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ORIGINAL PAPER

Stochastic solutions for fractional wave equations

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Abstract A fractional wave equation replaces the second time derivative by a Caputo derivative of order between one and two. In this paper, we show that the fractional wave equation governs a stochastic model for wave propagation, with deterministic time replaced by the inverse of a stable subordinator whose index is one-half the order of the fractional time derivative.

Keywords Fractional calculus · Wave equation · Inverse subordinator · Stochastic solution

1 Introduction

The traditional wave equation

$$\frac{\partial^2}{\partial t^2} p(x,t) = \Delta_x p(x,t) \tag{1.1}$$

models wave propagation in an ideal conducting medium. Assume a plane wave solution $p(x,t) = e^{-i\omega t+ikx}$ for some frequency $\omega > 0$ and substitute into (1.1) to see that we must have $(-i\omega)^2 = (ik)^2$ or in other words $k = \pm\omega$. This solution is a traveling wave at speed one (justified by a suitable choice of units), and the general solution to (1.1) can be written as a linear combination of plane waves.

In a complex inhomogeneous conducting medium, experimental evidence reveals that sound waves exhibit power-law attenuation, with an amplitude that falls off at an exponential rate $\alpha = \alpha(\omega) \approx \omega^p$ for some power-law index p (e.g., see Duck [12] for applications to medical ultrasound). A variety of modified wave equations have been proposed to model wave conduction in complex media [11,14,22,23]. We note here, apparently for the first time, that a simple time-fractional wave equation

$$\frac{\partial^\gamma}{\partial t^\gamma} p(x,t) = \Delta_x p(x,t), \qquad 1 < \gamma < 2, \tag{1.2}$$

which replaces the second time derivative by a fractional derivative, also exhibits power-law attenuation. Assuming the same plane wave solution, and using the well-known formula $\frac{\mathrm{d}^\gamma}{\mathrm{d}t^\gamma}[\mathrm{e}^{at}] = t^\gamma \mathrm{e}^{at}$ (e.g., see [18, Example 2.6]), we now have $(-i\omega)^\gamma = -k^2$, and a little algebra yields $k = \beta(\omega) + i\alpha(\omega)$ with attenuation coefficient $\alpha(\omega) = \alpha_0\omega^{\gamma/2}$. Hence, solutions to the time-fractional wave equation (1.2) also exhibit power-law attenuation with power-law index $p = \gamma/2$.

The goal of this paper is to develop a new stochastic solution to the time-fractional wave equation (1.2). Our

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stochastic solution is based on limit theory for random walks and, therefore, provides a simple and illuminating statistical physics model for wave conduction in complex media.

2 Background

In one spatial dimension, the general solution to (1.1) with initial conditions $p(x,0) = \phi(x)$ and $\frac{\partial}{\partial t}p(x,0) = \psi(x)$ is given by the d'Alembert formula

$$p(x,t) = \frac{1}{2}\left[\phi(x+t)+\phi(x-t)\right] + \int\limits_{x-t}^{x+t} \psi(y)\mathrm{d}y. \tag{2.1}$$

In fact, for all continuous and exponentially bounded functions ϕ, ψ the unique solution to the equivalent integral equation

$$p(x,t) = \phi(x) + t\psi(x) + \int\limits_0^t (t-s)\Delta_x p(x,s)\mathrm{d}s \tag{2.2}$$

is given by the d'Alembert formula. In this paper, we consider the following integral form of the fractional wave equation:

$$p(x,t) = \phi(x) + \frac{t^{\gamma/2}}{\Gamma(1+\gamma/2)}\psi(x) + \frac{1}{\Gamma(\gamma)}\int\limits_0^t (t-s)^{\gamma-1}\Delta_x p(x,s)\mathrm{d}s. \tag{2.3}$$

The differential form of equation (2.3) employs the Riemann–Liouville fractional derivative. The Riemann–Liouville fractional integral of non-integer order $\gamma > 0$ is defined by

$$\mathbb{I}_t^\gamma f(t) = \frac{1}{\Gamma(\gamma)}\int\limits_0^\infty (t-s)^{\gamma-1} f(s)\mathrm{d}s. \tag{2.4}$$

The Riemann–Liouville fractional derivative of non-integer order $\gamma > 0$ is defined by

$$\mathbb{D}_t^\gamma f(t) = \frac{1}{\Gamma(n-\gamma)}\frac{\mathrm{d}^n}{\mathrm{d}t^n}\int\limits_0^\infty (t-s)^{n-\gamma-1} f(s)\mathrm{d}s \tag{2.5}$$

where n is the smallest integer greater than γ.

Equation (2.3) corresponds to the following fractional differential equation:

$$\mathbb{D}_t^\gamma p(x,t) - \phi(x)\frac{t^{-\gamma}}{\Gamma(1-\gamma)} - \psi(x)\frac{t^{-\gamma/2}}{\Gamma(1-\gamma/2)} = \Delta_x p(x,t) \tag{2.6}$$

with initial conditions

$$p(x,0) = \phi(x) \quad \text{and} \quad \frac{\partial^{\gamma/2}}{\partial t^{\gamma/2}} p(x,0) = \psi(x).$$

In this paper, we will show that the solution to (2.3) is

$$p(x,t) = \frac{1}{2}\mathbb{E}\left[\phi(x+E_t)+\phi(x-E_t)\right] + \frac{1}{2}\mathbb{E}\left[\int\limits_{x-E_t}^{x+E_t}\psi(y)\mathrm{d}y\right] \tag{2.7}$$

where E_t is the inverse (hitting time or first passage time) of a standard stable subordinator with index $\gamma/2$. Then, using the general theory of second-order Cauchy problems, we will extend this result to a wide variety of fractional partial differential equations that model wave-like motions. Finally, we will develop random walk models that provide a physical explanation for these fractional wave equations.

3 Fractional wave equations

Let D_u be a standard stable subordinator with $D_0 = 0$ a.s. and Laplace transform $\mathbb{E}[\mathrm{e}^{-sD_u}] = \mathrm{e}^{-us^\beta}$ for some $0 < \beta < 1$. The random variable D_1 has a smooth density function $g_\beta(u)$. Define the *inverse subordinator* (generalized inverse, first passage time, or hitting time)

$$E_t = \inf\{u \geq 0 : D_u > t\} \tag{3.1}$$

for $t \geq 0$. Then, a simple computation [16, Corollary 3.1] shows that E_t has a smooth density

$$u \mapsto h(u,t) = \frac{t}{\beta}u^{-1-1/\beta}g_\beta(tu^{-1/\beta}), \quad u>0,\ t>0. \tag{3.2}$$

