^{\alpha/2} \varphi)(\mathbf{x}) + \mu \, \varphi(\mathbf{x}) = j(\mathbf{x}), \tag{13}$$

we apply the Fourier method, which is based on the relation

$$\mathcal{F}[(-\Delta)^{\alpha/2} \varphi(\mathbf{x})](\mathbf{k}) = |\mathbf{k}|^{\alpha} \, \hat{\varphi}(\mathbf{k}). \tag{14}$$

Applying the Fourier transform $\mathcal{F}$ to both sides of (13) and using (14), we have

$$(\mathcal{F}\varphi)(\mathbf{k}) = \left(g \, |\mathbf{k}|^{\alpha} + \mu\right)^{-1} (\mathcal{F} j)(\mathbf{k}). \tag{15}$$

The fractional analog of the response function that can be called the fractional Green function (see Sect. 5.5.1. in [5]) is given by

$$G^n_{\alpha}(\mathbf{x}) = \mathcal{F}^{-1}\Big[\left(g \, |\mathbf{k}|^{\alpha} + \mu\right)^{-1}\Big](\mathbf{x}) = \int_{\mathbb{R}^n} \left(g \, |\mathbf{k}|^{\alpha} + \mu\right)^{-1} \, e^{+i(\mathbf{k},\mathbf{x})} \, d^n\mathbf{k}. \tag{16}$$

The function (16) can be simplified (Lemma 25.1 of [4]) by using the relation

$$\int_{\mathbb{R}^n} e^{i(\mathbf{k},\mathbf{x})} \, f(|\mathbf{k}|) \, d^n\mathbf{k} = \frac{(2\pi)^{n/2}}{|\mathbf{x}|^{(n-2)/2}} \int_0^{\infty} f(\lambda) \, \lambda^{n/2} \, J_{(n-2)/2}(\lambda|\mathbf{x}|) \, \mathrm{d}\lambda, \tag{17}$$

where J_{ν} is the Bessel function of the first kind. As a result, the Fourier transform of a radial function is also a radial function.

Using relation (17), the fractional Green function (16) can be represented (see Theorem 5.22 in [5]) in the form of the integral with respect to one parameter λ by

$$G^n_{\alpha}(\mathbf{x}) = \frac{|\mathbf{x}|^{(2-n)/2}}{(2\pi)^{n/2}} \int_0^{\infty} \frac{\lambda^{n/2} J_{(n-2)/2}(\lambda|\mathbf{x}|)}{g\lambda^{\alpha} + \mu} \mathrm{d}\lambda, \tag{18}$$

where $n = 1, 2, 3$ and $\alpha > (n-1)/2$, and $J_{(n-2)/2}$ is the Bessel function of the first kind.

Using the Theorem 5.22 and Corollary from [5] for the case $\mu \neq 0$ and $\alpha > (n-1)/2$, we can state that Eq. (28) is solvable, and its particular solution is given by

$$\varphi(\mathbf{x}) = G^n_{\alpha} * j = \int_{\mathbb{R}^n} G^n_{\alpha}(\mathbf{x} - \mathbf{x}') \, j(\mathbf{x}') \mathrm{d}\mathbf{x}', \tag{19}$$

where $G^n_{\alpha}(\mathbf{x})$ is defined by (18), and the asterisk (or star) $*$ is the convolution operation.

For the 3-dimensional case, we can use

$$J_{1/2}(z) = \sqrt{\frac{2}{\pi z}} \, \sin(z), \tag{20}$$

and we have

$$G^3_{\alpha}(\mathbf{x}) = \frac{1}{2\pi^2 |\mathbf{x}|} \int_0^{\infty} \frac{\lambda \sin(\lambda|\mathbf{x}|)}{g\lambda^{\alpha} + \mu} \mathrm{d}\lambda. \tag{21}$$

For the 1-dimensional case, we use

$$J_{-1/2}(z) = \sqrt{\frac{2}{\pi z}} \, \cos(z). \tag{22}$$

Then we have (see Theorem 5.24 in [5]) the function

$$G^1_{\alpha}(\mathbf{x}) = \frac{1}{\pi} \int_0^{\infty} \frac{\cos(\lambda|\mathbf{x}|)}{g\lambda^{\alpha} + \mu} \mathrm{d}\lambda. \tag{23}$$

Let us consider the field $\varphi(\mathbf{x})$, appearing from a point source $j(\mathbf{x})$, that is placed to the origin of coordinates, such that

$$j(\mathbf{x}) = j_0 \delta(\mathbf{x}). \tag{24}$$

In the electrodynamics the point source means that we consider a point charge in the media [45]. In continuum mechanics the point source means that we consider the Thomson's problem (1848) [46]. This problems means that we should determine the deformation of an infinite continuum, when a force is applied to a small region in it [47,48].

