

Nonlinear transverse vibration of axially accelerating strings with exact internal resonances and longitudinally varying tensions

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Received: 14 August 2013 / Accepted: 22 December 2013 / Published online: 12 January 2014
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Abstract This work explores the steady-state periodic transverse responses with their stabilities of axially accelerating viscoelastic strings. Longitudinally varying tension due to the axial acceleration is recognized in the modeling, while the tension was approximately assumed to be longitudinally uniform in previous investigations. Exact internal resonances are highlighted in the analysis, while the resonances have been neglected in all available works. A governing equation of transverse nonlinear vibration is derived from the generalized Hamilton principle and the Kelvin viscoelastic model on the assumption that the string defor-

mation is not infinitesimal, but still small. The axial speed is supposed to be a small simple harmonic fluctuation about the constant mean axial speed. The method of multiple scales is applied to solve the governing equation in the parametric resonances when the axial speed fluctuation frequency approaches the first three natural frequencies of the linear generating system based on 1–3 term truncations. The amplitude, the existence conditions, and the stability are determined, and the effects of the viscosity, the mean axial speed, the axial speed fluctuation amplitude, and the axial support rigidity on the amplitude and the existence are examined via the numerical examples. It is found that the 1-term, the 2-term, and the 3-term truncations yield the qualitatively same and the quantitatively close results in the case that there exist the exact internal resonances among the first three frequencies.

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Keywords Nonlinear parametric vibration · Axially accelerating string · Exact internal resonance · Longitudinally varying tension · Method of multiple scales

1 Introduction

Axially moving strings can represent many engineering devices such as paper sheets and textile fibers, while transverse vibrations of the strings may degenerate production quality and generate noise. Knowledge of transverse vibrations of axially moving strings [1]

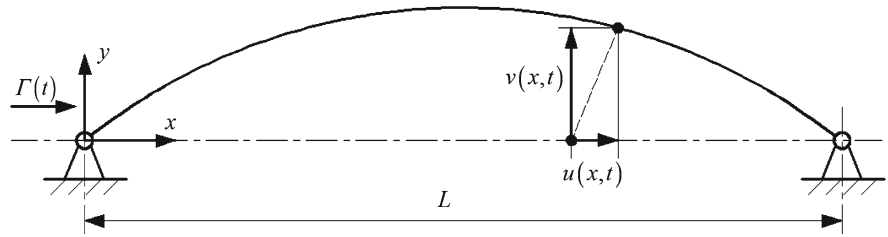
is significant for the design and the operation of the devices. An axially moving string may undergo transverse parametric vibration excited by its axial acceleration. Miranker established the equation governing transverse linear vibration of an axially accelerating string [2]. Mote pioneered the stability analysis of axially accelerating strings [3]. Within the framework of linear vibration, dynamic stability was investigated via the calculation based on the Galerkin discretization [4,5], the method of multiple scales based on the Galerkin discretization [6,7], the Floquet theory based on the finite element method [8], and the direct method of multiple scales [6,9,10]. If the string deformation is not infinitesimal, the geometric nonlinear terms should be included. Within the framework of nonlinear vibration, transient responses can be computed based on the Galerkin discretization for axially accelerating strings with different constitutive laws [11–13]. Steady-state periodical responses can be predicted via the method of multiple scales based on the Galerkin discretization [14], the direct method of multiple scales [15–17], and an asymptotic analysis approach [18]. In all above-mentioned works [2–18] on transverse vibration of axially accelerating strings, the string tensions were assumed to be spatially uniform (along the longitudinal direction), although there are many works that consider time-dependent variations of tension as parametric excitations [1]. As a consequence, the magnitudes of the tensions are equal and the directions are opposite at both ends of the string. The consequence contradicts the fact that the strings move with a nonzero acceleration, because Newton's law demands that the acceleration should be caused by a nonzero resultant force. Therefore, the assumption that the tension is independent of the longitudinal coordinate cannot be exactly held. It is only an approximation that reduces the mathematical difficulty to solve the governing equations, because, without the approximation, the coefficients of the governing equations depend not only on the temporal variable, but also on the spatial variable. Recent works on axially accelerating beams demonstrated that the tension longitudinal variation changes both the stability boundaries in linear parametrical vibration [19,20] and the steady-state response in nonlinear parametrical vibration [21]. However, there has yet been no investigation on the effects of longitudinally varying tensions on the transverse vibration of axially accelerating strings.

Approximate analytical methods are powerful tool to determine stability boundaries of linear vibration and steady-state responses of nonlinear vibration of axially moving strings [1]. Many significant results have been achieved by the application of the approximate analytical methods. In all available works via the approximate analytical methods [6–9,13–23], it was assumed that two modes with corresponding natural frequencies involved in summation resonance or a mode in principal parametric resonance contribute to dynamic behaviors. The effects of other modes that were truncated have not been considered. If there is no internal resonance, with two or more rationally commensurable frequencies of the linear generating system, the truncations seem physically sound. However, in the case of axially moving strings, all frequencies of the linear generating system are proportional to the first frequency [24], and thus there are infinite exact internal resonances. So far, the widespread practice of neglecting modes uninvolved in parametric resonance has not been validated by examining possible internal resonances. It should be remarked that the infinite exact internal resonances may occur in nonlinear vibration of other physical systems than axially moving strings. For example, all linear frequencies of an Euler-Bernoulli beam pinned at both ends are proportional.

To address the lack of investigations in above mentioned two aspects, this work revisits steady-state responses in the combination and the principal parametric resonances of axially accelerating viscoelastic strings with the emphasis on longitudinally varying tension in modeling and exact internal resonances in analysis. However, infinite mode analysis is infeasible at the present stage. Instead, a heuristic approach, based on finite mode analysis, to internal resonances will be proposed.

The paper is organized as follows. Section 2 derives the governing equations and the associated boundary conditions of coupled planar vibration from the generalized Hamilton principle and the Kelvin viscoelastic model, and reduces the equations to the governing equation of transverse motion in small, but finite stretching problems. Section 3 provides an application framework of the method of multiple scales and determines the possible parametric resonances. Sections 4–6 treat the first three parametric resonances with the account for the exact internal resonances. Section 6 ends the paper with the concluding remarks.

Fig. 1 The physical model of an axially accelerating viscoelastic string



2 Modeling with the recognition of longitudinally varying tensions

A uniform viscoelastic string of density ρ and cross-sectional area A moves axially between two eyelets separated by distance L . Assume that the deformation of the string is confined to a plane. The string is subjected to no external loads. A mixed Eulerian–Lagrangian description is adopted. At axial coordinate x and time t , the planar vibration of the string is specified, respectively, by the transverse displacement $v(x, t)$ related to a spatial frame and the longitudinal displacement $u(x, t)$ related to a translating frame moving with the string. The axial transport speed, denoted by $\Gamma(t)$, is time dependent. The physical model is shown in Fig. 1.

In the present investigation, the effects of the axial string motion on the string deformation are highlighted, while the effects of the string deformation on the axial string motion are neglected. Application of Newton’s second law to small string element dx leads to the longitudinal change rate of axial tension P

$$P_{,x} = \rho A \dot{\Gamma} \tag{1}$$

Integrating equation (1) with respect to x yields

$$P = \rho A \dot{\Gamma} x + C(t), \tag{2}$$

where $C(t)$ is the “constant of integration” with respect to x . That is, $C(t)$ may depend on time t and is independent of longitudinal coordinate x . At $x = L$, the right hand side term is equal to the right-end axial tension $P_0 + \eta \rho A \Gamma^2$, where P_0 is initial axial tension (tension in the string without the axial acceleration and the transverse vibration) and η is the axial support rigidity parameter varying between 0 (infinite rigidity) and 1 (no rigidity) [25]. Thus,

$$P = P_0 + \eta \rho A \Gamma^2 + (x - L) \rho A \dot{\Gamma}, \tag{3}$$

where the third term in Eq. (3) represents the longitudinally varying tension due to the axially accelerating motion.

The generalized Hamilton principle will be employed to formulate the governing equations and the associated boundary conditions. The kinetic energy of the axially moving string is

$$T = \frac{1}{2} \int_0^L \rho A \left[(\Gamma + u_{,t} + \Gamma u_{,x})^2 + (v_{,t} + \Gamma v_{,x})^2 \right] dx, \tag{4}$$

where a comma preceding t or x denotes partial differentiation with respect to t or x , and $\Gamma + u_{,t} + \Gamma u_{,x}$ and $v_{,t} + \Gamma v_{,x}$ are, respectively, the longitudinal and the transverse velocity projections of a point of the axially moving string. Then, the virtual work done by the axial tension during the deformation is

$$\delta W = - \int_0^L (P + \sigma_x A) \delta \varepsilon_x dx \tag{5}$$

If the string is elastic, the virtual work is the variation of the potential energy. Otherwise, it contains a nonconservative part. The material of the string is assumed to be constituted by the Kelvin model. That is, normal stress σ_x and normal strain ε_x are related by

$$\sigma_x = E \varepsilon_x + \alpha (\varepsilon_{x,t} + \Gamma \varepsilon_{x,x}), \tag{6}$$

where E and α are, respectively, the Young modulus and the viscosity of the string. The strain obeys the strain-displacement relation

$$\varepsilon_x = \sqrt{(1 + u_{,x})^2 + v_{,x}^2} - 1 \tag{7}$$

The generalized Hamilton principle takes the form

$$\delta \int_{t_1}^{t_2} T dt + \int_{t_1}^{t_2} \delta W dt = 0 \tag{8}$$

Substitution of Eqs. (4) and (5) with Eqs. (6) and (7) into Eq. (8) and application of the standard variation procedure in the resulting equation yield

$$\int_{t_1}^{t_2} \int_0^L \rho A \left(u_{,tt} + 2\Gamma u_{,xt} + \dot{\Gamma} + \dot{\Gamma} u_{,x} + \Gamma^2 u_{,xx} \right) - \frac{\partial}{\partial x} \left[\frac{(P + A\sigma_x)(1 + u_{,x})}{\sqrt{(1 + u_{,x})^2 + v_{,x}^2}} \right] \delta u dx dt = 0 \quad (9)$$

$$\int_{t_1}^{t_2} \int_0^L \left\{ \rho A \left(v_{,tt} + 2\Gamma v_{,xt} + \dot{\Gamma} v_{,x} + \Gamma^2 v_{,xx} \right) - \frac{\partial}{\partial x} \left[\frac{(P + A\sigma_x) v_{,x}}{\sqrt{(1 + u_{,x})^2 + v_{,x}^2}} \right] \right\} \delta v dx dt = 0 \quad (10)$$

$$\int_{t_1}^{t_2} \left[\frac{(P + A\sigma_x)(1 + u_{,x})}{\sqrt{(1 + u_{,x})^2 + v_{,x}^2}} - \rho A \Gamma (\Gamma + u_{,t} + \Gamma u_{,x}) \right] \delta u \Big|_0^L dt = 0, \quad (11)$$

$$\int_{t_1}^{t_2} \left\{ \frac{(P + A\sigma_x) v_{,x}}{\sqrt{(1 + u_{,x})^2 + v_{,x}^2}} - \rho A \Gamma (v_{,t} + \Gamma v_{,x}) \right\} \delta v \Big|_0^L dt = 0. \quad (12)$$

$$\int_0^L \rho A (\Gamma + u_{,t} + \Gamma u_{,x}) \delta u|_{t_1}^{t_2} dx = 0, \quad (13)$$

$$\int_0^L \rho A (v_{,t} + \Gamma v_{,x}) \delta v|_{t_1}^{t_2} dx = 0. \quad (14)$$

The governing equations of the planar motion for the axially accelerating viscoelastic string are derived from Eqs. (9) and (10) as

$$\rho A \left(u_{,tt} + 2\Gamma u_{,xt} + \dot{\Gamma} + \dot{\Gamma} u_{,x} + \Gamma^2 u_{,xx} \right) - \frac{\partial}{\partial x} \left[\frac{(P + A\sigma_x)(1 + u_{,x})}{\sqrt{(1 + u_{,x})^2 + v_{,x}^2}} \right] = 0, \quad (15)$$

$$\rho A \left(v_{,tt} + 2\Gamma v_{,xt} + \dot{\Gamma} v_{,x} + \Gamma^2 v_{,xx} \right) - \frac{\partial}{\partial x} \left[\frac{(P + A\sigma_x) v_{,x}}{\sqrt{(1 + u_{,x})^2 + v_{,x}^2}} \right] = 0. \quad (16)$$

Using a Taylor series, one can approximate the denominator of the last term in Eqs. (15) and (16)

$$\begin{aligned} & \left[(1 + u_{,x})^2 + v_{,x}^2 \right]^{-1/2} \\ &= 1 - u_{,x} - \frac{v_{,x}^2}{2} + u_{,x}^2 + \frac{3v_{,x}^2}{2} u_{,x} - v_{,x}^3 + O(4), \end{aligned} \quad (17)$$

where $O(4)$ stands for terms of power 4 or higher. The transverse motion is generally coupled with the longitudinal motion, but they both are small-amplitude motions in the practical circumstances [26]. In small, but finite stretching problems, one can only retain the lowest order nonlinear term and omit all higher order nonlinear terms. Assume that the longitudinal displacement is much smaller than the transverse displacement, that is $u = O(v^2)$. Equation (17) leads to

$$\begin{aligned} & \left[(1 + u_{,x})^2 + v_{,x}^2 \right]^{-1/2} \\ &= 1 - u_{,x} - \frac{v_{,x}^2}{2} - v_{,x}^3 + O(v^4) \end{aligned} \quad (18)$$

Substituting equation (18) into Eqs. (15) and (16) and neglecting terms $O(v^4)$, one obtains

$$\rho A \dot{\Gamma} = P_{,x} + EA \frac{\partial}{\partial x} \left(\frac{1}{2} v_{,x}^2 + u_{,x} \right), \quad (19)$$

$$\begin{aligned} & \rho A \left(v_{,tt} + 2\Gamma v_{,xt} + \dot{\Gamma} v_{,x} + \Gamma^2 v_{,xx} \right) \\ & - \frac{\partial}{\partial x} \left\{ \left[P + EA \left(\frac{1}{2} v_{,x}^2 + u_{,x} \right) + A\alpha (u_{,xt} + \Gamma u_{,xx}) \right. \right. \\ & \left. \left. + A\alpha (v_{,xt} + \Gamma v_{,xx}) v_{,x} \right] v_{,x} \right\} = 0. \end{aligned} \quad (20)$$

Substitution of Eq. (3) and into Eq. (19) yields

$$\frac{\partial}{\partial x} \left(\frac{1}{2} v_{,x}^2 + u_{,x} \right) = 0 \quad (21)$$

Integrating equation (21) twice with respect to x , one gets

$$u = x C_1(t) - \frac{1}{2} \int_0^x v_{,x}^2 dx + C_2(t), \quad (22)$$

where $C_1(t)$ and $C_2(t)$ are the “constants of integration” with respect to x . For the boundaries as shown in Fig. 1, it can be assumed that $u = 0$ at $x = 0$ and $x = L$, which satisfy Eq. (11). $C_1(t)$ and $C_2(t)$ can be determined, and Eq. (22) gives

$$u = \frac{x}{2L} \int_0^L v_{,x}^2 dx - \frac{1}{2} \int_0^x v_{,x}^2 dx \quad (23)$$

Substitution of Eqs. (3) and (23) into Eq. (20) yields

$$\rho A \left(v_{,tt} + 2\Gamma v_{,xt} + \Gamma^2 v_{,xx} \right) - \left[P_0 + \eta \rho A \Gamma^2 + (x - L) \rho A \dot{\Gamma} + \frac{EA}{2L} \int_0^L v_{,x}^2 dx \right. \\ \left. + A\alpha \left(\frac{1}{L} \int_0^L v_{,x} v_{,xt} dx \right) \right] v_{,xx} = 0 \tag{24}$$

Nonlinear integro-partial-differential equation (24) governs transverse motion of the axially accelerating viscoelastic string.