Write $\mathbb{R}^+ = [0,\infty)$, let $\mathcal{B}(\mathbb{R}\times\mathbb{R}^+)$ denote the set of real-valued continuous functions $p(x,t)$ on $\mathbb{R}\times\mathbb{R}^+$ such that $|p(x,t)| \leq A\mathrm{e}^{B(|x|+t)}$ for some constants $A, B > 0$, and denote by $\mathcal{B}(\mathbb{R})$ the set of real-valued continuous functions $\phi(x)$ on $\mathbb{R}$ such that $|\phi(x)| \leq A\mathrm{e}^{B|x|}$ for some constants $A, B > 0$. Denote by $\mathcal{B}^m(\mathbb{R})$

the set of real-valued continuously differentiable functions $\phi(x)$ such that $\frac{d^j}{dx^j}\phi \in \mathcal{B}(\mathbb{R})$ for all integers $0 \le j \le m$ and by $\mathcal{B}^{m,0}(\mathbb{R} \times \mathbb{R}_+)$ the set of real-valued functions $p(x,t)$ on $\mathbb{R} \times \mathbb{R}^+$ continuously differentiable in x with $\frac{d^j}{dx^j} p(x,t) \in \mathcal{B}(\mathbb{R} \times \mathbb{R}_+)$ for all integers $0 \le j \le m$.

Theorem 3.1 *Let $\Psi(x) = \int_0^x \psi(y)dy$. For any ϕ, ψ such that ϕ, $\Psi \in \mathcal{B}^2(\mathbb{R})$, the unique solution to the fractional wave equation (2.3) in $\mathcal{B}^{2,0}(\mathbb{R} \times \mathbb{R}^+)$ with $p(x,0) = \phi(x)$ and $\frac{\partial^\beta}{\partial t^\beta} p(x,0) = \psi(x)$ is given by the formula (2.7), where E_t is the inverse stable subordinator (3.1) with index $\beta = \gamma/2$.*

Proof The proof uses a result of Fujita [13] together with a duality result from [3]. Fujita considers a stable Lévy process $X_\gamma(t)$ with characteristic function

$$\mathbb{E}\left[e^{ikX_\gamma(t)}\right] = \exp\left[-t|k|^{2/\gamma} e^{-i(\pi/2)(2-2/\gamma)\,\mathrm{sgn}(k)}\right]. \tag{3.3}$$

with index $1 < 2/\gamma < 2$ along with its supremum process

$$Y_\gamma(t) = \sup_{0 \le u \le t} X_\gamma(u). \tag{3.4}$$

Fujita shows that for ϕ, ψ such that ϕ, $\Psi \in \mathcal{B}^2(\mathbb{R})$, the unique solution to (2.3) in $\mathcal{B}^{2,0}(\mathbb{R} \times \mathbb{R}^+)$ is

$$p(x,t) = \frac{1}{2}\mathbb{E}\left[\phi(x+Y_\gamma(t)) + \phi(x-Y_\gamma(t))\right] + \frac{1}{2}\mathbb{E}\left[\int_{x-Y_\gamma(t)}^{x+Y_\gamma(t)} \psi(y)\,dy\right]. \tag{3.5}$$

Using the parameterization of Samorodnitsky and Taqqu [21], the characteristic function of a generic stable process $\xi(t) = \xi_{\mu,\alpha,\sigma,\theta}(t)$ is

$$\mathbb{E}\left[e^{ik\xi(t)}\right] = \exp\left[ik\mu - \sigma^\alpha |k|^\alpha \times \{1 - i\theta\,\mathrm{sgn}(k)\tan(\pi\alpha/2)\}\right].$$

An elementary calculation (e.g., see [3, p. 1101]) shows that the process $X_\gamma(t)$ has stability index $\alpha = 2/\gamma$, skewness $\theta = -1$, scale $\sigma^\alpha = -t\cos(\pi\alpha/2) > 0$, and centering constant $\mu = 0$. Hence, $X_\gamma(t)$ is a *spectrally negative* stable Lévy process, with no positive jumps. Use the elementary formula (e.g., see [18, Eq. 5.5]) $(ik)^\alpha = |k|^\alpha \cos(\pi\alpha/2)(1 + i\,\mathrm{sgn}(k)\tan(\pi\alpha/2))$ to write $\mathbb{E}\left[e^{ikX_\gamma(t)}\right] = e^{t(ik)^\alpha}$ and then set $k = -is$ to see that

$$\mathbb{E}\left[e^{sX_\gamma(t)}\right] = e^{ts^\alpha}$$

for all $s \ge 0$ and $t \ge 0$. Now, it follows from [7, Theorem 1, p. 189] that the first-passage time process

$$D_u = \inf\{t \ge 0 : X_\gamma(t) > u\}$$

is a stable subordinator with Laplace transform $\mathbb{E}\left[e^{-sD_u}\right] = e^{-us^\beta}$ for all $u \ge 0$ and $s \ge 0$, where the stability index $\beta = 1/\alpha = \gamma/2$. Then, the inverse β-stable subordinator E_t in (3.1) is the generalized inverse of $X_\gamma(t)$, which equals the supremum of $X_\gamma(t)$. Hence, we have $E_t = Y_\gamma(t)$ pathwise, see also Proposition 1 in [9]. Then, the form of the solution follows from (3.5).

The integral form of the fractional wave equation (2.3) corresponds to the differential form (1.2) with the initial conditions $p(x,0) = \phi(0)$ and $\psi = \frac{\partial^\beta}{\partial x^\beta} p(x,t)$, $\beta = \gamma/2$. The first initial condition follows directly from (2.3). As for the second one, note that

$$\frac{1}{\Gamma(\gamma)} \int_0^t (t-s)^{\gamma-1} \Delta_x p(x,s) ds = \mathbb{I}_t^\gamma \Delta_x p(x,t)$$

and apply the Caputo derivative to both sides of (2.3) to get

$$\frac{\partial^\beta}{\partial t^\beta} p(x,t) = \psi(x) + \frac{\partial^\beta}{\partial t^\beta} \mathbb{I}_t^\gamma \Delta_x p(x,t).$$

Since Caputo and Riemann–Liouville derivatives of order $0 < \beta < 1$ are related by (e.g., see [18, p. 39])

$$\frac{\partial^\beta}{\partial t^\beta} f(t) = \mathbb{D}_t^\beta f(x) - f(0)\frac{t^{-\beta}}{\Gamma(1-\beta)},$$

and $\mathbb{I}_t^\gamma \Delta_x p(x,t)$ evaluated at $t = 0$ is zero, we have

$$\frac{\partial^\beta}{\partial t^\beta} \mathbb{I}_t^\gamma \Delta_x p(x,t) = \mathbb{D}_t^\beta \mathbb{I}_t^\gamma \Delta_x p(x,t) = \mathbb{I}_t^\beta \Delta_x p(x,t).$$

Since $p(x,t) \in \mathcal{B}^{2,0}(\mathbb{R} \times \mathbb{R}^+)$, $|\Delta_x p(x,s)| \le Ae^{B(|x|+t)}$ for some constants A, $B > 0$ and $0 \le s \le t$. Therefore,

$$\left|\mathbb{I}_t^\beta \Delta_x p(x,t)\right| \le \frac{Ae^{B(|x|+t)}}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} ds = \frac{Ae^{B(|x|+t)}}{\Gamma(\beta+1)} t^\beta \to 0 \text{ as } t \to 0.$$

Thus, the initial conditions corresponding to (2.3) are $p(x,0) = \phi(x)$ and $\frac{\partial^\beta}{\partial t^\beta} p(x,0) = \psi(x)$. □

In the remainder of this section, we discuss related results in the literature and give some alternative stochastic representations of the solution.