For the case (24), the scalar field $\varphi(\mathbf{x})$ has a simple form of the particular solution that is proportional to the Green's function

$$\varphi(\mathbf{x}) = j_0 \, G^n_{\alpha}(\mathbf{x}). \tag{25}$$

As a result, the field for the source at a point (24) has the form

$$\varphi(\mathbf{x}) = \frac{1}{2\pi^2} \frac{j_0}{|\mathbf{x}|} \int_0^{\infty} \frac{\lambda \sin(\lambda|\mathbf{x}|)}{g\lambda^{\alpha} + \mu} \mathrm{d}\lambda. \tag{26}$$

This is the solution of the linear fractional differential Eq. (18) for $n = 3$ and the point source of field $j(\mathbf{x})$.

5 Background field method: deviation from linear states

Suppose that $\varphi(\mathbf{x}) = \varphi_0(\mathbf{x})$ is a solution of Eq. (1) with $\varepsilon = 0$, i.e. $\varphi_0(\mathbf{x})$ is a solution of the linear equation

$$g((-\Delta)^{\alpha/2}\varphi_0)(\mathbf{x}) + \mu\varphi_0(\mathbf{x}) = j(\mathbf{x}). \tag{27}$$

This is the linear fractional differential equation. The solution of this equation has the form (19).

We will seek a solution of nonlinear Eq. (1) with $\varepsilon \neq 0$ in the form

$$\varphi(\mathbf{x}) = \varphi_0(\mathbf{x}) + \varepsilon\,\varphi_1(\mathbf{x}) + \cdots . \tag{28}$$

This means that we consider perturbations with respect to the background field $\varphi_0(\mathbf{x})$. It allows us to use the background field method (in general form this method is described in [49–51]).

In this case, Eq. (27) is an approximation of the zero order. The first order approximation with respect to ε gives the equation

$$g((-\Delta)^{\alpha/2}\varphi_1)(\mathbf{x}) + \mu\varphi_1(\mathbf{x}) + N(\varphi_0(\mathbf{x})) = 0. \tag{29}$$

This equation is equivalent to the linear equation

$$g((-\Delta)^{\alpha/2}\varphi_1)(\mathbf{x}) + \mu\varphi_1(\mathbf{x}) = j_{\text{eff}}(\mathbf{x}) \tag{30}$$

with the effective external field

$$j_{\text{eff}}(\mathbf{x}) = -N(\varphi_0(\mathbf{x})). \tag{31}$$

The solution of Eq. (29) has the form

$$\begin{aligned}\varphi(\mathbf{x}) &= \varphi_0(\mathbf{x}) + \varphi_1(\mathbf{x}) = G^n_\alpha * j + \varepsilon G^n_\alpha * j_{\text{eff}}\\ &= G^n_\alpha * j - \varepsilon G^n_\alpha * N\left(G^n_\alpha * j\right),\end{aligned} \tag{32}$$

where the asterisk (or star) $*$ denotes the convolution operation. As a result, we have

$$\varphi(\mathbf{x}) = G^n_\alpha * j - \varepsilon G^n_\alpha * N\left(G^n_\alpha * j\right). \tag{33}$$

For the case of point source (25) Eq. (33) has the form

$$\varphi(\mathbf{x}) = j_0 G^n_\alpha(\mathbf{x}) - \varepsilon\Big(G^n_\alpha * N\left(j_0 G^n_\alpha\right)\Big)(\mathbf{x}). \tag{34}$$

For the non-linearity (2), we have

$$\varphi(\mathbf{x}) = j_0\, G^n_\alpha(\mathbf{x}) - \varepsilon j_0^3\Big(G^n_\alpha * \left(G^n_\alpha(\mathbf{x})\right)^3\Big)(\mathbf{x}). \tag{35}$$