Both the longitudinal variation of the tension and the finite axial support rigidity are accounted for by Eq. (24). If the support is completely rigid, then $\eta = 0$. Equation (24) reduces to

$$\rho A \left(v_{,tt} + 2\Gamma v_{,xt} + \Gamma^2 v_{,xx} \right) - \left[P_0 + (x - L) \rho A \dot{\Gamma} + \frac{EA}{2L} \int_0^L v_{,x}^2 dx \right. \\ \left. + A\alpha \left(\frac{1}{L} \int_0^L v_{,x} v_{,xt} dx \right) \right] v_{,xx} = 0 \tag{25}$$

If the tension variation is neglected, then the tension can be assumed as $P = P_0 + \eta \rho A \Gamma^2$. Substitution of the tension and Eqs. (23) into (20) gives

$$\rho A \left(v_{,tt} + 2\Gamma v_{,xt} + \dot{\Gamma} v_{,x} + \Gamma^2 v_{,xx} \right) - \left[P_0 + \eta \rho A \Gamma^2 + \frac{EA}{2L} \int_0^L v_{,x}^2 dx \right. \\ \left. + A\alpha \left(\frac{1}{L} \int_0^L v_{,x} v_{,xt} dx \right) \right] v_{,xx} = 0 \tag{26}$$

In the special case of infinite rigid supports, Eq. (26) reduces to

$$\rho A \left(v_{,tt} + 2\Gamma v_{,xt} + \dot{\Gamma} v_{,x} + \Gamma^2 v_{,xx} \right) - \left[P_0 + \frac{EA}{2L} \int_0^L v_{,x}^2 dx + A\alpha \left(\frac{1}{L} \int_0^L v_{,x} v_{,xt} dx \right) \right] v_{,xx} \\ = 0 \tag{27}$$

The following boundary conditions of the axially accelerating viscoelastic string satisfy Eq. (12)

$$v(0, t) = 0, v(L, t) = 0 \tag{28}$$

Equations (24) and (28) can be cast into the dimensionless forms by using the following coordinates or parameters

$$v \leftrightarrow \frac{v}{\sqrt{\varepsilon L}}, x \leftrightarrow \frac{x}{L}, t \leftrightarrow \frac{t}{L} \sqrt{\frac{P_0}{\rho A}}, \gamma = \Gamma \sqrt{\frac{\rho A}{P_0}}, \\ \alpha \leftrightarrow \frac{\alpha}{EL} \sqrt{\frac{P_0}{\rho A}}, k_1 = \sqrt{\frac{EA}{P_0}}, \tag{29}$$

where bookkeeping device ε is introduced as a small dimensionless parameter accounting for the fact that the transverse displacement is very small. Substituting Equation (29) into (24) and (28), one gets the dimensionless forms

$$v_{,tt} + 2\gamma v_{,xt} + \left[\kappa \gamma^2 - (x - 1) \dot{\gamma} - 1 \right] v_{,xx} \\ = \varepsilon k_1^2 v_{,xx} \left(\frac{1}{2} \int_0^1 v_{,x}^2 dx + \alpha \int_0^1 v_{,x} v_{,xt} dx \right), \tag{30}$$

$$v(0, t) = 0, v(1, t) = 0, \tag{31}$$

where $\kappa = 1 - \eta$. Equation (30) is essentially identical to Eq. (33) in [21] for axially accelerating beams except for two changes. All bending-related terms in beam case disappear here and a damping term appears because the damping coefficient is no longer assumed to be small.

3 Analysis with the recognition of internal resonances

In the present investigation, the axial speed is supposed to be a small simple harmonic fluctuation about the constant mean axial speed,

$$\gamma(t) = \gamma_0 + \varepsilon \gamma_1 \sin(\omega t), \tag{32}$$

where γ_0 is the constant mean axial speed, and γ_1 and ω are, respectively, the amplitude and the frequency of the axial speed fluctuation, all in the dimensionless forms. Substitution of Eqs. (32) into (30) leads to

$$v_{,tt} + 2\gamma_0 v_{,xt} + (\kappa \gamma_0^2 - 1) v_{,xx} \\ = -\varepsilon \left\{ 2\gamma_1 \sin(\omega t) (v_{,xt} + \kappa \gamma_0 v_{,xx}) \right. \\ \left. + (1 - x) \omega \gamma_1 \cos(\omega t) v_{,xx} \right.$$

$$\begin{aligned}
 & \left. -k_1^2 v_{,xx} \left(\frac{1}{2} \int_0^1 v_{,x}^2 dx + \alpha \int_0^1 v_{,x} v_{,xt} dx \right) \right\} \\
 & -O(\varepsilon^2) \tag{33}
 \end{aligned}$$

The method of multiple scales will be employed to analyze parametric vibration. The solutions to Eq. (33) can be assumed as

$$\begin{aligned}
 v(x, t; \varepsilon) &= v_0(x, T_0, T_1) + \varepsilon v_1(x, T_0, T_1) \\
 &+ O(\varepsilon^2), \tag{34}
 \end{aligned}$$

where $T_0 = t$ and $T_1 = \varepsilon t$ are, respectively, the fast and the slow time scales. Substitution of Eq. (34) and the following relationship

$$\begin{aligned}
 \frac{\partial}{\partial t} &= \frac{\partial}{\partial T_0} + \varepsilon \frac{\partial}{\partial T_1}, \quad \frac{\partial^2}{\partial t^2} = \frac{\partial^2}{\partial T_0^2} + 2\varepsilon \frac{\partial^2}{\partial T_0 \partial T_1} \\
 &+ O(\varepsilon^2) \tag{35}
 \end{aligned}$$

into Eqs. (33) and (31) and then equalization of coefficients of ε^0 and ε^1 in the resulting equations lead to, at the order ε^0 ,

$$v_{0,T_0 T_0} + 2\gamma_0 v_{0,x T_0} + (\kappa \gamma_0^2 - 1) v_{0,xx} = 0, \tag{36}$$

$$v_0(0, T_0, T_1) = 0, \quad v_0(1, T_0, T_1) = 0; \tag{37}$$

and, at the order ε^1 ,

$$\begin{aligned}
 \varepsilon^1 : \quad & v_{1,T_0 T_0} + 2\gamma_0 v_{1,x T_0} + (\kappa \gamma_0^2 - 1) v_{1,xx} \\
 &= - \left[2(v_{0,T_0 T_1} + \gamma_0 v_{0,x T_1}) + 2\gamma_1 \sin(\omega t) \right. \\
 &\quad \times (v_{0,x T_0} + \kappa \gamma_0 v_{0,xx}) + (1-x) \omega \gamma_1 \cos(\omega t) v_{0,xx} \\
 &\quad \left. - k_1^2 v_{0,xx} \left(\frac{1}{2} \int_0^1 v_{0,x}^2 dx + \alpha \int_0^1 v_{0,x} v_{0,x T_0} dx \right) \right], \tag{38}
 \end{aligned}$$

$$v_1(0, T_0, T_1) = 0, \quad v_1(1, T_0, T_1) = 0; \tag{39}$$

Since the damping term due to the viscoelasticity appears only in Eq. (38), the linear generating system (36) is a gyroscopic continuous system with pure imaginary eigenvalues. The solution is [24]

$$v_0(x, T_0, T_1) = \sum_{n=1}^{\infty} A_n(T_1) \varphi_n(x) e^{i\omega_n T_0} + cc, \tag{40}$$

where

$$\begin{aligned}
 \varphi_n &= \sin(n\pi x) e^{ia_n x}, \quad a_n = \frac{n\pi \gamma_0}{\sqrt{\gamma_0^2 - \kappa \gamma_0^2 + 1}}, \\
 \omega_n &= \frac{n\pi(1 - \kappa \gamma_0^2)}{\sqrt{\gamma_0^2 - \kappa \gamma_0^2 + 1}} \quad (n = 1, 2, \dots). \tag{41}
 \end{aligned}$$

A_n denotes a complex function to be determined later, φ_n and ω_n are, respectively, the n th mode function and natural frequency, and cc stands for the complex conjugate of the preceding terms. It should be noted that, under boundary conditions (37), there exists the strict proportional relationship $\omega_n = n\omega_1$ or $m\omega_n = n\omega_m$ among any nature frequencies. The fact has been neglected in all previous investigations on transverse vibration of axially moving strings with the same conditions.

Here the internal resonances will be treated in a heuristic way. Parametric resonances in the cases of ω near $\omega_1, 2\omega_1$, and $3\omega_1$ are investigated with the presence of the exact internal resonances. In each case, solution (40) is, respectively, retained 1–3 terms. If the different term truncations lead to qualitatively and quantitatively similar results, the results are convincingly applicable even if the effects of infinite high-order internal resonances are neglected. Otherwise, if the different term truncations yield totally diverse results, more sophisticated approaches should be developed to derive the applicable solutions.

The sum of first k terms of the infinite expansion in equation (40) will be referred as the k -term truncated solution or simply the k -term truncation. Then, the 3-term truncated solution to Eq. (36) is expressed by

$$\begin{aligned}
 v_0(x, T_0, T_1) &= \varphi_1(x) A_1(T_1) e^{i\omega_1 T_0} \\
 &+ \varphi_2(x) A_2(T_1) e^{i\omega_2 T_0} \\
 &+ \varphi_3(x) A_3(T_1) e^{i\omega_3 T_0} + cc \tag{42}
 \end{aligned}$$

Substitution of Eqs. (42) into (38) yields

$$\begin{aligned}
 v_{1,T_0 T_0} + 2\gamma_0 v_{1,x T_0} + [(1-\eta)\gamma_0^2 - 1] v_{1,xx} \\
 = -\eta_1 \dot{A}_1 e^{i\omega_1 T_0} - \eta_2 \dot{A}_2 e^{i\omega_2 T_0} - \eta_3 \dot{A}_3 e^{i\omega_3 T_0} \\
 - \bar{\eta}_1 \dot{A}_1 e^{-i\omega_1 T_0} - \bar{\eta}_2 \dot{A}_2 e^{-i\omega_2 T_0} - \bar{\eta}_3 \dot{A}_3 e^{-i\omega_3 T_0} \\
 - \gamma_1 \chi_1 A_1 e^{i(\omega+\omega_1)T_0} - \gamma_1 \chi_2 A_2 e^{i(\omega+\omega_2)T_0} \\
 - \gamma_1 \chi_3 A_3 e^{i(\omega+\omega_3)T_0} - \gamma_1 \bar{\chi}_1 \bar{A}_1 e^{-i(\omega+\omega_1)T_0} \\
 - \gamma_1 \bar{\chi}_2 \bar{A}_2 e^{-i(\omega+\omega_2)T_0} - \gamma_1 \bar{\chi}_3 \bar{A}_3 e^{-i(\omega+\omega_3)T_0} \\
 - \gamma_1 \delta_1 A_1 e^{i(-\omega+\omega_1)T_0} - \gamma_1 \delta_2 A_2 e^{i(-\omega+\omega_2)T_0} \\
 - \gamma_1 \delta_3 A_3 e^{i(-\omega+\omega_3)T_0} - \gamma_1 \bar{\delta}_1 \bar{A}_1 e^{i(\omega-\omega_1)T_0}
 \end{aligned}$$