Remark 3.2 Define the *reflected stable process*

$$Z_t = X_\gamma(t) - \inf\{X_\gamma(s) : 0 \le s \le t\}, \tag{3.6}$$

where $X_\gamma(t)$ is the spectrally negative stable process with index $1 < \gamma \le 2$ and characteristic function (3.3). Apply [4, Lemma 4.5] to see that Z_t has the same one-dimensional distributions as the inverse (3.1) of a standard β-stable subordinator with $\beta = \gamma/2$. Then, it follows from Theorem 3.1 that for any ϕ, ψ such that $\phi, \Psi \in \mathcal{B}^2(\mathbb{R})$, the unique solution to the fractional wave equation (1.2) in $\mathcal{B}^{2,0}(\mathbb{R}\times\mathbb{R}^+)$ with $p(x,0) = \phi(x)$ and $\frac{\partial^\beta}{\partial t^\beta} p(x,0) = \psi(x)$ is given by the formula

$$p(x,t) = \frac{1}{2}\mathbb{E}\left[\phi(x+Z_t) + \phi(x-Z_t)\right] + \frac{1}{2}\mathbb{E}\left[\int_{x-Z_t}^{x+Z_t} \psi(y)\,dy\right]. \tag{3.7}$$

The advantage to this representation is that Z_t is a Markov process.

Remark 3.3 Mainardi [15, Sect. 6.3] considers a version of (1.2) that employs the Caputo fractional derivative

$$\frac{\partial^\gamma}{\partial t^\gamma} p(x,t) = \frac{1}{\Gamma(2-\gamma)} \int_0^t \frac{\partial^2}{\partial u^2} p(x,u)(t-u)^{1-\gamma} du \tag{3.8}$$

of order $1 < \gamma < 2$. Mainardi [15, Sect. 6.4] derives the fractional wave equation (1.2) from a viscoelastic model with a power-law stress–strain relationship. He notes that the Green's function solution to the fractional wave equation (1.2) can be also expressed in terms of stable densities. He considers the fractional wave equation (1.2) subject to the initial conditions $p(x,0) = \delta(x)$ and $\frac{\partial}{\partial t} p(x,0) = 0$. Since the Caputo and Riemann-Liouville fractional derivatives of order $1 < \gamma < 2$ are related by (e.g., see [5, p. 11])

$$\frac{\partial^\gamma}{\partial t^\gamma} f(t) = \mathbb{D}_t^\gamma f(t) - f(0)\frac{t^{-\gamma}}{\Gamma(1-\gamma)} - f'(0)\frac{t^{1-\gamma}}{\Gamma(2-\gamma)}, \tag{3.9}$$

when $\frac{\partial}{\partial t} p(x,0) = 0$ Eq. (2.6) with initial conditions $p(x,0) = \phi(x)$ and $\frac{\partial^\beta}{\partial t^\beta} p(x,0) = 0$ has the same integral form as Eq. (1.2) with $p(x,0) = \phi(x)$ and $\frac{\partial}{\partial t} p(x,0) = 0$.

Letting $L_\alpha^\eta(x)$ be the stable probability density function with characteristic function

$$\int_{-\infty}^{\infty} e^{ikx} L_\alpha^\eta(x) dx = \exp\left[-|k|^\alpha e^{i\,\mathrm{sgn}(k)\eta\pi/2}\right]$$

Mainardi shows that

$$L_\alpha^{\alpha-2}(x) = \frac{1}{\alpha}\Phi_{1/\alpha}(x) \quad \text{for any } x \in \mathbb{R} \text{ and } 1 < \alpha \le 2,$$

where

$$\Phi_\beta(z) = \sum_{n=0}^{\infty} \frac{(-z)^n}{n!\Gamma(1-n\beta-\beta)}, \quad 0 < \beta < 1 \tag{3.10}$$

is the Wright function. It follows [15, Eq. 6.37] that the solution to the fractional wave equation (1.2) with initial conditions $p(x,0) = \delta(x)$ and $\frac{\partial}{\partial t} p(x,0) = 0$ can be written in the form

$$p(x,t) = \frac{1}{2\beta t^\beta} L_{1/\beta}^{(1/\beta)-2}\left(\frac{|x|}{t^\beta}\right)$$

with $\beta = \gamma/2$. Hence, the solution to the fractional wave equation (1.2) with initial conditions $p(x,0) = \phi(x)$ and $\frac{\partial}{\partial t} p(x,0) = 0$ is given by

$$p(x,t) = \int_0^\infty \frac{1}{2}\Big[\phi(x-u) + \phi(x+u)\Big] \times \frac{1}{\beta t^\beta} L_{1/\beta}^{(1/\beta)-2}\left(\frac{u}{t^\beta}\right) du. \tag{3.11}$$

The solution (3.11) to the fractional wave equation (1.2) involves a stable density with index $\alpha = 1/\beta = 2/\gamma$, whereas the solution in Theorem 3.1 uses the inverse of a stable law with index $\beta = \gamma/2$. This can be explained using the Zolotarev duality formula for stable densities [3, Theorem 2.1]. Baeumer et al. [3, Theorem 4.1] use Zolotarev duality to prove that $\beta h(u,t) = q(u,t)$ for all $t > 0$ and $u \ge 0$, where $h(u,t)$ is the density (3.2) of the standard inverse β-stable subordinator on the set $u \ge 0$, $q(u,t)$ is the density of the spectrally negative stable process $X_\gamma(t)$ with index $1 < \gamma \le 2$ on the set $-\infty < u < \infty$, and $\alpha = 1/\beta$. For $u > 0$, the self-similarity argument shows that the function

$$q(u,t) = \frac{1}{t^\beta} L_{1/\beta}^{(1/\beta)-2}\left(\frac{u}{t^\beta}.\right)$$

Then, the solution (3.11) reduces to a special case of (2.7) with $\psi(x) = 0$ since

$$p(x,t) = \int_0^\infty \frac{1}{2}\left[\phi(x-u) + \phi(x+u)\right] \frac{1}{\beta t^\beta} L_{1/\beta}^{(1/\beta)-2}\left(\frac{u}{t^\beta}\right) \mathrm{d}u$$
$$= \int_0^\infty \frac{1}{2}\left[\phi(x-u)+\phi(x+u)\right] \frac{q(u,t)}{\beta}\mathrm{d}u$$
$$= \int_0^\infty \frac{1}{2}\left[\phi(x-u)+\phi(x+u)\right] h(u,t)\mathrm{d}u.$$