6 Mean field method: deviation from equilibrium state

Equilibrium value of $\varphi_0 = \text{const}$ (where $(-\Delta)^{\alpha/2}\varphi_0 = 0$) and $j(\mathbf{x}) = h = \text{const}$ is defined by the condition

$$\mu\varphi_0 + \varepsilon N(\varphi_0) = h. \tag{36}$$

For example, if the non-linear function has the form (2), then we have the equation

$$\mu\varphi_0 + \varepsilon\varphi_0^3 = h. \tag{37}$$

For $h \neq 0$, there is no solution $\varphi_0 = 0$. For $\mu > 0$ and the weak external fields $h \ll h_c$ with respect to the critical value $h_c = \sqrt{\mu^3/\varepsilon}$, there exists only one solution

$$\varphi_0 \approx h/\mu. \tag{38}$$

For $\mu < 0$ and in the absence of an external field $h = 0$, we have three solution

$$\varphi_0 \approx \pm\sqrt{|\mu|/\varepsilon}, \quad \varphi_0 = 0. \tag{39}$$

For the values $h < (2\sqrt{3}/9)h_c$, also exist three solutions. For strong external fields $h \gg h_c$, we can neglect the first term ($\mu \approx 0$),

$$\varepsilon\varphi_0^3 \approx h, \tag{40}$$

and we get

$$\varphi_0 \approx (h/\varepsilon)^{1/3} = \sqrt[3]{h/\varepsilon}. \tag{41}$$

In any cases the equilibrium values φ_0 are solutions of the algebraic Eq. (36).

Let us consider a deviation $\varphi_1(\mathbf{x})$ of the field $\varphi(\mathbf{x})$ from the equilibrium value φ_0. For this purpose we will seek a solution in the form

$$\varphi(\mathbf{x}) = \varphi_0 + \varphi_1(\mathbf{x}). \tag{42}$$

Since the external field is generally not constant $j(\mathbf{x}) \neq h$, we get the equation for the first approximation

$$\begin{aligned}&g\left((-\Delta)^{\alpha/2}\varphi_1\right)(\mathbf{x})\\ &\quad+\Big(\mu + \varepsilon\, N'_\varphi(\varphi_0)\Big)\varphi_1(\mathbf{x}) = j(\mathbf{x}),\end{aligned} \tag{43}$$

where $N'_\varphi = \partial N(\varphi)/\partial\varphi$. Equation (43) is equivalent to the linear fractional differential equation

$$g((-\Delta)^{\alpha/2}\varphi_1)(\mathbf{x}) + \mu_{\text{eff}}\varphi_1(\mathbf{x}) = j(\mathbf{x}) \tag{44}$$

with the effective parameter

$$\mu_{\text{eff}} = \mu + \varepsilon N'_\varphi(\varphi_0). \tag{45}$$

If $N(\varphi) = \varphi^3$, then

$$\mu_{\text{eff}} = \mu + 3\varepsilon\varphi_0.$$

The solution of Eq. (44) has the form (19), where μ is replaced by μ_{eff}. For the case of point source (25) Eq. (33) has the form

$$\varphi(\mathbf{x}) = \frac{1}{2\pi^2}\frac{j_0}{|\mathbf{x}|}\int\limits_0^\infty \frac{2g\,\lambda^\alpha + \mu + \mu_{\text{eff}}}{(g\lambda^\alpha + \mu)(g\lambda^\alpha + \mu_{\text{eff}})} \sin(\lambda|\mathbf{x}|)d\lambda. \tag{46}$$

Let us consider the field $\varphi_1(\mathbf{x})$, appearing from a point source of field $j(\mathbf{x}) = j_0\delta(\mathbf{x})$, that is placed to the origin of coordinates. The solution of Eq. (42) with $\alpha = 2$ for the external field (25) has the form

$$\varphi_1(\mathbf{x}) = \frac{j_0}{4\pi g|\mathbf{x}|} \exp\Big(-|\mathbf{x}|/r_c\Big), \tag{47}$$

where the value r_c is called the correlation radius and

$$r_c^2 = \frac{g}{\mu + \varepsilon N'_{\varphi}(\varphi_0)}. \tag{48}$$

Note that $\varphi_1(\mathbf{x})$ coincides with correlator $\langle\varphi(\mathbf{x})\varphi(\mathbf{x})\rangle = \varphi_1(\mathbf{x})$ in the fluctuation theory of phase transitions [24]. In the electrodynamics the field $\varphi_1(\mathbf{x})$ describes the Coulomb potential with the Debye's screening. For fractional differential field equation ($\alpha \neq 2$), we have the power-law type of screening that is described in the paper [45]. The electrostatic potential for media with power-law spatial dispersion differs from the Coulomb's potential by the factor

$$C_{\alpha,0}(|\mathbf{r}|) = \frac{2}{\pi} \int\limits_0^{\infty} \frac{\lambda \sin(\lambda|\mathbf{r}|)}{g\lambda^{\alpha} + \mu_{\text{eff}}} d\lambda. \tag{49}$$

Note that the Debye's potential differs from the Coulomb's potential by the exponential factor $C_D(|\mathbf{r}|) = \exp(-|\mathbf{r}|/r_D)$ (for details see [45]).