$$\begin{aligned}
 & -\gamma_1 \bar{\delta}_2 \bar{A}_2 e^{i(\omega-\omega_2)T_0} - \gamma_1 \bar{\delta}_3 \bar{A}_3 e^{i(\omega-\omega_3)T_0} \\
 & - \left[(\kappa_{11} + \alpha \kappa_{12}) A_1^2 \bar{A}_1 + (\mu_{121} + \alpha \mu_{122}) A_1 A_2 \bar{A}_2 \right. \\
 & \left. + (\mu_{131} + \alpha \mu_{132}) A_1 A_3 \bar{A}_3 \right] e^{i\omega_1 T_0} \\
 & - (\tau_{11} + \alpha \tau_{12}) A_3 \bar{A}_1^2 e^{i(-2\omega_1 + \omega_3)T_0} \\
 & - (\varsigma_{11} + \alpha \varsigma_{12}) A_2^2 \bar{A}_3 e^{i(2\omega_2 - \omega_3)T_0} \\
 & - \left[(\kappa_{21} + \alpha \kappa_{22}) A_2^2 \bar{A}_2 + (\mu_{211} + \alpha \mu_{212}) A_1 A_2 \bar{A}_1 \right. \\
 & \left. + (\mu_{231} + \alpha \mu_{232}) A_2 A_3 \bar{A}_3 \right] e^{i\omega_2 T_0} \\
 & - (\xi_{21} + \alpha \xi_{22}) A_1 A_3 \bar{A}_2 e^{i(\omega_1 - \omega_2 + \omega_3)T_0} \\
 & - \left[(\kappa_{31} + \alpha \kappa_{32}) A_3^2 \bar{A}_3 + (\mu_{311} + \alpha \mu_{312}) A_1 A_3 \bar{A}_1 \right. \\
 & \left. + (\mu_{321} + \alpha \mu_{322}) A_2 A_3 \bar{A}_2 \right] e^{i\omega_3 T_0} \\
 & - (\nu_{31} + \alpha \nu_{32}) A_1^3 e^{3i\omega_1 T_0} \\
 & - (\varsigma_{31} + \alpha \varsigma_{32}) A_2^2 \bar{A}_1 e^{i(-\omega_1 + 2\omega_2)T_0} \\
 & + NSNT + cc, \tag{43}
 \end{aligned}$$

where

$$\eta_h = 2(i\omega_h \varphi_h + \gamma_0 \varphi'_h) \quad (h = 1, 2, 3), \tag{44a}$$

$$\begin{aligned}
 \chi_h &= 0.5(1-x)\omega\varphi''_h + \omega_h\varphi'_h - i\kappa\gamma_0\varphi''_h \\
 & \quad (h = 1, 2, 3), \tag{44b}
 \end{aligned}$$

$$\begin{aligned}
 \delta_h &= 0.5(1-x)\omega\varphi''_h - \omega_h\varphi'_h + i\kappa\gamma_0\varphi''_h \\
 & \quad (h = 1, 2, 3), \tag{44c}
 \end{aligned}$$

$$\begin{aligned}
 \kappa_{h1} &= -0.5k_1^2 \left(2\varphi''_h \int_0^1 \varphi'_h \bar{\varphi}'_h dx + \bar{\varphi}''_h \int_0^1 \varphi_h^2 dx \right) \\
 & \quad (h = 1, 2, 3), \tag{44d}
 \end{aligned}$$

$$\kappa_{h2} = -ik_1^2 \omega_h \bar{\varphi}''_h \int_0^1 \varphi_h^2 dx \quad (h = 1, 2, 3), \tag{44e}$$

$$\begin{aligned}
 \mu_{hs1} &= -k_1^2 \left(\varphi''_h \int_0^1 \varphi'_s \bar{\varphi}'_s dx + \varphi''_s \int_0^1 \varphi'_h \bar{\varphi}'_s dx \right. \\
 & \quad \left. + \bar{\varphi}''_s \int_0^1 \varphi'_h \varphi'_s dx \right), \tag{44f}
 \end{aligned}$$

$$\begin{aligned}
 \mu_{hs2} &= -ik_1^2 \left[(\omega_h - \omega_s) \varphi''_s \int_0^1 \varphi'_h \bar{\varphi}'_s dx \right. \\
 & \quad \left. + (\omega_h + \omega_s) \bar{\varphi}''_s \int_0^1 \varphi'_h \varphi'_s dx \right] \\
 & \quad (h = 1, 2, 3; s \neq h), \tag{44g}
 \end{aligned}$$

$$\tau_{11} = -0.5k_1^2 \left(\varphi''_3 \int_0^1 \bar{\varphi}_1^2 dx + 2\bar{\varphi}''_1 \int_0^1 \varphi'_3 \bar{\varphi}'_1 dx \right), \tag{44h}$$

$$\tau_{12} = ik_1^2 \left[\omega_1 \varphi''_3 \int_0^1 \bar{\varphi}_1^2 dx + (\omega_1 - \omega_3) \bar{\varphi}''_1 \int_0^1 \varphi'_3 \bar{\varphi}'_1 dx \right], \tag{44i}$$

$$\begin{aligned}
 s_{h1} &= -0.5k_1^2 \left(2\varphi''_2 \int_0^1 \varphi'_2 \bar{\varphi}'_s dx + \bar{\varphi}''_s \int_0^1 \varphi_2^2 dx \right) \\
 & \quad (\text{if } h = 1, s = 3; \text{ if } h = 3, s = 1), \tag{44j}
 \end{aligned}$$

$$\begin{aligned}
 s_{h2} &= -ik_1^2 \left[(\omega_2 - \omega_s) \varphi''_2 \int_0^1 \varphi'_2 \bar{\varphi}'_s dx + \omega_2 \bar{\varphi}''_s \int_0^1 \varphi_2^2 dx \right] \\
 & \quad (\text{if } h = 1, s = 3; \text{ if } h = 3, s = 1), \tag{44k}
 \end{aligned}$$

$$\begin{aligned}
 \xi_{21} &= -k_1^2 \left(\varphi''_1 \int_0^1 \varphi'_3 \bar{\varphi}'_2 dx + \varphi''_3 \int_0^1 \varphi'_1 \bar{\varphi}'_2 dx + \bar{\varphi}''_2 \int_0^1 \varphi'_1 \varphi'_3 dx \right), \\
 & \tag{44l}
 \end{aligned}$$

$$\begin{aligned}
 \xi_{22} &= -ik_1^2 \left[(\omega_3 - \omega_2) \varphi''_1 \int_0^1 \varphi'_3 \bar{\varphi}'_2 dx \right. \\
 & \quad \left. + (\omega_1 - \omega_2) \varphi''_3 \int_0^1 \varphi'_1 \bar{\varphi}'_2 dx + (\omega_1 + \omega_3) \bar{\varphi}''_2 \int_0^1 \varphi'_1 \varphi'_3 dx \right], \\
 & \tag{44m}
 \end{aligned}$$

$$\nu_{31} = -0.5k_1^2 \varphi''_1 \int_0^1 \varphi_1^2 dx, \tag{44n}$$

$$\nu_{32} = -i\omega_1 k_1^2 \varphi''_1 \int_0^1 \varphi_1^2 dx. \tag{44o}$$

and *NSNT* stands for nonsecular nonlinear terms. They cannot give rise to secular terms for the first three truncations, such as $\exp(3i\omega_2 T_0)$, $\exp(3i\omega_3 T_0)$, $\exp[i(\omega_1 + 2\omega_2)T_0]$, $\exp[i(\omega_1 + 2\omega_3)T_0]$, $\exp[i(\omega_2 + 2\omega_3)T_0]$, $\exp[i(2\omega_1 + \omega_2)T_0]$, $\exp[i(2\omega_1 + \omega_3)T_0]$, $\exp[i(2\omega_2 + \omega_3)T_0]$, $\exp[i(-\omega_1 + 2\omega_3)T_0]$, $\exp[i(-\omega_2 + 2\omega_3)T_0]$, $\exp[i(-2\omega_1 + \omega_2)T_0]$, $\exp[i(\omega_1 + \omega_2 + \omega_3)T_0]$, $\exp[i(-\omega_1 + \omega_2 + \omega_3)T_0]$, and $\exp[i(\omega_1 + \omega_2 - \omega_3)T_0]$.

According to Eq. (43), if the axial speed variation frequency ω approaches integer times of the first natural frequency of the linear generating system (36), parametric resonance may occur.

4 Parametric resonance in the case of ω near ω_1

This case is the secondary parametric resonance in the first mode. The secondary parametric resonance in a

mode means the excitation frequency is close to the natural frequency corresponding to the mode. A detuning parameter σ is introduced to quantify the deviation of ω from ω_1 , and ω is described by

$$\omega = \omega_1 + \varepsilon\sigma \tag{45}$$

The solvability conditions of Eq. (43) demand the orthogonal relationships [27]

$$\begin{aligned} & \left\langle -\eta_1 \dot{A}_1 - \gamma_1 \delta_2 A_2 e^{-i\sigma T_1} - (\kappa_{11} + \alpha\kappa_{12}) A_1^2 \bar{A}_1 \right. \\ & \quad - (\mu_{121} + \alpha\mu_{122}) A_1 A_2 \bar{A}_2 \\ & \quad - (\mu_{131} + \alpha\mu_{132}) A_1 A_3 \bar{A}_3 - (\tau_{11} + \alpha\tau_{12}) A_3 \bar{A}_1^2 \\ & \quad \left. - (\varsigma_{11} + \alpha\varsigma_{12}) A_2^2 \bar{A}_3, \varphi_1 \right\rangle = 0, \end{aligned} \tag{46a}$$

$$\begin{aligned} & \left\langle -\eta_2 \dot{A}_2 - \gamma_1 \chi_1 A_1 e^{i\sigma T_1} - \gamma_1 \delta_3 A_3 e^{-i\sigma T_1} \right. \\ & \quad - (\kappa_{21} + \alpha\kappa_{22}) A_2^2 \bar{A}_2 - (\mu_{211} + \alpha\mu_{212}) A_1 A_2 \bar{A}_1 \\ & \quad - (\mu_{231} + \alpha\mu_{232}) A_2 A_3 \bar{A}_3 \\ & \quad \left. - (\xi_{21} + \alpha\xi_{22}) A_1 A_3 \bar{A}_2, \varphi_2 \right\rangle = 0, \end{aligned} \tag{46b}$$

$$\begin{aligned} & \left\langle -\eta_3 \dot{A}_3 - \gamma_1 \chi_2 A_2 e^{i\sigma T_1} \right. \\ & \quad - (\kappa_{31} + \alpha\kappa_{32}) A_3^2 \bar{A}_3 - (\mu_{311} + \alpha\mu_{312}) A_1 A_3 \bar{A}_1 \\ & \quad - (\mu_{321} + \alpha\mu_{322}) A_2 A_3 \bar{A}_2 - (\nu_{31} + \alpha\nu_{32}) A_1^3 \\ & \quad \left. - (\varsigma_{31} + \alpha\varsigma_{32}) A_2^2 \bar{A}_1, \varphi_3 \right\rangle = 0. \end{aligned} \tag{46c}$$

Application of the distributive law of the inner product for complex functions to Eqs. (46a–c) leads to

$$\begin{aligned} & \dot{A}_1 + \gamma_1 \delta_2 A_2 e^{-i\sigma T_1} + (\kappa_{11} + \alpha\kappa_{12}) A_1^2 \bar{A}_1 \\ & \quad + (\mu_{121} + \alpha\mu_{122}) A_1 A_2 \bar{A}_2 \\ & \quad + (\mu_{131} + \alpha\mu_{132}) A_1 A_3 \bar{A}_3 + (\tau_{11} + \alpha\tau_{12}) A_3 \bar{A}_1^2 \\ & \quad + (\varsigma_{11} + \alpha\varsigma_{12}) A_2^2 \bar{A}_3 = 0, \end{aligned} \tag{47a}$$

$$\begin{aligned} & \dot{A}_2 + \gamma_1 \chi_1 A_1 e^{i\sigma T_1} + \gamma_1 \delta_3 A_3 e^{-i\sigma T_1} \\ & \quad + (\kappa_{21} + \alpha\kappa_{22}) A_2^2 \bar{A}_2 + (\mu_{211} + \alpha\mu_{212}) A_1 A_2 \bar{A}_1 \\ & \quad + (\mu_{231} + \alpha\mu_{232}) A_2 A_3 \bar{A}_3 \\ & \quad + (\xi_{21} + \alpha\xi_{22}) A_1 A_3 \bar{A}_2 = 0, \end{aligned} \tag{47b}$$

$$\begin{aligned} & \dot{A}_3 + \gamma_1 \chi_2 A_2 e^{i\sigma T_1} + (\kappa_{31} + \alpha\kappa_{32}) A_3^2 \bar{A}_3 \\ & \quad + (\mu_{311} + \alpha\mu_{312}) A_1 A_3 \bar{A}_1 \\ & \quad + (\mu_{321} + \alpha\mu_{322}) A_2 A_3 \bar{A}_2 + (\nu_{31} + \alpha\nu_{32}) A_1^3 \\ & \quad + (\varsigma_{31} + \alpha\varsigma_{32}) A_2^2 \bar{A}_1 = 0. \end{aligned} \tag{47c}$$

where

$$\begin{aligned} \ell_h \leftrightarrow & \int_0^1 \ell_h \bar{\varphi}_h dx / \int_0^1 \eta_h \bar{\varphi}_h dx \\ & \times \begin{pmatrix} h = 1, 2, 3; \\ \ell_1 = \delta_2, \kappa_{11}, \kappa_{12}, \mu_{121}, \mu_{122}, \mu_{131}, \mu_{132}, \tau_{11}, \tau_{12}, \varsigma_{11}, \varsigma_{12}; \\ \ell_2 = \chi_1, \delta_3, \kappa_{21}, \kappa_{22}, \mu_{211}, \mu_{212}, \mu_{231}, \mu_{232}, \xi_{21}, \xi_{22}; \\ \ell_3 = \chi_2, \kappa_{31}, \kappa_{32}, \mu_{311}, \mu_{312}, \mu_{321}, \mu_{322}, \nu_{31}, \nu_{32}, \varsigma_{31}, \varsigma_{32} \end{pmatrix} \end{aligned} \tag{48}$$

The detailed expressions of these coefficients can be found in Appendixes 1 and 2. The expressions show that for $h = 1, 2, 3$, $\kappa_{h1}, \mu_{h1}, \tau_{h1}, \xi_{h1}, \nu_{h1}$ and $-\varsigma_{h1}$ are pure imaginary numbers with negative imaginary parts, $\kappa_{h2}, \mu_{h2}, \tau_{h2}, \xi_{h2}, \nu_{h2}$ and $-\varsigma_{h2}$ are positive real numbers, while δ_h and χ_h are complex numbers.