Theorem 4.1 in [3] also shows that the conditional distribution of $X_\gamma(t)$ given $X_\gamma(t) > 0$ is identical to the distribution of E_t. Hence, for any ϕ, ψ such that $\phi, \Psi \in \mathcal{B}^2(\mathbb{R})$, the unique solution to the fractional wave equation (2.6) in $\mathcal{B}^{2,0}(\mathbb{R}\times\mathbb{R}^+)$ with $p(x,0) = \phi(x)$ and $\frac{\partial^\beta}{\partial t^\beta} p(x,0) = \psi(x)$ can be written as

$$p(x,t) = \frac{1}{2}\mathbb{E}\left[\phi(x+X_\gamma(t)) + \phi(x-X_\gamma(t)) + \int_{x-X_\gamma(t)}^{x+X_\gamma(t)} \psi(y)\mathrm{d}y \,\middle|\, X_\gamma(t) > 0\right]. \qquad (3.12)$$

An extension of the well-known D. André reflection principle (see Appendix) shows that $\mathbb{P}[Y_\gamma(t) \geq x] = \mathbb{P}[X_\gamma(t) \geq x \mid X_\gamma(t) \geq 0]$, and this together with (3.5) gives another proof of (3.12).

4 General wave equations

Given a closed operator L on a Banach space X of functions, consider the second-order Cauchy problem

$$\frac{\partial^2}{\partial t^2} p(x,t) = Lp(x,t);$$
$$p(x,0) = \phi(x),\ \frac{\partial}{\partial t} p(x,0) = \psi(x). \qquad (4.1)$$

The traditional wave equation (1.1) is a special case where $L = \Delta_x$. Bajlekova [5,6] developed the theory of fractional order Cauchy problems

$$\frac{\partial^\gamma}{\partial t^\gamma} p(x,t) = Lp(x,t);$$
$$p(x,0) = \phi(x),\ \frac{\partial}{\partial t} p(x,0) = 0 \qquad (4.2)$$

using a Caputo fractional derivative of order $1 < \gamma < 2$.

The general theory of second-order Cauchy problems is laid out in [2, Sects. 3.14–3.16]. A strongly continuous (i.e., continuous in the Banach space norm) family of linear operators $(\mathrm{Cos}(t))_{t\geq 0}$ is called a *cosine family* if $\mathrm{Cos}(0) = I$ and $2\,\mathrm{Cos}(t)\,\mathrm{Cos}(s) = \mathrm{Cos}(t+s) + \mathrm{Cos}(t-s)$ for all $s, t \geq 0$. The *generator* L of the cosine family is defined by

$$Lf(x) = \lim_{t\downarrow 0} \frac{2}{t^2}\left[\mathrm{Cos}(t)f(x) - f(x)\right],$$

and the domain $\mathrm{Dom}(L)$ is the set of functions $f \in X$ for which this limit exists strongly. If the operator L in (4.1) is a generator of a cosine family $(\mathrm{Cos}(t))_{t\geq 0}$, then for any $\phi, \psi \in X$, the unique mild solution to the second-order Cauchy problem (4.1) is given by

$$p(x,t) = \mathrm{Cos}(t)\phi(x) + \int_0^t \mathrm{Cos}(s)\psi(x)\mathrm{d}s. \qquad (4.3)$$

That is, we have

$$p(x,t) = \phi(x) + t\psi(x) + L\int_0^t (t-s)p(x,s)\mathrm{d}s$$

for all $t \geq 0$, the integrated version of the second-order Cauchy problem. Furthermore, (4.3) is the unique (classical) solution to the second-order Cauchy problem (4.1) for any $\phi, \psi \in \mathrm{Dom}(L)$ [2, Theorem 3.14.11].

Theorem 4.1 *Suppose that the operator L in (4.2) is a generator of a cosine family $(\mathrm{Cos}(t))_{t\geq 0}$. Then for any $\phi \in X$, the unique mild solution to the fractional Cauchy problem (4.2) is given by the formula*

$$p(x,t) = \mathbb{E}\left[\mathrm{Cos}(E_t)\phi(x)\right], \qquad (4.4)$$

where $\mathrm{Cos}(t)\phi(x)$ is the unique mild solution to the second-order Cauchy problem (4.1) with $\psi = 0$, and E_t is the inverse (3.1) of the standard stable subordinator with index $\beta = \gamma/2$. Furthermore, equation (4.4) gives the unique classical solution to (4.2) for any $\phi \in \mathrm{Dom}(L)$.

Proof Bajlekova [5, Theorem 3.1] proves that if $\mathrm{Cos}(t)\phi(x) = S_2(t)\phi(x)$ solves the second-order Cauchy problem (4.1) with $\psi = 0$, then the unique solution to the fractional Cauchy problem (4.2) is $p(x,t) = S_\gamma(t)\phi(x)$ where the family of solution operators $S_\gamma(t)$ is given by the subordination formula

$$S_\gamma(t) = \int_0^\infty S_2(s)t^{-\gamma/2}\Phi_{\gamma/2}(st^{-\gamma/2})\mathrm{d}s, \qquad (4.5)$$

using the Wright function defined in (3.10). Recall the identity (e.g., see [5, Eq. 1.31])

$$\int_0^\infty \mathrm{e}^{zt}\Phi_\beta(z)\mathrm{d}z = \mathcal{E}_\beta(t) \tag{4.6}$$

where the Mittag-Leffler function

$$\mathcal{E}_\beta(z) = \sum_{n=0}^\infty \frac{z^n}{\Gamma(1+\beta n)}$$

for $\beta > 0$ and $z \in \mathbb{C}$. Bingham [8] and Bondesson et al. [10] show that the inverse E_t of a β-stable subordinator has a Mittag-Leffler distribution with

$$\mathbb{E}(\mathrm{e}^{-sE_t}) = \int_0^\infty \mathrm{e}^{-su}h(u,t)\mathrm{d}u = \mathcal{E}_\beta(-st^\beta).$$

But, it follows from (4.6) along with a substitution $z = u/t^\beta$ that we also have

$$\int_0^\infty \mathrm{e}^{-su}\frac{1}{t^\beta}\Phi_\beta\left(\frac{u}{t^\beta}\right)\mathrm{d}u = \int_0^\infty \mathrm{e}^{-szt^\beta}\Phi_\beta(z)\,\mathrm{d}z = \mathcal{E}_\beta(-st^\beta)$$

where $\beta = \gamma/2$. Then, it follows from the uniqueness of the Laplace transform that the standard inverse β-stable density (3.2) is related to the Wright function by

$$h(u,t) = \frac{1}{t^\beta}\Phi_\beta\left(\frac{u}{t^\beta}\right). \tag{4.7}$$

Hence, Bajlekova's solution (4.5) to the fractional wave equation is equivalent to the formula (4.4). □

Remark 4.2 Mainardi [15, Sect. 6.3] shows that the solution to the fractional wave equation (1.2) with initial conditions $p(x,0) = \phi(x)$ and $\frac{\partial}{\partial t}p(x,0) = 0$ is given by the convolution formula

$$p(x,t) = \int_0^\infty \frac{1}{2}\left[\phi(x-u)+\phi(x+u)\right]\frac{1}{t^\beta}\Phi_\beta\left(\frac{u}{t^\beta}\right)\mathrm{d}u.$$

Using (4.7), this reduces to (2.7) with $\psi(x) \equiv 0$.