7 Some applications in physics

Let us briefly describe the possible applications of the suggested method for physical models.

The Ising model can be defined by the Hamiltonian

$$H_I = \frac{I}{2}\sum_{\mathbf{x},\mathbf{a}}\Big(\varphi(\mathbf{x})-\varphi(\mathbf{x}+\mathbf{a})\Big)^2+\lambda\sum_{\mathbf{x}}\Big(\varphi^2(\mathbf{x})-\varphi_0^2\Big)^2. \tag{50}$$

Continuum analog of the Ising model is defined by the Hamiltonian

$$H_c = \int d\mathbf{x}\Big(\frac{c}{2}(\nabla\varphi)^2 + \lambda'\Big(\varphi^2-\varphi_0^2\Big)^2\Big)$$
$$= \int d\mathbf{x}\Big(\frac{c}{2}(\nabla\varphi)^2 \frac{b}{2}\varphi^2 + \lambda'\varphi^4 + \text{const}, \tag{51}$$

where

$$c = Ia^{2-n}, \quad b = -2\lambda\varphi_0^2 a^{-n}, \quad \lambda' = \lambda a^{-n}$$

Here $a = |\mathbf{a}|$ is the lattice constant, n is the dimension of the space. For $n > 2$ exists the phase transition [24].

Fractional generalizations of continuum Ising model (51) allows us to take into account a power-law non-locality which is caused by long-range interactions of lattice particles. For fractional continuum Ising model with external forces, we should use the mean field method and the solutions that are defined by (46).

We can consider the Landau theory of phase transitions with the free energy functional [24]

$$F\{\varphi\} = F\{\varphi_0\}\frac{1}{2}\int d^n\mathbf{x}\Big(c(\nabla\varphi)^2$$
$$+ b\varphi^2 + \frac{c}{2}\varphi^4 - 2h\varphi\Big) \tag{52}$$

where φ describes the field of order parameters. A fractional generalization of the Landau theory of phase transitions to describe transition for non-local continuum. Using the mean field method, we can get the solutions that have the form (46).

To study non-local effect for wide class of magnetic materials, we can consider the fractional generalization of Ginzburg–Landau model, which was devised to provide a simple general form of the effective Hamiltonian for magnetic systems (see Chap. 5 in [52]). Note that fractional generalization of Ginzburg–Landau models are described in [35,38–40]. For the fractional Ginzburg-Landau models, the suggested methods can be applied for the case of the weak non-linearity.

8 Conclusion

A classical field model of distributed system with power-law non-locality and weak non-linearity are suggested. The scalar field $\varphi(\mathbf{x})$ in the n-dimensional space $\mathbb{R}^n$ can describe the ordered field in the fluctuation theory of phase transitions [24,25], thermodynamic field in non-equilibrium thermodynamics [26], or the field functions in the continuum mechanics [27,28].

The suggested fractional nonlinear model can allow us to describe phase transitions for the non-local media with power-law long-range interactions in the framework of the fluctuation theory of phase transitions [24]. It allow us to describe non-linear effects in the elasticity and plasticity models of materials with power-law non-locality [47,48,53]. We also assume that suggested approach allows us to describe a weak non-linear effects in the dielectric materials and plasma-like media with power-law spatial dispersion [45]. The suggested approach can be used for nonlinear generalizations of fractional diffusion equations for open quantum systems [54]. Using the Lorentz invariant definition of the Riesz fractional derivatives suggested in [55] is possible to generalize suggested consideration for relativistic field theory. It is important to generalize a controllability of nonlinear fractional field and distributed systems [56], where fractional orders of derivatives are considered as control parameters. For the case $x \in \mathbb{R}^1$, we can consider the coordinate as a time variable $x = t$, and apply the suggested approach to mechanical systems [57–59]. The suggested fractional field theory can be generalized on the case of statistical field theory. To describe fluctuation processes in the distributed nonlocal continuum it is important to generalize the perturbation theory and diagram technique in the framework of the statistical field theory (see for example Sect. 5.4 in [52]).

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