4.1 The 1-term truncation

The 1-term truncated solution to Eq. (36) is expressed by

$$v_0(x, T_0, T_1) = \varphi_1(x) A_1(T_1) e^{i\omega_1 T_0} + cc \tag{49}$$

The solvability condition reduces to

$$\dot{A}_1 + (\kappa_{11} + \alpha\kappa_{12}) A_1^2 \bar{A}_1 = 0 \tag{50}$$

Express the solution to Eq. (50) in the polar form

$$A_1 = \alpha_1(T_1) e^{i\beta_1(T_1)}, \tag{51}$$

where α_1 and β_1 , both real functions of T_1 , are, respectively, the amplitudes and the phase angles of the responses in the 1-term truncation. Substitution of equation (51) into Eq. (50) and separation of real and imaginary parts in the resulting equation give

$$\dot{\alpha}_1 = -\alpha\kappa_{12}\alpha_1^3, \tag{52a}$$

$$\alpha_1 \dot{\beta}_1 = -\kappa_{11}^I \alpha_1^3, \tag{52b}$$

where the superscript I denotes the imaginary part of the corresponding parameter. Equations (52a) and (52b) possess a fixed point at the origin (trivial zero solution). Thus, the only steady-state response is the straight equilibrium configuration.

4.2 The 2-term truncation

The 2-term truncated solution to Eq. (36) is expressed by

$$\begin{aligned} v_0(x, T_0, T_1) = & \varphi_1(x) A_1(T_1) e^{i\omega_1 T_0} \\ & + \varphi_2(x) A_2(T_1) e^{i\omega_2 T_0} + cc \end{aligned} \tag{53}$$

The solvability conditions are simplified to

$$\dot{A}_1 + \gamma_1 \delta_2 A_2 e^{-i\sigma T_1} + (\kappa_{11} + \alpha \kappa_{12}) A_1^2 \bar{A}_1 + (\mu_{121} + \alpha \mu_{122}) A_1 A_2 \bar{A}_2 = 0, \tag{54a}$$

$$\dot{A}_2 + \gamma_1 \chi_1 A_1 e^{i\sigma T_1} + (\kappa_{21} + \alpha \kappa_{22}) A_2^2 \bar{A}_2 + (\mu_{211} + \alpha \mu_{212}) A_1 A_2 \bar{A}_1 = 0. \tag{54b}$$

Express the solution to Eqs. (54a) and (54b) in the polar form

$$A_1 = \alpha_1(T_1) e^{i\beta_1(T_1)}, \quad A_2 = \alpha_2(T_1) e^{i\beta_2(T_1)}, \tag{55}$$

where α_2 and β_2 , both real functions of T_1 , are, respectively, the amplitudes and the phase angles of the responses in the second mode. Substitution of Eq. (55) into Eqs. (54a) and (54b) and separation of real and imaginary parts in the resulting equation yield

$$\dot{\alpha}_1 = -\alpha \alpha_1 (\kappa_{12} \alpha_1^2 + \mu_{122} \alpha_2^2) - \gamma_1 \alpha_2 (\delta_2^R \cos \theta_1 + \delta_2^I \sin \theta_1), \tag{56a}$$

$$\alpha_1 \dot{\beta}_1 = -\alpha_1 (\kappa_{11}^I \alpha_1^2 + \mu_{121}^I \alpha_2^2) + \gamma_1 \alpha_2 (\delta_2^R \sin \theta_1 - \delta_2^I \cos \theta_1), \tag{56b}$$

$$\dot{\alpha}_2 = -\alpha \alpha_2 (\mu_{212} \alpha_1^2 + \kappa_{22} \alpha_2^2) - \gamma_1 \alpha_1 (\chi_1^R \cos \theta_1 - \chi_1^I \sin \theta_1), \tag{56c}$$

$$\alpha_2 \dot{\beta}_2 = -\alpha_2 (\mu_{211}^I \alpha_1^2 + \kappa_{21}^I \alpha_2^2) - \gamma_1 \alpha_1 (\chi_1^R \sin \theta_1 + \chi_1^I \cos \theta_1), \tag{56d}$$

where the superscripts R and I denote real part and imaginary part of the corresponding parameter, respec-

$$\frac{\alpha_2}{\alpha_1} = \sqrt{\frac{H_1 \mu_{122} - \mu_{212} + \sqrt{4H_1 \kappa_{12} \kappa_{22} + (H_1 \mu_{122} - \mu_{212})^2}}{2\kappa_{22}}} = \sqrt{\frac{\sqrt{128^2 a_1^8 (1 + \cos 3a_1)^2 - 18 (9a_1^2 - \pi^2)^4 \cos^2 a_1 \sin^4 a_1} - 128 a_1^4 (1 + \cos 3a_1)}{6 (9a_1^2 - \pi^2)^2 \sin^2 a_1}}, \tag{61}$$

tively, and $\theta_1 = \sigma T_1 + \beta_1 - \beta_2$. Equations (56a)–(56d) possess a fixed point at the origin. From Eqs. (56b) and (56d), one can obtain

$$\theta_1 = \sigma + \alpha_1^2 (\mu_{211}^I - \kappa_{11}^I) + \alpha_2^2 (\kappa_{21}^I - \mu_{121}^I) + \frac{\gamma_1}{\alpha_1 \alpha_2} [(\alpha_1^2 \chi_1^R + \alpha_2^2 \delta_2^R) \sin \theta_1 + (\alpha_1^2 \chi_1^I - \alpha_2^2 \delta_2^I) \cos \theta_1] \tag{57}$$

If there is a nonzero solution, the amplitudes α_1 and α_2 and the new phase angle θ_1 should be constant for the steady-state response,

$$0 = -\alpha \alpha_1 (\kappa_{12} \alpha_1^2 + \mu_{122} \alpha_2^2) + \gamma_1 \alpha_2 (\delta_2^R \cos \theta_1 + \delta_2^I \sin \theta_1), \tag{58a}$$

$$0 = -\alpha \alpha_2 (\mu_{212} \alpha_1^2 + \kappa_{22} \alpha_2^2) - \gamma_1 \alpha_1 (\chi_1^R \cos \theta_1 - \chi_1^I \sin \theta_1), \tag{58b}$$

$$0 = \sigma + \alpha_1^2 (\mu_{211}^I - \kappa_{11}^I) + \alpha_2^2 (\kappa_{21}^I - \mu_{121}^I) + \frac{\gamma_1}{\alpha_1 \alpha_2} [(\alpha_1^2 \chi_1^R + \alpha_2^2 \delta_2^R) \sin \theta_1 + (\alpha_1^2 \chi_1^I - \alpha_2^2 \delta_2^I) \cos \theta_1] \tag{58c}$$

When $\alpha = 0$, for nonzero α_1 and α_2 , Eqs. (58a) and (58b) yield

$$\frac{\chi_1^R}{\delta_2^R} = -\frac{\chi_1^I}{\delta_2^I} \tag{59}$$

It is noticed that δ_2 and χ_1 are dependent of γ_0 and η , but independent of α . Therefore, Eq. (59) still hold for $\alpha \neq 0$. Thus, there is a constant H_1 (possibly dependent of γ_0 and η) such that

$$\frac{\chi_1^R}{\delta_2^R} = -\frac{\chi_1^I}{\delta_2^I} = H_1 \tag{60}$$

Actually, Eqs. (134)–(137) yield $H_1 = -1/2$. Substitution of Eq. (60) into (58) leads to

where Eqs. (107), (109), (117), and (119) are used. Under the square root, the denominator is positive as a square, and the numerator is negative because

$$\sqrt{128^2 a_1^8 (1 + \cos 3a_1)^2 - 18 (9a_1^2 - \pi^2)^4 \cos^2 a_1 \sin^4 a_1} < \sqrt{128^2 a_1^8 (1 + \cos 3a_1)^2} = 128 a_1^4 (1 + \cos 3a_1).$$

Therefore, the right hand of equation (105) is not a real number. Thus, the nontrivial solution does not exist.

4.3 The 3-term truncation

Express the solution to Eqs. (47a)–(47c) in the polar form

$$\begin{aligned} A_1 &= \alpha_1(T_1) e^{i\beta_1(T_1)}, \quad A_2 = \alpha_2(T_1) e^{i\beta_2(T_1)}, \\ A_3 &= \alpha_3(T_1) e^{i\beta_3(T_1)}, \end{aligned} \tag{62}$$

where α_3 and β_3 , both real functions of T_1 , are, respectively, the amplitudes and the phase angles of the responses in the third mode. Substitution of Eqs. (62) into (47) and separation of real and imaginary parts in the resulting equation give

$$\begin{aligned} \dot{\alpha}_1 &= -\alpha\alpha_1(\kappa_{12}\alpha_1^2 + \mu_{122}\alpha_2^2 + \mu_{132}\alpha_3^2) \\ &\quad -\gamma_1\alpha_2(\delta_2^R \cos \theta_1 + \delta_2^I \sin \theta_1) \\ &\quad -\alpha_2^2\alpha_3(\zeta_{11}^I \sin \theta_3 + \alpha\zeta_{12} \cos \theta_3) \\ &\quad -\alpha_1^2\alpha_3(\tau_{11}^I \sin \theta_4 + \alpha\tau_{12} \cos \theta_4), \end{aligned} \tag{63a}$$

$$\begin{aligned} \alpha_1\dot{\beta}_1 &= -\alpha_1(\kappa_{11}^I\alpha_1^2 + \mu_{121}^I\alpha_2^2 + \mu_{131}^I\alpha_3^2) \\ &\quad +\gamma_1\alpha_2(\delta_2^R \sin \theta_1 - \delta_2^I \cos \theta_1) \\ &\quad -\alpha_2^2\alpha_3(\zeta_{11}^I \cos \theta_3 - \alpha\zeta_{12} \sin \theta_3) \\ &\quad -\alpha_1^2\alpha_3(\tau_{11}^I \cos \theta_4 - \alpha\tau_{12} \sin \theta_4), \end{aligned} \tag{63b}$$

$$\begin{aligned} \dot{\alpha}_2 &= -\alpha\alpha_2(\mu_{212}\alpha_1^2 + \kappa_{22}\alpha_2^2 + \mu_{232}\alpha_3^2) \\ &\quad -\gamma_1\alpha_1(\chi_1^R \cos \theta_1 - \chi_1^I \sin \theta_1) \\ &\quad -\gamma_1\alpha_3(\delta_3^R \cos \theta_2 + \delta_3^I \sin \theta_2) \\ &\quad +\alpha_1\alpha_2\alpha_3(\xi_{21}^I \sin \theta_3 - \alpha\xi_{22} \cos \theta_3), \end{aligned} \tag{63c}$$

$$\begin{aligned} \alpha_2\dot{\beta}_2 &= -\alpha_2(\mu_{211}^I\alpha_1^2 + \kappa_{21}^I\alpha_2^2 + \mu_{231}^I\alpha_3^2) \\ &\quad -\gamma_1\alpha_1(\chi_1^R \sin \theta_1 + \chi_1^I \cos \theta_1) \\ &\quad +\gamma_1\alpha_3(\delta_3^R \sin \theta_2 - \delta_3^I \cos \theta_2) \\ &\quad -\alpha_1\alpha_2\alpha_3(\xi_{21}^I \cos \theta_3 + \alpha\xi_{22} \sin \theta_3), \end{aligned} \tag{63d}$$

$$\begin{aligned} \dot{\alpha}_3 &= -\alpha\alpha_3(\mu_{312}\alpha_1^2 + \mu_{322}\alpha_2^2 + \kappa_{32}\alpha_3^2) \\ &\quad -\gamma_1\alpha_2(\chi_2^R \cos \theta_2 - \chi_2^I \sin \theta_2) \\ &\quad -\alpha_1\alpha_2^2(\zeta_{31}^R \sin \theta_3 + \alpha\zeta_{32}^I \cos \theta_3) \\ &\quad +\alpha_1^3(v_{31}^I \sin \theta_4 - \alpha v_{32} \cos \theta_4), \end{aligned} \tag{63e}$$

$$\begin{aligned} \alpha_3\dot{\beta}_3 &= -\alpha_3(\mu_{311}^I\alpha_1^2 + \mu_{321}^I\alpha_2^2 + \kappa_{31}^I\alpha_3^2) \\ &\quad -\gamma_1\alpha_2(\chi_2^R \sin \theta_2 + \chi_2^I \cos \theta_2) \\ &\quad -\alpha_1\alpha_2^2(\zeta_{31}^R \cos \theta_3 - \alpha\zeta_{32}^I \sin \theta_3) \\ &\quad -\alpha_1^3(v_{31}^I \cos \theta_4 + \alpha v_{32} \sin \theta_4), \end{aligned} \tag{63f}$$

where $\theta_1 = \sigma T_1 + \beta_1 - \beta_2$, $\theta_2 = \sigma T_1 + \beta_2 - \beta_3$, $\theta_3 = \beta_1 - 2\beta_2 + \beta_3 = \theta_1 - \theta_2$, and $\theta_4 = 3\beta_1 - \beta_3$. Equations (63a–f) possess a fixed point at the origin. From Eqs. (63b), (63d), and (63f), one can obtain

$$\begin{aligned} \dot{\theta}_1 &= \sigma + \alpha_1^2(\mu_{211}^I - \kappa_{11}^I) + \alpha_2^2(\kappa_{21}^I - \mu_{121}^I) \\ &\quad +\alpha_3^2(\mu_{231}^I - \mu_{131}^I) \\ &\quad +\frac{\gamma_1}{\alpha_1\alpha_2}[(\alpha_1^2\chi_1^I - \alpha_2^2\delta_2^I) \cos \theta_1 \\ &\quad +(\alpha_1^2\chi_1^R + \alpha_2^2\delta_2^R) \sin \theta_1] \\ &\quad +\frac{\gamma_1\alpha_3}{\alpha_2}(\delta_3^I \cos \theta_2 - \delta_3^R \sin \theta_2) \\ &\quad +\frac{\alpha_3}{\alpha_1}[(\alpha_1^2\xi_{21}^I - \alpha_2^2\zeta_{11}^I) \cos \theta_3 \\ &\quad +\alpha(\alpha_1^2\xi_{22} + \alpha_2^2\zeta_{12}) \sin \theta_3] \\ &\quad -\alpha_1\alpha_3(\tau_{11}^I \cos \theta_4 - \alpha\tau_{12} \sin \theta_4), \end{aligned} \tag{64a}$$