Example 4.3 Given an open subset D of $\mathbb{R}^d$, consider the Laplace operator $L = \Delta_x$ on $L^2(D)$ with Dirichlet boundary conditions [2, Example 7.2.1]. For any $\phi \in \mathrm{Dom}(L)$ there exists a unique solution $p(x,t)$ to the wave equation

$$\frac{\partial^2}{\partial t^2}p(x,t) = \Delta_x p(x,t); \quad p(x,0) = \phi(x);$$
$$\frac{\partial}{\partial t}p(x,0) = 0; \; p(x,t) = 0 \quad \forall x \notin D$$

by [2, Theorem 7.2.2]. Then, it follows from Theorem 4.1 that the function $p_\gamma(x,t) = \mathbb{E}[p(x,E_t)]$ solves the corresponding fractional wave equation

$$\frac{\partial^\gamma}{\partial t^\gamma}p(x,t) = \Delta_x p(x,t); \quad p(x,0) = \phi(x);$$
$$\frac{\partial}{\partial t}p(x,0) = 0; \; p(x,t) = 0 \quad \forall x \notin D$$

on this bounded domain for $1 < \gamma < 2$, where E_t is the inverse stable subordinator (3.1) with index $\beta = \gamma/2$.

Example 4.4 If $L = B^2$, where B is a generator of a C_0-semigroup $(A(t))_{t\geq 0}$ on a Banach space of functions, then L is a generator of a cosine family given by

$$\mathrm{Cos}(t) = \frac{1}{2}\left(A(t) + A(-t)\right), \quad t \in \mathbb{R},$$

see [2, Example 3.14.15]. When $B = \frac{\partial}{\partial x}$, $(A(t))_{t\geq 0}$ is a shift semigroup, and equation (4.1) becomes the traditional wave equation (1.1). Equation (4.3) giving the solution becomes the d'Alembert formula (2.1). Theorem 4.1 gives the solution to the fractional wave equation (1.2) with the initial conditions $p(x,0) = 0$, $\frac{\partial}{\partial t}p(x,0) = 0$.

Example 4.5 If L is a self-adjoint linear operator on some Hilbert space such that $(Lx,x)_H \leq \omega\|x\|_H^2$ for some $\omega > 0$ and all $x \in \mathrm{Dom}(L)$, then L generates a cosine family [2, Example 3.14.16], and hence, (4.3) is the unique classical solution to the wave equation

$$\frac{\partial^2}{\partial t^2}p(x,t) = Lp(x,t);$$
$$p(x,0) = \phi(x); \; \frac{\partial}{\partial t}p(x,t) = 0$$

for any $\phi \in \mathrm{Dom}(L)$. Then, Theorem 4.1 implies that the function $p_\gamma(x,t) = \mathbb{E}[p(x,E_t)]$ solves the corresponding fractional wave equation

$$\frac{\partial^\gamma}{\partial t^\gamma}p(x,t) = Lp(x,t);$$
$$p(x,0) = \phi(x); \; \frac{\partial}{\partial t}p(x,t) = 0,$$

where E_t is the inverse stable subordinator (3.1) with index $\beta = \gamma/2$.

5 Random walk models

In this section, we will develop a random walk model for the fractional wave equation (1.2). First, we decompose the fractional wave equation into simpler parts. Using the notation (2.4) for the Riemann–Liouville fractional integral, the integral form (2.3) of the fractional wave equation with $\psi \equiv 0$ can be written as

$$p(x,t) = \phi(x) + \mathbb{I}_t^{\gamma} \Delta_x p(x,t). \tag{5.1}$$

Using the property $\mathbb{D}_t^{\gamma} = \mathbb{D}_t^{n} \mathbb{I}_t^{n-\gamma}$ for the Riemann–Liouville factional derivative and integral, and the semigroup property $\mathbb{I}_t^{\alpha}\mathbb{I}_t^{\beta} = \mathbb{I}_t^{\alpha+\beta}$, it follows easily that $\mathbb{D}_t^{\gamma}\mathbb{I}_t^{\gamma} f(t) = f(t)$ [5, Theorem 1.5]. Apply the operator $\mathbb{D}_t^{\gamma}$ to both sides of (5.1) to get the equivalent form

$$\mathbb{D}_t^{\gamma} p(x,t) = \Delta_x p(x,t) + \mathbb{D}_t^{\gamma}\phi(x). \tag{5.2}$$

An easy computation [18, Example 2.8] shows that $\mathbb{D}_t^{\gamma} 1 = t^{-\gamma}/\Gamma(1-\gamma)$, and then, (5.2) becomes

$$\mathbb{D}_t^{\gamma} p(x,t) - \phi(x)\frac{t^{-\gamma}}{\Gamma(1-\gamma)} = \Delta_x p(x,t). \tag{5.3}$$

Since the Caputo and Riemann–Liouville fractional derivatives of order $1 < \gamma < 2$ are related by (3.9), Eq. (5.3) is equivalent to the fractional wave equation (1.2) with initial conditions $p(x,0) = \phi(x)$ and $\frac{\partial}{\partial t} p(x,0) = 0$.

Now, consider the *one way fractional wave equations*

$$\mathbb{D}_t^{\gamma/2} p^+(x,t) = -\nabla_x p^+(x,t) + \frac{1}{2}\phi(x)\frac{t^{-\gamma/2}}{\Gamma(1-\gamma/2)}$$
$$\mathbb{D}_t^{\gamma/2} p^-(x,t) = \nabla_x p^-(x,t) + \frac{1}{2}\phi(x)\frac{t^{-\gamma/2}}{\Gamma(1-\gamma/2)} \tag{5.4}$$

where again $1 < \gamma < 2$. Apply the operator $\mathbb{I}_t^{\gamma/2}$ to both sides to obtain the integral forms

$$(I + \mathbb{I}_t^{\gamma/2}\nabla_x) p^+(x,t) = \frac{1}{2}\phi(x)$$
$$(I - \mathbb{I}_t^{\gamma/2}\nabla_x) p^-(x,t) = \frac{1}{2}\phi(x) \tag{5.5}$$

where I is the identity operator.