$$\begin{aligned} \dot{\theta}_2 &= \sigma + \alpha_1^2(\mu_{311}^I - \mu_{211}^I) \\ &\quad +\alpha_2^2(\mu_{321}^I - \kappa_{21}^I) + \alpha_3^2(\kappa_{31}^I - \mu_{231}^I) \\ &\quad -\frac{\gamma_1\alpha_1}{\alpha_2}(\chi_1^I \cos \theta_1 + \chi_1^R \sin \theta_1) \\ &\quad +\frac{\gamma_1}{\alpha_2\alpha_3}[(\alpha_2^2\chi_2^I - \alpha_3^2\delta_3^I) \cos \theta_2 \\ &\quad +(\alpha_2^2\chi_2^R + \alpha_3^2\delta_3^R) \sin \theta_2] \\ &\quad +\frac{\alpha_1}{\alpha_3}[(\alpha_2^2\zeta_{31}^I - \alpha_3^2\xi_{21}^I) \cos \theta_3 \\ &\quad -\alpha(\alpha_2^2\zeta_{32} + \alpha_3^2\xi_{22}) \sin \theta_3] \\ &\quad +\frac{\alpha_1^3}{\alpha_3}(v_{31}^I \cos \theta_4 + \alpha v_{32} \sin \theta_4), \end{aligned} \tag{64b}$$

$$\begin{aligned} \dot{\theta}_4 &= \alpha_1^2(\mu_{311}^I - 3\kappa_{11}^I) + \alpha_2^2(\mu_{321}^I - 3\mu_{121}^I) \\ &\quad +\alpha_3^2(\kappa_{31}^I - 3\mu_{131}^I) \\ &\quad +\frac{3\gamma_1\alpha_2}{\alpha_1}(\delta_2^I \cos \theta_1 - \delta_2^R \sin \theta_1) \\ &\quad +\frac{\gamma_1\alpha_2}{\alpha_3}(\chi_2^I \cos \theta_2 + \chi_2^R \sin \theta_2) \end{aligned}$$

$$\begin{aligned}
 & + \frac{\alpha_2^2}{\alpha_1 \alpha_3} \left[\left(\alpha_1^2 \zeta_{31}^I - 3\alpha_3^2 \zeta_{11}^I \right) \cos \theta_3 \right. \\
 & \left. - \alpha \left(\alpha_1^2 \zeta_{32} - 3\alpha_3^2 \zeta_{12} \right) \sin \theta_3 \right] \\
 & + \frac{\alpha_1}{\alpha_3} \left[\left(\alpha_1^2 \nu_{31}^I - 3\alpha_3^2 \tau_{11}^I \right) \cos \theta_4 \right. \\
 & \left. + \alpha \left(\alpha_1^2 \nu_{32} + 3\alpha_3^2 \tau_{12} \right) \sin \theta_4 \right]. \tag{64c}
 \end{aligned}$$

If there is a nonzero solution, the amplitudes α_1 and α_2 and the new phase angles $\theta_1, \theta_2,$ and θ_4 should be constant for the steady-state response. Numerical calculations for different sets of parameters do not locate the nonzero solution. Thus, numerical results strongly indicate the nonexistence of nontrivial steady-state response.

In the secondary parametric resonance of the first mode, 1-term, 2-term, and 3-term truncations all reveal the existence of the zero steady-state response and the nonexistence of nontrivial steady-state response.

5 Parametric resonance in the case of ω near $2\omega_1$

This case can be regarded as the principal parametric resonance in the first mode and the secondary parametric resonance in the second mode, since there is internal resonance $\omega_2 = 2\omega_1$. The principal parametric resonance in a mode means the excitation frequency is close to the 2 times of the natural frequency corresponding to the mode. A detuning parameter σ is introduced to quantify the deviation of ω from $2\omega_1$, and ω is described by

$$\omega = 2\omega_1 + \varepsilon\sigma \tag{65}$$

The solvability conditions demand the orthogonal relationships [27]

$$\begin{aligned}
 & \left\langle -\eta_1 \dot{A}_1 - \gamma_1 \delta_3 A_3 e^{-i\sigma T_1} - \gamma_1 \bar{\delta}_1 \bar{A}_1 e^{i\sigma T_1} \right. \\
 & \quad - (\kappa_{11} + \alpha\kappa_{12}) A_1^2 \bar{A}_1 - (\mu_{121} + \alpha\mu_{122}) A_1 A_2 \bar{A}_2 \\
 & \quad - (\mu_{131} + \alpha\mu_{132}) A_1 A_3 \bar{A}_3 - (\tau_{11} + \alpha\tau_{12}) A_3 \bar{A}_1^2 \\
 & \quad \left. - (\zeta_{11} + \alpha\zeta_{12}) A_2^2 \bar{A}_3, \varphi_1 \right\rangle = 0, \tag{66a}
 \end{aligned}$$

$$\begin{aligned}
 & \left\langle -\eta_2 \dot{A}_2 - (\kappa_{21} + \alpha\kappa_{22}) A_2^2 \bar{A}_2 \right. \\
 & \quad - (\mu_{211} + \alpha\mu_{212}) A_1 A_2 \bar{A}_1 \\
 & \quad - (\mu_{231} + \alpha\mu_{232}) A_2 A_3 \bar{A}_3 \\
 & \quad \left. - (\xi_{21} + \alpha\xi_{22}) A_1 A_3 \bar{A}_2, \varphi_2 \right\rangle = 0, \tag{66b}
 \end{aligned}$$

$$\begin{aligned}
 & \left\langle -\eta_3 \dot{A}_3 - \gamma_1 \chi_1 A_1 e^{i\sigma T_1} - (\kappa_{31} + \alpha\kappa_{32}) A_3^2 \bar{A}_3 \right. \\
 & \quad - (\mu_{311} + \alpha\mu_{312}) A_1 A_3 \bar{A}_1 \\
 & \quad - (\mu_{321} + \alpha\mu_{322}) A_2 A_3 \bar{A}_2 - (\nu_{31} + \alpha\nu_{32}) A_1^3 \\
 & \quad \left. - (\zeta_{31} + \alpha\zeta_{32}) A_2^2 \bar{A}_1, \varphi_3 \right\rangle = 0. \tag{66c}
 \end{aligned}$$

Application of the distributive law of the inner product for complex functions to Eqs. (66a–c) leads to

$$\begin{aligned}
 & \dot{A}_1 + \gamma_1 \delta_3 A_3 e^{-i\sigma T_1} + \gamma_1 \bar{\delta}_1 \bar{A}_1 e^{i\sigma T_1} \\
 & \quad + (\kappa_{11} + \alpha\kappa_{12}) A_1^2 \bar{A}_1 + (\mu_{121} + \alpha\mu_{122}) A_1 A_2 \bar{A}_2 \\
 & \quad + (\mu_{131} + \alpha\mu_{132}) A_1 A_3 \bar{A}_3 + (\tau_{11} + \alpha\tau_{12}) A_3 \bar{A}_1^2 \\
 & \quad + (\zeta_{11} + \alpha\zeta_{12}) A_2^2 \bar{A}_3 = 0, \tag{67a}
 \end{aligned}$$

$$\begin{aligned}
 & \dot{A}_2 + (\kappa_{21} + \alpha\kappa_{22}) A_2^2 \bar{A}_2 + (\mu_{211} + \alpha\mu_{212}) A_1 A_2 \bar{A}_1 \\
 & \quad + (\mu_{231} + \alpha\mu_{232}) A_2 A_3 \bar{A}_3 \\
 & \quad + (\xi_{21} + \alpha\xi_{22}) A_1 A_3 \bar{A}_2 = 0, \tag{67b}
 \end{aligned}$$

$$\begin{aligned}
 & \dot{A}_3 + \gamma_1 \chi_1 A_1 e^{i\sigma T_1} + (\kappa_{31} + \alpha\kappa_{32}) A_3^2 \bar{A}_3 \\
 & \quad + (\mu_{311} + \alpha\mu_{312}) A_1 A_3 \bar{A}_1 \\
 & \quad + (\mu_{321} + \alpha\mu_{322}) A_2 A_3 \bar{A}_2 \\
 & \quad + (\nu_{31} + \alpha\nu_{32}) A_1^3 + (\zeta_{31} + \alpha\zeta_{32}) A_2^2 \bar{A}_1 = 0, \tag{67c}
 \end{aligned}$$

where

$$\begin{aligned}
 \ell_h \leftrightarrow & \int_0^1 \ell_h \bar{\varphi}_h dx / \int_0^1 \eta_h \bar{\varphi}_h dx \\
 & \times \begin{pmatrix} h = 1, 2, 3; \\ \ell_1 = \delta_3, \bar{\delta}_1, \kappa_{11}, \kappa_{12}, \mu_{121}, \mu_{122}, \mu_{131}, \mu_{132}, \tau_{11}, \tau_{12}, \zeta_{11}, \zeta_{12}; \\ \ell_2 = \kappa_{21}, \kappa_{22}, \mu_{211}, \mu_{212}, \mu_{231}, \mu_{232}, \xi_{21}, \xi_{22}; \\ \ell_3 = \chi_1, \kappa_{31}, \kappa_{32}, \mu_{311}, \mu_{312}, \mu_{321}, \mu_{322}, \nu_{31}, \nu_{32}, \zeta_{31}, \zeta_{32} \end{pmatrix} \tag{68}
 \end{aligned}$$

The detailed expressions of these coefficients can be found in Appendixes 1 and 3. The expressions show that, for $h = 1, 2, 3, \kappa_{h1}, \mu_{h1}, \tau_{h1}, \xi_{h1}, \nu_{h1}$ and $-\zeta_{h1}$ are pure imaginary numbers with negative imaginary parts, $\kappa_{h2}, \mu_{h2}, \tau_{h2}, \xi_{h2}, \nu_{h2}$ and $-\zeta_{h2}$ are positive real numbers, while δ_h and χ_h are complex numbers.

5.1 The 1-term truncation

The solvability condition reduces to

$$\dot{A}_1 + \gamma_1 \bar{\delta}_1 \bar{A}_1 e^{i\sigma T_1} + (\kappa_{11} + \alpha\kappa_{12}) A_1^2 \bar{A}_1 = 0 \tag{69}$$

Express the solution to Eq. (69) in the polar form (51). Substitution of Eq. (51) into Eq. (69) and separation of real and imaginary parts in the resulting equation yield

$$\dot{\alpha}_1 = -\alpha\kappa_{12}\alpha_1^3 - \gamma_1\alpha_1 \left(\bar{\delta}_1^R \cos \theta_2 - \bar{\delta}_1^I \sin \theta_2 \right), \tag{70a}$$

$$\alpha_1\dot{\beta}_1 = -\kappa_{11}^I\alpha_1^3 - \gamma_1\alpha_1 \left(\bar{\delta}_1^I \cos \theta_2 + \bar{\delta}_1^R \sin \theta_2 \right), \tag{70b}$$

where $\theta_2 = \sigma T_1 - 2\beta_1$. Equations (70a) and (70b) possess a fixed point at the origin. From equation (70b), one obtains

$$\dot{\theta}_2 = \sigma + 2\alpha_1^2\kappa_{11}^I - 2\gamma_1 \left(\bar{\delta}_1^I \cos \theta_2 + \bar{\delta}_1^R \sin \theta_2 \right) \tag{71}$$

If there is a nonzero steady-state response, the amplitude α_1 and the new phase angle θ_2 should be constant,

$$0 = -\alpha\kappa_{12}\alpha_1^2 - \gamma_1 \left(\bar{\delta}_1^R \cos \theta_2 - \bar{\delta}_1^I \sin \theta_2 \right), \tag{72a}$$

$$0 = \sigma + 2\alpha_1^2\kappa_{11}^I - 2\gamma_1 \left(\bar{\delta}_1^I \cos \theta_2 + \bar{\delta}_1^R \sin \theta_2 \right). \tag{72b}$$

Eliminating θ_2 from Eqs. (72a) and (72b), one obtains the amplitude of the steady-state response

$$\begin{aligned} &(\alpha_1)_{1,2} \\ &= \sqrt{\frac{-\sigma\kappa_{11}^I \pm \sqrt{4\left[(\kappa_{11}^I)^2 + (\alpha\kappa_{12})^2\right]\gamma_1^2|\bar{\delta}_1|^2 - \alpha^2\kappa_{12}^2\sigma^2}}{2\left[(\kappa_{11}^I)^2 + (\alpha\kappa_{12})^2\right]}} \end{aligned} \tag{73a}$$

Inserting equations (106), (107), (144), and (145) into Eq. (73) can express explicitly the amplitude $(\alpha_1)_{1,2}$ in $k_1, \kappa, \alpha, \gamma_0, \gamma_1, a_1, \omega_1$, and σ . Equation (73) leads to the existence conditions of the steady-state responses as

$$\begin{aligned} \sigma &\leq 2\gamma_1|\bar{\delta}_1| \frac{\sqrt{(\kappa_{11}^I)^2 + (\alpha\kappa_{12})^2}}{\alpha\kappa_{12}} \\ &= \frac{\gamma_1}{8\pi^2\alpha\omega_1 a_1^2 \sin^2 a_1 (\omega_1 + a_1\gamma_0)} \\ &\quad \times \sqrt{\left[2a_1^2 (a_1^2 + \pi^2)^2 + \pi^4 \sin^2 a_1\right]^2 + 4\pi^8\alpha^2\omega_1^2 \sin^4 a_1} \\ &\quad \times \sqrt{4\sin^4 a_1 (\omega_1 - 2a_1\kappa\gamma_0)^2 + [(\omega_1 - 2a_1\kappa\gamma_0) \sin 2a_1 - 2a_1\omega_1]^2}, \end{aligned} \tag{74a}$$

and

$$\begin{aligned} \sigma &\geq \pm 2\gamma_1|\bar{\delta}_1| \\ &= \pm \frac{\pi^2\gamma_1\sqrt{4\sin^4 a_1 (\omega_1 - 2a_1\kappa\gamma_0)^2 + [(\omega_1 - 2a_1\kappa\gamma_0) \sin 2a_1 - 2a_1\omega_1]^2}}{4a_1^2(\omega_1 + a_1\gamma_0)}, \end{aligned} \tag{74b}$$

where Eqs. (106), (107), (144), and (145) are used. The Jacobian matrix of the right hand functions of Eqs. (70a) and (71) calculated at the steady-state response is