Theorem 5.1 *For any $\phi \in \mathcal{B}^1(\mathbb{R})$, the unique solutions to the one way fractional wave equations (5.5) in $\mathcal{B}^{1,0}(\mathbb{R}\times\mathbb{R}^+)$ are given by the formulae*

$$p^+(x,t) = \frac{1}{2}\mathbb{E}[\phi(x - E_t)]$$
$$p^-(x,t) = \frac{1}{2}\mathbb{E}[\phi(x + E_t)] \tag{5.6}$$

where E_t is the generalized inverse (3.1) of the standard stable subordinator with index $\beta = \gamma/2$. Furthermore, the unique solution to the fractional wave equation (2.3) in $\mathcal{B}^{1,0}(\mathbb{R}\times\mathbb{R}^+)$ with $p(x,0) = \phi(x)$ and $\frac{\partial}{\partial t} p(x,0) = \psi(x) \equiv 0$ is then given by $p(x,t) = p^+(x,t) + p^-(x,t)$.

Proof Fujita [13] proves the same result with E_t replaced by the supremum process (3.4). As noted in the proof of Theorem 3.1, these two processes have the same one-dimensional distributions. Then, the result follows. □

Remark 5.2 A direct proof of Theorem 5.1 uses an idea from Fujita [13]. Apply [17, Theorem 4.1] to see that the density (3.2) of the inverse stable subordinator with index $\beta = \gamma/2$ solves equation

$$\mathbb{D}_t^{\gamma/2} h(x,t) = -\nabla_x h(x,t) + \delta(x)\frac{t^{-\gamma/2}}{\Gamma(1-\gamma/2)}. \tag{5.7}$$

It follows using the principle of superposition that

$$p^+(x,t) = \frac{1}{2}\mathbb{E}[\phi(x - E_t)] = \frac{1}{2}\int_0^{\infty} \phi(x-u)h(u,t)\mathrm{d}y$$

solves the positive one way fractional wave equation. Then, a simple change of coordinates shows that $p^-(x,t) = \mathbb{E}[\phi(x + E_t)]/2$ solves the negative one way fractional wave equation. Now, write

$$\begin{aligned}(I - \mathbb{I}_t^{\gamma}\Delta_x)(p^+ + p^-) &= (I - \mathbb{I}_t^{\gamma/2}\nabla_x)(I + \mathbb{I}_t^{\gamma/2}\nabla_x)p^+ \\ &\quad + (I + \mathbb{I}_t^{\gamma/2}\nabla_x)(I - \mathbb{I}_t^{\gamma/2}\nabla_x)p^- \\ &= (I - \mathbb{I}_t^{\gamma/2}\nabla_x)\frac{1}{2}\phi + (I + \mathbb{I}_t^{\gamma/2}\nabla_x)\frac{1}{2}\phi = \phi\end{aligned}$$

which is equivalent to the fractional wave equation (2.3) with $p(x,0) = \phi(x)$ and $\frac{\partial}{\partial t} p(x,0) = 0$. One can also prove Theorem 3.1 in the same manner. Just apply the same argument again with $\phi(x)$ replaced by the function $\Psi(x) = \int_0^x \psi(y)\mathrm{d}y$, and then add the two solutions.

Remark 5.3 Here, we indicate an alternative proof of Theorem 4.1 using Riemann–Liouville fractional derivatives and an idea from [19]. Suppose that $p(x,t)$ solves the second-order Cauchy problem (4.1) with initial conditions $p(x,0) = \phi(x)$ and $\frac{\partial}{\partial t} p(x,0) = 0$. Let $h(u,t)$ be the density (3.2) of the inverse stable subordinator with index $\beta = \gamma/2$. From (5.7), it follows that $\mathbb{D}_t^{\beta} h(u,t) = -\partial_u h(u,t)$ on $t > 0$ and $u > 0$. It follows from (3.2) and the asymptotic behavior of stable densities that $h(0+, t) = t^{-\beta}/\Gamma(1-\beta)$ and $h(0-, t) = 0$ for all $t > 0$ (e.g., see [20, Eq. 20]). Write

$$p_\gamma(x,t) = \int_0^\infty p(x,u)h(u,t)\mathrm{d}u$$

and integrate by parts to get

$$\begin{aligned}\mathbb{D}_t^\beta p_\gamma(x,t) &= \int_0^\infty p(x,u)\mathbb{D}_t^\beta h(u,t)\mathrm{d}u\\ &= \int_0^\infty p(x,u)\left[-\partial_u h(u,t)\right]\mathrm{d}u\\ &= p(x,0)h(0+,t)\\ &\quad+\int_0^\infty \partial_u p(x,u)h(u,t)\mathrm{d}u.\end{aligned}$$

Integrate by parts again and use (4.1) to get

$$\begin{aligned}\mathbb{D}_t^{2\beta} p_\gamma(x,t) &= p(x,0)\mathbb{D}_t^\beta h(0+,t)\\ &\quad+\int_0^\infty \partial_u p(x,u)\left[-\partial_u h(u,t)\right]\mathrm{d}u\\ &= p(x,0)\mathbb{D}_t^\beta h(0+,t)\\ &\quad+\int_0^\infty \partial_u^2 p(x,u)h(u,t)\mathrm{d}u\\ &= Lp_\gamma(x,t)+\phi(x)\frac{t^{-2\beta}}{\Gamma(1-2\beta)}\end{aligned}$$

using the general formula $\mathbb{D}_t^\beta[t^\alpha] = \Gamma(1+\alpha)t^{\alpha-\beta}/\Gamma(1+\alpha-\beta)$ [18, Example 2.7]. This equation for $p_\gamma(x,t)$ is equivalent to the second-order Cauchy problem (4.1) with initial conditions $p_\gamma(x,0) = \phi(x)$ and $\frac{\partial}{\partial t}p_\gamma(x,0) = 0$, since the Caputo and Riemann–Liouville fractional derivatives of order $\gamma = 2\beta \in (1,2)$ are related by (3.9).

Finally, we develop a simple particle tracking method for solving the fractional wave equation (2.3), using a continuous-time random walk [16,17] that converges to the stochastic solution of the fractional wave equation. The main idea is to construct a random walk model that converges to the inverse stable subordinator E_t and use the fact that the density (3.2) of E_t solves the positive one way fractional wave equation.