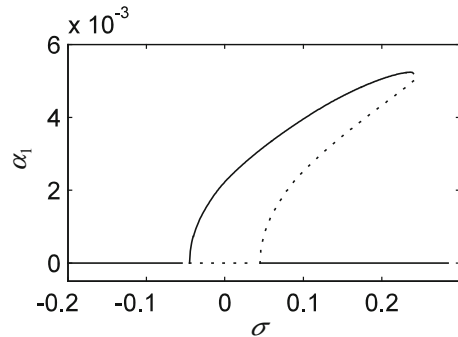


Fig. 2 The steady-state responses in parametric resonance ($\omega \approx 2\omega_1$) based on the 1-term truncation

$$\mathbf{J} = \begin{pmatrix} -2\alpha\kappa_{12}(\alpha_1)_{1,2}^2 - (\alpha_1)_{1,2} [0.5\sigma + \kappa_{11}^I(\alpha_1)_{1,2}^2] & \\ 4\kappa_{11}^I(\alpha_1)_{1,2} & -2\alpha\kappa_{12}(\alpha_1)_{1,2}^2 \end{pmatrix} \tag{75}$$

The characteristic equation of the Jacobian matrix is

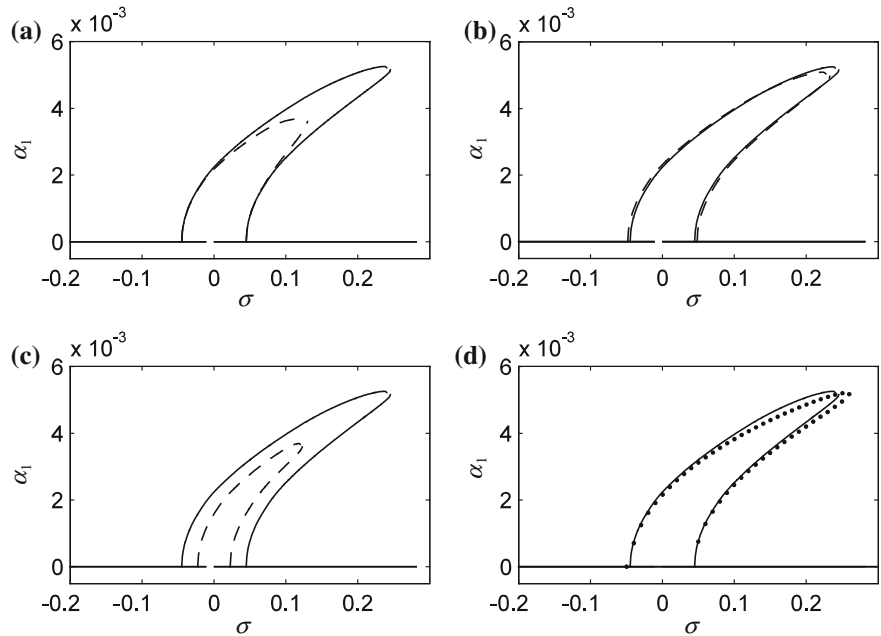
$$\begin{aligned} \lambda^2 + 4\alpha\kappa_{12}(\alpha_1)_{1,2}^2\lambda + 2\left\{\kappa_{11}^I\sigma + 2\left[(\kappa_{11}^I)^2 + (\alpha\kappa_{12})^2\right](\alpha_1)_{1,2}^2\right\}(\alpha_1)_{1,2}^2 &= 0 \end{aligned} \tag{76}$$

According to the Routh–Hurwitz criterion, the first nontrivial solution (positive sign chosen in Eq. (73)) is always stable, and the second nontrivial solution (negative sign chosen in Eq. (73)) is always unstable.

To demonstrate the analytical results, consider a string with $\rho = 7680\text{kg/m}^3$, $E = 0.3 \times 10^{11}\text{Pa}$, $\alpha = 3.0358 \times 10^7\text{Pa} \cdot \text{s}$, $A = 0.04 \times 0.02\text{m}^2$, $L = 1\text{m}$, and $P_0 = 60\text{kN}$. The string axially moves in constant mean speed $\gamma_0 = 19.7642\text{m/s}$ plus simple harmonic variation with the amplitude $\gamma_1 = 0.9882\text{m/s}$. The corresponding dimensionless values are $\alpha = 0.1$, $\gamma_0 = 0.2$, $\gamma_1 = 0.01$, and $k_1 = 20$. Choose $\eta = 0.5$. The set of the parameters will be used in the calculation unless other values are assigned. The first three natural frequencies of the linear generating system are $\omega_1 = 3.048427442$, $\omega_2 = 6.096854884$, and $\omega_3 = 9.145282326$.

Figure 2 depicts the relationship between the amplitude and the detuning parameter for the steady-state responses based on the 1-term truncation. The solid line denotes stable response and the dot line denotes unstable response. If the detuning parameter becomes larger and approaches to zero enough from the negative, the zero solution loses its stability and bifurcates into a nonzero stable response. If the detuning parameter becomes larger and leaves the zero far enough

Fig. 3 The effects of parameters on the responses ($\omega \approx 2\omega_1$) based on the 1-term truncation: **a** the viscosity coefficients (solid $\alpha = 0.1$; dashed $\alpha = 0.2$), **b** the mean axial speeds ($\gamma_0 = 0.2$; dashed $\gamma_0 = 0.1$), **c** the axial speed fluctuation amplitudes (solid $\gamma_1 = 0.01$; dashed $\gamma_1 = 0.005$), **d** the axial support rigidity parameter (dot $\eta = 0.0$; solid $\eta = 0.5$)



to the positive, the zero solution becomes stable again and bifurcates an unstable nonzero response. The interval between two bifurcations will be referred as the instability range. The amplitudes of both the stable and unstable responses increase with the increasing detuning parameter.

The effects of relevant parameters on the steady-state response can be numerically revealed. The effects of the viscosity coefficients are shown in Fig. 3a. The smaller viscosity coefficient leads to the larger stable amplitude and the larger instability range. The effects of the mean axial speeds are demonstrated in Fig. 3b. The smaller mean axial speed leads to the larger amplitude and the larger instability range. The effects of the axial speed fluctuation amplitudes are exhibited in Fig. 3c. The larger axial speed fluctuation amplitude leads to the larger amplitude in the responses but the larger instability range. The effects of the support rigidity parameters are displayed in Fig. 3d. The smaller support rigidity parameter leads to the slightly smaller amplitude in the responses and the slightly larger instability range.

5.2 The 2-term truncation

The solvability conditions are simplified to

$$\begin{aligned} \dot{A}_1 + \gamma_1 \bar{\delta}_1 \bar{A}_1 e^{i\sigma T_1} + (\kappa_{11} + \alpha \kappa_{12}) A_1^2 \bar{A}_1 \\ + (\mu_{121} + \alpha \mu_{122}) A_1 A_2 \bar{A}_2 = 0, \end{aligned} \tag{77a}$$

$$\begin{aligned} \dot{A}_2 + (\kappa_{21} + \alpha \kappa_{22}) A_2^2 \bar{A}_2 \\ + (\mu_{211} + \alpha \mu_{212}) A_1 A_2 \bar{A}_1 = 0. \end{aligned} \tag{77b}$$

Express the solution to Eqs. (77a) and (77b) in the polar form (55). Substitution of Eq. (55) into Eqs. (77a), (77b) and separation of real and imaginary parts in the resulting equation give

$$\begin{aligned} \dot{\alpha}_1 = -\alpha \alpha_1 (\kappa_{12} \alpha_1^2 + \mu_{122} \alpha_2^2) \\ - \gamma_1 \alpha_1 (\bar{\delta}_1^R \cos \theta_2 - \bar{\delta}_1^I \sin \theta_2), \end{aligned} \tag{78a}$$

$$\begin{aligned} \alpha_1 \dot{\beta}_1 = -\alpha_1 (\kappa_{11}^I \alpha_1^2 + \mu_{121}^I \alpha_2^2) \\ - \gamma_1 \alpha_1 (\bar{\delta}_1^I \cos \theta_2 + \bar{\delta}_1^R \sin \theta_2), \end{aligned} \tag{78b}$$

$$\dot{\alpha}_2 = -\alpha \alpha_2 (\mu_{212} \alpha_1^2 + \kappa_{22} \alpha_2^2), \tag{78c}$$

$$\alpha_2 \dot{\beta}_2 = -\alpha_2 (\mu_{211}^I \alpha_1^2 + \kappa_{21}^I \alpha_2^2), \tag{78d}$$

where $\theta_2 = \sigma T_1 - 2\beta_1$. Equations (78a)–(78d) possess a fixed point at the origin. If there is a nonzero steady-state response in the second mode, the amplitude α_2 and the phase angle β_2 should be constant. Equations (78c,d) yield

$$\mu_{212} \alpha_1^2 + \kappa_{22} \alpha_2^2 = 0, \tag{79a}$$

$$\mu_{211}^I \alpha_1^2 + \kappa_{21}^I \alpha_2^2 = 0. \tag{79b}$$

However, Eqs. (116)–(119) imply

$$\frac{\kappa_{22}}{\mu_{212}} \neq \frac{\kappa_{21}^I}{\mu_{211}^I} \tag{80}$$

So Eqs. (79a, b) lead to a contradiction if $\alpha_2 \neq 0$. Therefore, the amplitudes α_2 must be equal to zero. The second mode has actually no effect on the first mode in the 2-term truncation. Thus, the 2-term truncation yields the same result as that of the 1-term truncation.

5.3 The 3-term truncation

Express the solution to Eqs. (67a–c) in the polar form (62). Substitution of Eq. (62) into them and separation of real and imaginary parts in the resulting equation give

$$\begin{aligned} \dot{\alpha}_1 = & -\alpha\alpha_1 \left(\kappa_{12}\alpha_1^2 + \mu_{122}\alpha_2^2 + \mu_{132}\alpha_3^2 \right) \\ & -\gamma_1\alpha_3 \left(\delta_3^R \cos \theta_1 + \delta_3^I \sin \theta_1 \right) \\ & -\gamma_1\alpha_1 \left(\bar{\delta}_1^R \cos \theta_2 - \bar{\delta}_1^I \sin \theta_2 \right) \\ & -\alpha_2^2\alpha_3 \left(\varsigma_{11}^I \sin \theta_3 + \alpha\varsigma_{12} \cos \theta_3 \right) \\ & -\alpha_1^2\alpha_3 \left(\tau_{11}^I \sin \theta_4 + \alpha\tau_{12} \cos \theta_4 \right), \end{aligned} \tag{81a}$$

$$\begin{aligned} \alpha_1\dot{\beta}_1 = & -\alpha_1 \left(\kappa_{11}^I\alpha_1^2 + \mu_{121}^I\alpha_2^2 + \mu_{131}^I\alpha_3^2 \right) \\ & -\gamma_1\alpha_3 \left(\delta_3^I \cos \theta_1 - \delta_3^R \sin \theta_1 \right) \\ & -\gamma_1\alpha_1 \left(\bar{\delta}_1^I \cos \theta_2 + \bar{\delta}_1^R \sin \theta_2 \right) \\ & -\alpha_2^2\alpha_3 \left(\varsigma_{11}^I \cos \theta_3 - \alpha\varsigma_{12} \sin \theta_3 \right) \\ & -\alpha_1^2\alpha_3 \left(\tau_{11}^I \cos \theta_4 - \alpha\tau_{12} \sin \theta_4 \right), \end{aligned} \tag{81b}$$

$$\begin{aligned} \dot{\alpha}_2 = & -\alpha\alpha_2 \left(\mu_{212}\alpha_1^2 + \kappa_{22}\alpha_2^2 + \mu_{232}\alpha_3^2 \right) \\ & +\alpha_1\alpha_2\alpha_3 \left(\xi_{21}^I \sin \theta_3 - \alpha\xi_{22} \cos \theta_3 \right), \end{aligned} \tag{81c}$$

$$\begin{aligned} \alpha_2\dot{\beta}_2 = & -\alpha_2 \left(\mu_{211}^I\alpha_1^2 + \kappa_{21}^I\alpha_2^2 + \mu_{231}^I\alpha_3^2 \right) \\ & -\alpha_1\alpha_2\alpha_3 \left(\xi_{21}^I \cos \theta_3 + \alpha\xi_{22} \sin \theta_3 \right), \end{aligned} \tag{81d}$$

$$\begin{aligned} \dot{\alpha}_3 = & -\alpha\alpha_3 \left(\mu_{312}\alpha_1^2 + \mu_{322}\alpha_2^2 + \kappa_{32}\alpha_3^2 \right) \\ & -\gamma_1\alpha_1 \left(\chi_1^R \cos \theta_1 - \chi_1^I \sin \theta_1 \right) \\ & -\alpha_1\alpha_2^2 \left(\varsigma_{31}^R \sin \theta_3 + \alpha\varsigma_{32}^I \cos \theta_3 \right) \\ & +\alpha_1^3 \left(\upsilon_{31}^I \sin \theta_4 - \alpha\upsilon_{32} \cos \theta_4 \right), \end{aligned} \tag{81e}$$

$$\begin{aligned} \alpha_3\dot{\beta}_3 = & -\alpha_3 \left(\mu_{311}^I\alpha_1^2 + \mu_{321}^I\alpha_2^2 + \kappa_{31}^I\alpha_3^2 \right) \\ & -\gamma_1\alpha_1 \left(\chi_1^R \sin \theta_1 + \chi_1^I \cos \theta_1 \right) \\ & -\alpha_1\alpha_2^2 \left(\varsigma_{31}^R \cos \theta_3 - \alpha\varsigma_{32}^I \sin \theta_3 \right) \\ & -\alpha_1^3 \left(\upsilon_{31}^I \cos \theta_4 + \alpha\upsilon_{32} \sin \theta_4 \right), \end{aligned} \tag{81f}$$

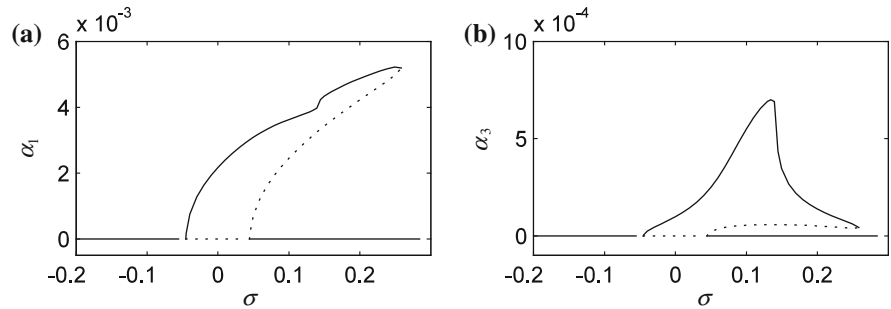
where $\theta_1 = \sigma T_1 + \beta_1 - \beta_3$, $\theta_2 = \sigma T_1 - 2\beta_1$, $\theta_3 = \beta_1 - 2\beta_2 + \beta_3$, and $\theta_4 = 3\beta_1 - \beta_3 = \theta_1 - \theta_2$. Equations (81a–f) possess a fixed point at the origin. From Eqs. (81b), (81d), and (81f), one gets

$$\begin{aligned} \dot{\theta}_1 = & \sigma + \alpha_1^2 \left(\mu_{311}^I - \kappa_{11}^I \right) + \alpha_2^2 \left(\mu_{321}^I - \mu_{121}^I \right) \\ & + \alpha_3^2 \left(\kappa_{31}^I - \mu_{131}^I \right) \\ & + \frac{\gamma_1}{\alpha_1\alpha_3} \left[\left(\alpha_1^2\chi_1^I - \alpha_3^2\delta_3^I \right) \cos \theta_1 \right. \\ & \left. + \left(\alpha_1^2\chi_1^R + \alpha_3^2\delta_3^R \right) \sin \theta_1 \right] \\ & -\gamma_1 \left(\bar{\delta}_1^I \cos \theta_2 + \bar{\delta}_1^R \sin \theta_2 \right) \\ & + \frac{\alpha_2^2}{\alpha_1\alpha_3} \left[\left(\alpha_1^2\varsigma_{31}^I - \alpha_3^2\varsigma_{11}^I \right) \cos \theta_3 \right. \\ & \left. - \alpha \left(\alpha_1^2\varsigma_{32} - \alpha_3^2\varsigma_{12} \right) \sin \theta_3 \right] \\ & + \frac{\alpha_1}{\alpha_3} \left[\left(\alpha_1^2\upsilon_{31}^I - \alpha_3^2\tau_{11}^I \right) \cos \theta_4 \right. \\ & \left. + \alpha \left(\alpha_1^2\upsilon_{32} + \alpha_3^2\tau_{12} \right) \sin \theta_4 \right], \end{aligned} \tag{82a}$$

$$\begin{aligned} \dot{\theta}_2 = & \sigma + 2 \left(\alpha_1^2\kappa_{11}^I + \alpha_2^2\mu_{121}^I + \alpha_3^2\mu_{131}^I \right) \\ & + \frac{2\gamma_1\alpha_3}{\alpha_1} \left(\delta_3^I \cos \theta_1 - \delta_3^R \sin \theta_1 \right) \\ & + 2\gamma_1 \left(\bar{\delta}_1^I \cos \theta_2 + \bar{\delta}_1^R \sin \theta_2 \right) \\ & + \frac{2\alpha_2^2\alpha_3}{\alpha_1} \left(\varsigma_{11}^I \cos \theta_3 - \alpha\varsigma_{12} \sin \theta_3 \right) \\ & + 2\alpha_1\alpha_3 \left(\tau_{11}^I \cos \theta_4 - \alpha\tau_{12} \sin \theta_4 \right), \end{aligned} \tag{82b}$$

$$\begin{aligned} \dot{\theta}_3 = & -\alpha_2^2 \left(\kappa_{11}^I - 2\mu_{211}^I + \mu_{311}^I \right) \\ & + \alpha_2^2 \left(2\kappa_{21}^I - \mu_{121}^I - \mu_{321}^I \right) \\ & -\alpha_3^2 \left(\kappa_{31}^I - 2\mu_{231}^I + \mu_{131}^I \right) \\ & - \frac{\gamma_1}{\alpha_1\alpha_3} \left[\left(\alpha_1^2\chi_1^I + \alpha_3^2\delta_3^I \right) \cos \theta_1 \right. \\ & \left. + \left(\alpha_1^2\chi_1^R - \alpha_3^2\delta_3^R \right) \sin \theta_1 \right] \\ & -\gamma_1 \left(\bar{\delta}_1^I \cos \theta_2 + \bar{\delta}_1^R \sin \theta_2 \right) \end{aligned}$$

Fig. 4 The steady-state responses in parametric resonance ($\omega \approx 2\omega_1$) based on the 3-term truncation: **a** the first mode and **b** the third mode



$$\begin{aligned}
 & -\frac{1}{\alpha_1\alpha_3} \left\{ \left[\alpha_2^2 \left(\alpha_1^2 \zeta_{31}^I + \alpha_3^2 \zeta_{11}^I \right) \right. \right. \\
 & \left. \left. - 2\alpha_1^2 \alpha_3^2 \xi_{21}^I \right] \cos \theta_3 \right. \\
 & \left. - \alpha \left[\alpha_2^2 \left(\alpha_1^2 \zeta_{32} + \alpha_3^2 \zeta_{12} \right) + 2\alpha_1^2 \alpha_3^2 \xi_{22} \right] \sin \theta_3 \right\} \\
 & -\frac{\alpha_1}{\alpha_3} \left[\left(\alpha_1^2 \nu_{31}^I + \alpha_3^2 \tau_{11}^I \right) \cos \theta_4 \right. \\
 & \left. + \alpha \left(\alpha_1^2 \nu_{32} - \alpha_3^2 \tau_{12} \right) \sin \theta_4 \right]. \tag{82c}
 \end{aligned}$$

If one assumes that there is a nonzero steady-state response, then the amplitudes α_1 , α_2 , and α_3 and the new phase angles θ_1 , θ_2 , and θ_3 should be constant. Numerical calculations for different sets of parameters strongly indicate that α_2 is equal to zero. To reduce the amount of computation, α_2 will be omitted in the following investigation. Even if $\alpha_2 = 0$ is only ad hoc for numerical simplification, at least the calculation results will not be affected. Then, the solvability conditions reduce

$$\begin{aligned}
 \dot{A}_1 + \gamma_1 \delta_3 A_3 e^{-i\sigma T_1} + \gamma_1 \bar{\delta}_1 \bar{A}_1 e^{i\sigma T_1} \\
 + (\kappa_{11} + \alpha \kappa_{12}) A_1^2 \bar{A}_1 + (\mu_{131} + \alpha \mu_{132}) A_1 A_3 \bar{A}_3 \\
 + (\tau_{11} + \alpha \tau_{12}) A_3 \bar{A}_1^2 = 0, \tag{83a}
 \end{aligned}$$

$$\begin{aligned}
 \dot{A}_3 + \gamma_1 \chi_1 A_1 e^{i\sigma T_1} + (\kappa_{31} + \alpha \kappa_{32}) A_3^2 \bar{A}_3 \\
 + (\mu_{311} + \alpha \mu_{312}) A_1 A_3 \bar{A}_1 + (\nu_{31} + \alpha \nu_{32}) A_1^3 = 0. \tag{83b}
 \end{aligned}$$

Separation of real and imaginary parts in the resulting equation gives

$$\begin{aligned}
 \dot{\alpha}_1 = & -\alpha \alpha_1 \left(\kappa_{12} \alpha_1^2 + \mu_{132} \alpha_3^2 \right) \\
 & - \gamma_1 \alpha_3 \left(\delta_3^R \cos \theta_1 + \delta_3^I \sin \theta_1 \right) \\
 & - \gamma_1 \alpha_1 \left(\bar{\delta}_1^R \cos \theta_2 - \bar{\delta}_1^I \sin \theta_2 \right) \\
 & - \alpha_1^2 \alpha_3 \left(\tau_{11}^I \sin \theta_4 + \alpha \tau_{12} \cos \theta_4 \right), \tag{84a}
 \end{aligned}$$

$$\begin{aligned}
 \dot{\alpha}_3 = & -\alpha \alpha_3 \left(\mu_{312} \alpha_1^2 + \kappa_{32} \alpha_3^2 \right) \\
 & - \gamma_1 \alpha_1 \left(\chi_1^R \cos \theta_1 - \chi_1^I \sin \theta_1 \right) \\
 & + \alpha_1^3 \left(\nu_{31}^I \sin \theta_4 - \alpha \nu_{32} \cos \theta_4 \right), \tag{84b}
 \end{aligned}$$

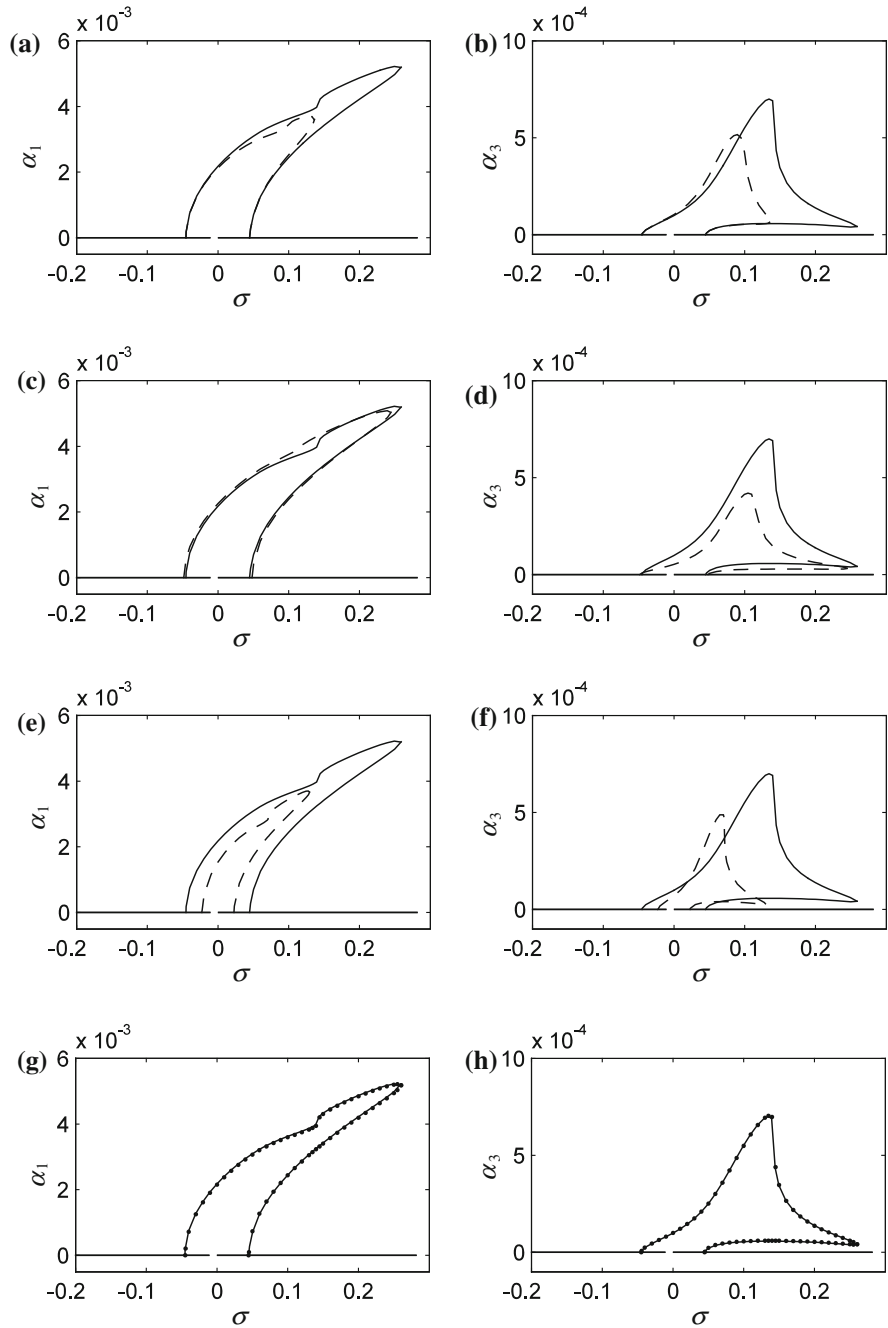
$$\begin{aligned}
 \dot{\theta}_1 = & \sigma + \alpha_1^2 \left(\mu_{311}^I - \kappa_{11}^I \right) \\
 & + \alpha_3^2 \left(\kappa_{31}^I - \mu_{131}^I \right) \\
 & + \frac{\gamma_1}{\alpha_1 \alpha_3} \left[\left(\alpha_1^2 \chi_1^I - \alpha_3^2 \delta_3^I \right) \cos \theta_1 \right. \\
 & \left. + \left(\alpha_1^2 \chi_1^R + \alpha_3^2 \delta_3^R \right) \sin \theta_1 \right] \\
 & - \gamma_1 \left(\bar{\delta}_1^I \cos \theta_2 + \bar{\delta}_1^R \sin \theta_2 \right) \\
 & + \frac{\alpha_1}{\alpha_3} \left[\left(\alpha_1^2 \nu_{31}^I - \alpha_3^2 \tau_{11}^I \right) \cos \theta_4 \right. \\
 & \left. + \alpha \left(\alpha_1^2 \nu_{32} + \alpha_3^2 \tau_{12} \right) \sin \theta_4 \right], \tag{84c}
 \end{aligned}$$

$$\begin{aligned}
 \dot{\theta}_2 = & \sigma + 2 \left(\alpha_1^2 \kappa_{11}^I + \alpha_3^2 \mu_{131}^I \right) \\
 & + \frac{2\gamma_1 \alpha_3}{\alpha_1} \left(\delta_3^I \cos \theta_1 - \delta_3^R \sin \theta_1 \right) \\
 & + 2\gamma_1 \left(\bar{\delta}_1^I \cos \theta_2 + \bar{\delta}_1^R \sin \theta_2 \right) \\
 & + 2\alpha_1 \alpha_3 \left(\tau_{11}^I \cos \theta_4 - \alpha \tau_{12} \sin \theta_4 \right), \tag{84d}
 \end{aligned}$$

The amplitudes of the steady-state oscillating response in the 3-term truncation can be solved from Eqs. (84a)–(84d). The stability of the nontrivial solutions can be determined by eigenvalues of the Jacobian matrix of Eqs. (84a)–(84d) calculated at the nontrivial solutions.