Theorem 5.4 *Given a continuous probability density function $\phi(x)$ on $\mathbb{R}$, let X_0 be a random variable with density $\phi(x)$. Let X_1 be a Bernoulli random variable independent of X_0 such that $\mathbb{P}[X_1 = 1] = \mathbb{P}[X_1 = -1] = 1/2$, set $X_n = X_1$ for $n > 1$, and let $S(n) = X_1 + \cdots + X_n = nX_1$. Let W_n be iid random variables independent of X_0, X_1 with $\mathbb{P}[W_n > t] = Ct^{-\beta}$ for $t > C^{1/\beta}$, where $0 < \beta < 1$ and $C = 1/\Gamma(1-\beta)$. Let $T_0 = 0$, $T_n = W_1 + \cdots + W_n$ for $n \ge 1$, and $N_t = \max\{n \ge 0 : T_n \le t\}$ for $t \ge 0$. Then,*

$$X_0 + c^{-\beta}S(N_{ct}) \Rightarrow U_t \quad \text{as } c\to\infty \tag{5.8}$$

in $\mathcal{D}[0,\infty)$ with the Skorokhod J_1 topology, where the random variable U_t has density

$$p(x,t) = \frac{1}{2}\mathbb{E}\left[\phi(x+E_t)+\phi(x-E_t)\right], \tag{5.9}$$

and E_t is the inverse stable subordinator (3.1) with index $\beta = \gamma/2$. Hence, $p(x,t)$ is the unique solution to the fractional wave equation (2.3) in $\mathcal{B}^{2,0}(\mathbb{R}\times\mathbb{R}^+)$ with $p(x,0) = \phi(x)$ and $\frac{\partial}{\partial t}p(x,0) = \psi(x) \equiv 0$.

Proof For any $c > 0$, it follows from [18, Theorem 3.41 and Eq. 4.29] that $c^{-1/\beta}T_{[ct]} \Rightarrow D_t$ as $c \to \infty$ in J_1 topology in $\mathcal{D}[0,\infty)$, where D_t is a β-stable subordinator with characteristic function $\mathbb{E}[\mathrm{e}^{ikD_t}] = \exp[-tC\Gamma(1-\beta)(-ik)^\beta]$. Taking $C = 1/\Gamma(1-\beta)$, the limit is a standard stable subordinator with Laplace transform $\mathbb{E}[\mathrm{e}^{-sD_t}] = \mathrm{e}^{-ts^\beta}$, and then, [16, Theorem 3.2 and Corollary 3.4] implies that $c^{-\beta}N_{ct} \Rightarrow E_t$ as $c\to\infty$, where E_t is the inverse (3.1) of the standard stable subordinator D_t. Since $S(n) = nX_1$ it follows easily that

$$X_0 + c^{-\beta}S(N_{ct}) = X_0 + c^{-\beta}N_{ct}X_1 \Rightarrow X_0 + E_tX_1$$

as $c\to\infty$. Then, (5.8) holds with $U_t = X_0 + E_tX_1$, and a simple conditioning argument yields (5.9). Then, Theorem 3.1 shows that $p(x,t)$ is the unique solution to the fractional wave equation (2.3) in $\mathcal{B}^{2,0}(\mathbb{R}\times\mathbb{R}^+)$ with $p(x,0) = \phi(x)$ and $\frac{\partial}{\partial t}p(x,0) = \psi(x) \equiv 0$. □

Theorem 5.4 provides a physical model for the fractional wave equation. Each sample path represents a packet of wave energy moving out from its initial position X_0 at unit speed, represented by the process $S(n)$. For the traditional wave equation, this is the correct particle model. In the fractional case, time delays with a power-law probability distribution occur between movements, and this retards the progress of the wave outward from the starting point. These delays are related to the heterogeneous structure of the conducting medium, see Mainardi [15, Sect. 6.4].

Remark 5.5 Theorem 5.4 implies that the histogram of a large number M of identical continuous-time random-walk processes $X_0 + c^{-\beta} S(N_{ct})$ gives an approximate solution to the fractional wave equation, which gains accuracy at $M \to \infty$ and $c \to \infty$. It is a simple matter to simulate the waiting times using the formula $W_n = (U_n/C)^{-1/\beta}$ where U_n are iid uniform random variables on (0, 1). Theorem 5.4 remains true for any iid waiting times $W_n > 0$ in the domain of attraction of the β-stable subordinator, except that the norming constants $c^{-\beta}$ need to be adjusted as in [16, Theorem 3.2].

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Appendix

Reflection principle

The goal of this appendix is to establish the following variation of the D. André reflection principle for Brownian motion, which may also be useful in other contexts. Since this extension is not completely standard, we include its simple proof.

Theorem 6.1 (Reflection principle) *Suppose that Y_t is a Lévy process started at the origin, with no positive jumps, and let $S_t = \sup\{Y_u : 0 \le u \le t\}$. Assume that $\mathbb{P}(Y_t > 0) = \mathbb{P}(Y_1 \ge 0)$, for all $t > 0$. Then,*

$$\mathbb{P}(S_t \ge x) = \mathbb{P}(Y_t \ge x \mid Y_t \ge 0) = \frac{\mathbb{P}(Y_t \ge x)}{\mathbb{P}(Y_t \ge 0)} \tag{6.1}$$

for all $t, x > 0$.

Proof Let $\tau_x := \inf\{u > 0 : Y_u > x\}$ denote the first-passage time process. Since $(Y_t)_{t\ge0}$ has stationary independent increments, it follows that $(Y_{t+\tau_x} - Y_{\tau_x})_{t\ge0}$ is a Lévy process, which is independent of the σ-algebra generated by $(Y_t)_{t\le\tau_x}$, and it has the same finite-dimensional distributions as $(Y_t)_{t\ge0}$. Consequently,

$$\mathbb{P}\left(\tau_x \le t, Y_t < Y_{\tau_x}\right) = \mathbb{P}\left(\tau_x \le t, Y_t - Y_{\tau_x} < 0\right) \tag{6.2}$$
$$= \mathbb{P}(\tau_x \le t)\,\mathbb{P}(Y_1 < 0)$$
$$= \frac{\mathbb{P}(Y_1 < 0)}{\mathbb{P}(Y_1 \ge 0)}\mathbb{P}(\tau_x \le t)\,\mathbb{P}(Y_1 \ge 0)$$
$$= \frac{\mathbb{P}(Y_1 < 0)}{\mathbb{P}(Y_1 \ge 0)}\mathbb{P}\left(\tau_x \le t, Y_t \ge Y_{\tau_x}\right).$$

Observe that we have $\{\tau_x < t\} \subset \{S_t > x\} \subset \{\tau_x \le t\} \subset \{S_t \ge x\}$ for all t and $x > 0$. Therefore,

$$\begin{aligned}\mathbb{P}(S_t > x) &\le \mathbb{P}(\tau_x \le t) = \mathbb{P}(\tau_x \le t, Y_t \ge Y_{\tau_x}) \\ &\quad + \mathbb{P}(\tau_x \le t, Y_t < Y_{\tau_x}) \\ &= \left(1 + \frac{\mathbb{P}(Y_1 < 0)}{\mathbb{P}(Y_1 \ge 0)}\right)\mathbb{P}(\tau_x \le t, Y_t \ge Y_{\tau_x}) \\ &= \frac{1}{\mathbb{P}(Y_1 \ge 0)}\mathbb{P}(\tau_x \le t, Y_t \ge Y_{\tau_x}) \\ &= \frac{1}{\mathbb{P}(Y_t \ge 0)}\mathbb{P}(\tau_x \le t, Y_t \ge Y_{\tau_x}, Y_t \ge 0) \\ &= \mathbb{P}(\tau_x \le t, Y_t \ge Y_{\tau_x} \mid Y_t \ge 0).\end{aligned}$$

On the other hand, and with similar arguments,

$$\begin{aligned}\mathbb{P}(S_t > x) &\ge \mathbb{P}(\tau_x < t) = \mathbb{P}(\tau_x < t, Y_t \ge Y_{\tau_x}) \\ &\quad + \mathbb{P}(\tau_x < t, Y_t < Y_{\tau_x}) \\ &= \mathbb{P}(\tau_x < t, Y_t \ge Y_{\tau_x} \mid Y_t \ge 0) \\ &\ge \mathbb{P}(\tau_x < t, Y_t > Y_{\tau_x} \mid Y_t \ge 0).\end{aligned}$$

Therefore,

$$\begin{aligned}&\mathbb{P}(\tau_x < t, Y_t > Y_{\tau_x} \mid Y_t \ge 0) \le \mathbb{P}(S_t > x) \\ &\quad \le \mathbb{P}(\tau_x \le t, Y_t \ge Y_{\tau_x} \mid Y_t \ge 0).\end{aligned}$$

Since Y_t has no upward jumps, $Y_{\tau_x} = x$ and $\{\tau_x < t\} \cap \{Y_t > x\} = \{Y_t > x\}$. Therefore,

$$\begin{aligned}&\mathbb{P}(Y_t > x \mid Y_t \ge 0) \le \mathbb{P}(S_t > x) \\ &\quad \le \mathbb{P}(Y_t \ge x \mid Y_t \ge 0).\end{aligned}$$

We can now use standard approximation techniques:

$$\{X \ge x\} = \bigcap_n \left\{X > x - 1/n\right\} = \bigcap_n \left\{X \ge x - 1/n\right\}$$

and

$$\{X > x\} = \bigcup_n \{X > x + 1/n\}$$

to get

$$\begin{aligned}\mathbb{P}(Y_t \geq x \mid Y_t \geq 0) &= \lim_{n\to\infty} \mathbb{P}(Y_t > x - 1/n \mid Y_t \geq 0) \\ &\leq \lim_{n\to\infty} \mathbb{P}(S_t > x - 1/n) \\ &= \mathbb{P}(S_t \geq x)\end{aligned}$$

and

$$\begin{aligned}\mathbb{P}(S_t \geq x) &= \lim_{n\to\infty} \mathbb{P}(S_t > x - 1/n) \\ &\leq \lim_{n\to\infty} \mathbb{P}(Y_t \geq x - 1/n \mid Y_t \geq 0) \\ &= \mathbb{P}(Y_t \geq x \mid Y_t \geq 0)\end{aligned}$$

which proves $\mathbb{P}(S_t \geq x) = \mathbb{P}(Y_t \geq x \mid Y_t \geq 0)$. □

Remark 6.2 If $(Y_t)_{t\geq 0}$ is a Brownian motion, then (6.1) becomes the classical reflection principle: $\mathbb{P}(Y_t \geq 0) = 1/2$, so that (6.1) is equivalent to $\mathbb{P}(S_t \geq x) = 2\mathbb{P}(Y_t \geq x)$.

The proof of Theorem 6.1 relies essentially on local symmetry and the strong Markov property. Let Y_t be a strong Markov process with càdlàg paths and transition function $p_t(z, \mathrm{d}y) = \mathbb{P}^z(Y_t \in \mathrm{d}y)$. Write $\tau_x^z = \inf\{u > 0 : Y_u - z > x\}$ for the first passage time above the level $x + z$ for the process Y_t started at z; observe that, in general, $Y_{\tau_x^z} \geq x + z$. We can use the strong Markov property in (6.2) to get for any starting point z

$$\begin{aligned}\mathbb{P}^z\left(\tau_x^z \leq t, Y_t < Y_{\tau_x^z}\right) &= \mathbb{P}^z\left(\tau_x^z \leq t, Y_t - Y_{\tau_x^z} < 0\right) \\ = &\int_{\{\tau_x^z \leq t\}} \mathbb{P}^{Y_{\tau_x^z}(\omega)}\left(Y_{t-\tau_x^z(\omega)} - Y_0 < 0\right) \mathbb{P}^z(\mathrm{d}\omega).\end{aligned}$$

If we assume, in addition, some local "symmetry," i.e., that for some constant $c \in (0, \infty)$ we have

$$\frac{\mathbb{P}^z(Y_t - z < 0)}{\mathbb{P}^z(Y_t - z \geq 0)} = c \quad \text{for all } t > 0,\ z \in \mathbb{R}, \tag{6.3}$$

then we get $\mathbb{P}^{Y_{\tau_x^z}(\omega)}\left(Y_{t-\tau_x^z(\omega)} - Y_0 < 0\right) = c\,\mathbb{P}^{Y_{\tau_x^z}(\omega)}\left(Y_{t-\tau_x^z(\omega)} - Y_0 \geq 0\right)$ and, with a similar argument,

$$\mathbb{P}^z\left(\tau_x^z \leq t, Y_t < Y_{\tau_x^z}\right) = c\,\mathbb{P}^z\left(\tau_x^z \leq t, Y_t \geq Y_{\tau_x^z}\right).$$

This means that we can follow the lines of the proof of Theorem 6.1 to derive the following general result.

Theorem 6.3 (Markov reflection principle) *Suppose $(Y_t, \mathbb{P}^z)$ is a strong Markov process satisfying the local symmetry condition (6.3). Set $S_t = \sup\{Y_u - Y_0 : 0 \leq u \leq t\}$. Then, we have for all $t, x > 0$ and $z \in \mathbb{R}$*

$$\begin{aligned}\mathbb{P}^z(S_t > x) &\leq \mathbb{P}^z(S_t \geq x, Y_t \geq Y_{\tau_x^z} \mid Y_t \geq z) \\ &\leq \mathbb{P}^z(Y_t - z \geq x \mid Y_t \geq z).\end{aligned} \tag{6.4}$$

If Y_t has only non-positive jumps, then $Y_{\tau_x^z} = x + z$ a.s., and we get for all $t, x > 0$ and $z \in \mathbb{R}$

$$\mathbb{P}^z(S_t \geq x) = \mathbb{P}^z(Y_t - z \geq x \mid Y_t \geq z). \tag{6.5}$$

Remark 6.4 It is also possible to prove (6.1) using relation (3.6) in Alili and Chaumont [1].

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