Figure 4 depicts the relationship between the amplitude and the detuning parameter for the responses based on the 3-term truncation for the parameters used in 5.2, namely $\alpha = 0.1$, $\gamma_0 = 0.2$, $\gamma_1 = 0.01$, $k_1 = 20$, and $\eta = 0.5$. In Fig. 4, the solid line denotes stable solutions and the dot line denotes unstable solutions. The behaviors of the first mode are similar to the outcomes based on the 1-term truncation shown in Fig. 2. However, the

Fig. 5 The effects of parameters on the responses ($\omega \approx 2\omega_1$) based on the 3-term truncation: **a** the viscosity coefficients (*solid* $\alpha = 0.1$, *dashed* $\alpha = 0.2$): the first mode, **b** the viscosity coefficients (*solid* $\alpha = 0.1$, *dashed* $\alpha = 0.2$): the third mode, **c** the mean axial speeds (*solid* $\gamma_0 = 0.2$; *dashed* $\gamma_0 = 0.1$): the first mode, **d** the mean axial speeds (*solid* $\gamma_0 = 0.2$; *dashed* $\gamma_0 = 0.1$): the third mode, **e** the axial speed fluctuation amplitudes (*solid* $\gamma_1 = 0.01$; *dashed* $\gamma_1 = 0.005$): the first mode, **f** the axial speed fluctuation amplitudes (*solid* $\gamma_1 = 0.01$; *dashed* $\gamma_1 = 0.005$): the third mode, **g** the axial support rigidity parameter (*dot* $\eta = 0.0$; *solid* $\eta = 0.5$): the first mode, **h** the axial support rigidity parameter (*dot* $\eta = 0.0$; *solid* $\eta = 0.5$): the third mode

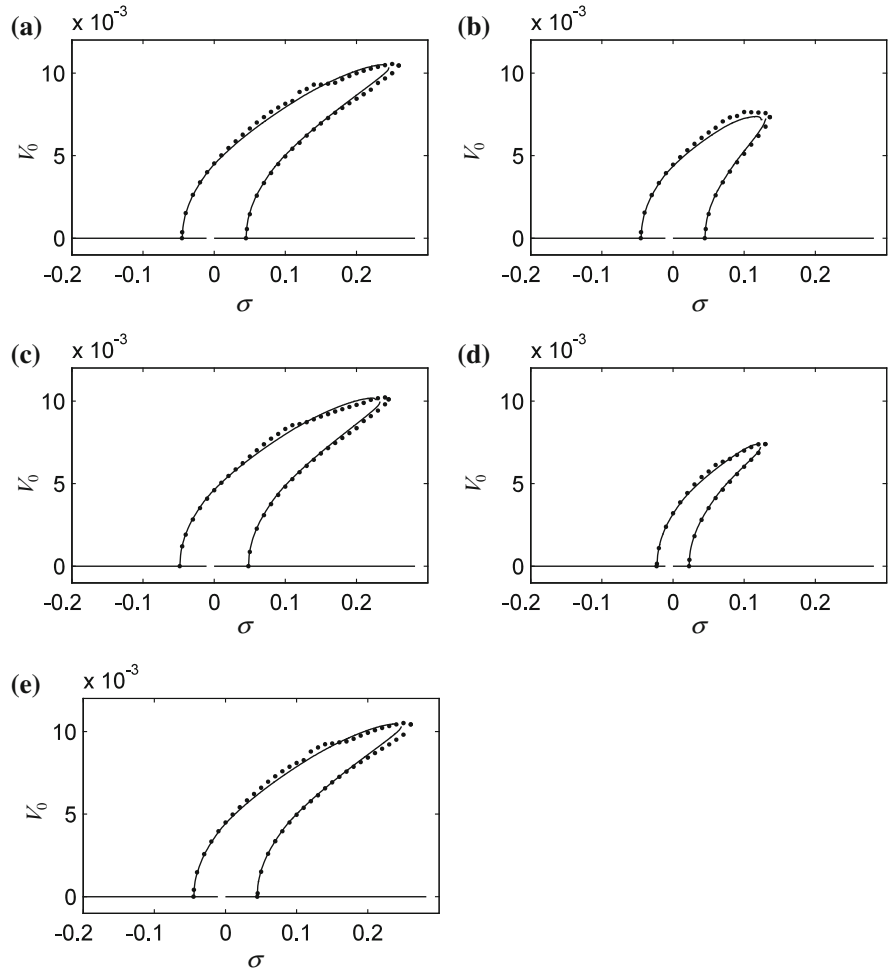


3-term truncation gives additionally the response of the third mode, but the response of the third mode is much smaller than that of the first mode. The effects of the viscosity, the mean axial speed, the axial speed fluctuation amplitude, and the axial support rigidity are illustrated in Fig. 5. The changing trends of the responses in the first mode are the same as those obtained via the 1-term truncation.

5.4 Comparisons of the results in the 1-term and the 3-term truncations

The previous two subsections demonstrate that the amplitudes of steady-state periodic responses in the first mode predicted based on the 1-term truncation change in the qualitatively same way as those on the 3-term truncation do, in despite the existence of the

Fig. 6 The comparison of the 1-term and the 3-term truncations for different parameters: **a** $\alpha = 0.1, \gamma_0 = 0.2, \gamma_1 = 0.01,$ and $\eta = 0.5,$ **b** $\alpha = 0.2, \gamma_0 = 0.2, \gamma_1 = 0.01,$ and $\eta = 0.5,$ **c** $\alpha = 0.1, \gamma_0 = 0.1, \gamma_1 = 0.01,$ and $\eta = 0.5,$ **d** $\alpha = 0.1, \gamma_0 = 0.2, \gamma_1 = 0.005,$ and $\eta = 0.5,$ **e** $\alpha = 0.1, \gamma_0 = 0.2, \gamma_1 = 0.01,$ and $\eta = 0.0$



steady-state response in the third mode yielded via 3-term truncation. In the following, the results based on the 1-term truncation and the 3-term truncation will be quantitatively contrasted.

To examine the quantitative differences with the account of the steady-state in the third mode, the vibration of the string center is calculated based on the 1-term truncation and the 3-term truncation, respectively. In the 1-term truncation, the amplitudes of the string center steady-state responses are $V_0 = 2\alpha_1|\varphi_1(0.5)|$, where α_1 is defined by Eq. (73) and modal function $\varphi_1(x)$ is defined by Eq. (41). In the 3-term truncation, substitution of Eqs. (62) into (42) yields

$$v_0 = \alpha_1 \left[\varphi_1 e^{i(\beta_1 + \omega_1 T_0)} + \bar{\varphi}_1 e^{-i(\beta_1 + \omega_1 T_0)} \right] + \alpha_3 \left[\varphi_3 e^{i(\beta_3 + \omega_3 T_0)} + \bar{\varphi}_3 e^{-i(\beta_3 + \omega_3 T_0)} \right] \quad (85)$$

As $\cos(\omega t) = (e^{\omega t} + e^{-\omega t})/2$ and $\sin(\omega t) = (e^{\omega t} - e^{-\omega t})/2i$, Eq. (85) can be expressed by

$$v_0 = V_1 \cos \delta_1 + V_3 \cos \delta_3, \quad (86)$$

where

$$V_1 = 2\alpha_1 |\varphi_1|, \delta_1 = \left(\omega_1 + \frac{\sigma}{2} \right) T_0 + \arccos \frac{\varphi_1^R}{|\varphi_1|} - \theta_2, \\ V_3 = 2\alpha_3 |\varphi_3|, \delta_3 = 3 \left(\omega_1 + \frac{\sigma}{2} \right) T_0 + \arccos \frac{\varphi_3^R}{|\varphi_3|} - \frac{\theta_2}{2} - \theta_1. \quad (87)$$

Then, the amplitude of the stable steady-state response V_0 of the 3-term truncation is the maximum of v_0 in equation (86). Figure 6 depicts the steady-state responses of the string center changing with the detuning parameter based on the 1-term truncation (dots) and the 3-term truncation (solid lines), respectively.

The comparisons demonstrate that they have a good agreement for the different 5 sets of the parameters.

6 Parametric resonance in the case of ω near $3\omega_1$

A detuning parameter σ is introduced to quantify the deviation of ω from $3\omega_1$, and ω is described by

$$\omega = 3\omega_1 + \varepsilon\sigma \tag{88}$$

The solvability conditions demand the orthogonal relationships [27]

$$\begin{aligned} & \left\langle -\eta_1 \dot{A}_1 - \gamma_1 \bar{\delta}_2 \bar{A}_2 e^{i\sigma T_1} - (\kappa_{11} + \alpha\kappa_{12}) A_1^2 \bar{A}_1 \right. \\ & - (\mu_{121} + \alpha\mu_{122}) A_1 A_2 \bar{A}_2 \\ & - (\mu_{131} + \alpha\mu_{132}) A_1 A_3 \bar{A}_3 - (\tau_{11} + \alpha\tau_{12}) A_3 \bar{A}_1^2 \\ & \left. - (\varsigma_{11} + \alpha\varsigma_{12}) A_2^2 \bar{A}_3, \varphi_1 \right\rangle = 0, \end{aligned} \tag{89a}$$

$$\begin{aligned} & \left\langle -\eta_2 \dot{A}_2 - \gamma_1 \bar{\delta}_1 \bar{A}_1 e^{i\sigma T_1} - (\kappa_{21} + \alpha\kappa_{22}) A_2^2 \bar{A}_2 \right. \\ & - (\mu_{211} + \alpha\mu_{212}) A_1 A_2 \bar{A}_1 \\ & - (\mu_{231} + \alpha\mu_{232}) A_2 A_3 \bar{A}_3 \\ & \left. - (\xi_{21} + \alpha\xi_{22}) A_1 A_3 \bar{A}_2, \varphi_2 \right\rangle = 0, \end{aligned} \tag{89b}$$

$$\begin{aligned} & \left\langle -\eta_3 \dot{A}_3 - (\kappa_{31} + \alpha\kappa_{32}) A_3^2 \bar{A}_3 \right. \\ & - (\mu_{311} + \alpha\mu_{312}) A_1 A_3 \bar{A}_1 \\ & - (\mu_{321} + \alpha\mu_{322}) A_2 A_3 \bar{A}_2 - (\nu_{31} + \alpha\nu_{32}) A_1^3 \\ & \left. - (\varsigma_{31} + \alpha\varsigma_{32}) A_2^2 \bar{A}_1, \varphi_3 \right\rangle = 0. \end{aligned} \tag{89c}$$

Application of the distributive law of the inner product for complex functions to Eqs. (89a–c) leads to

$$\begin{aligned} & \dot{A}_1 + \gamma_1 \bar{\delta}_2 \bar{A}_2 e^{i\sigma T_1} + (\kappa_{11} + \alpha\kappa_{12}) A_1^2 \bar{A}_1 \\ & + (\mu_{121} + \alpha\mu_{122}) A_1 A_2 \bar{A}_2 \\ & + (\mu_{131} + \alpha\mu_{132}) A_1 A_3 \bar{A}_3 + (\tau_{11} + \alpha\tau_{12}) A_3 \bar{A}_1^2 \\ & + (\varsigma_{11} + \alpha\varsigma_{12}) A_2^2 \bar{A}_3 = 0, \end{aligned} \tag{90a}$$

$$\begin{aligned} & \dot{A}_2 + \gamma_1 \bar{\delta}_1 \bar{A}_1 e^{i\sigma T_1} + (\kappa_{21} + \alpha\kappa_{22}) A_2^2 \bar{A}_2 \\ & + (\mu_{211} + \alpha\mu_{212}) A_1 A_2 \bar{A}_1 \\ & + (\mu_{231} + \alpha\mu_{232}) A_2 A_3 \bar{A}_3 + (\xi_{21} + \alpha\xi_{22}) A_1 A_3 \bar{A}_2 \\ & = 0, \end{aligned} \tag{90b}$$

$$\begin{aligned} & \dot{A}_3 + (\kappa_{31} + \alpha\kappa_{32}) A_3^2 \bar{A}_3 + (\mu_{311} + \alpha\mu_{312}) A_1 A_3 \bar{A}_1 \\ & + (\mu_{321} + \alpha\mu_{322}) A_2 A_3 \bar{A}_2 + (\nu_{31} + \alpha\nu_{32}) A_1^3 \\ & + (\varsigma_{31} + \alpha\varsigma_{32}) A_2^2 \bar{A}_1 = 0, \end{aligned} \tag{90c}$$

where

$$\begin{aligned} \ell_h \leftrightarrow & \int_0^1 \ell_h \bar{\varphi}_h dx / \int_0^1 \eta_h \bar{\varphi}_h dx \\ & \times \begin{pmatrix} h = 1, 2, 3; \\ \ell_1 = \bar{\delta}_2, \kappa_{11}, \kappa_{12}, \mu_{121}, \mu_{122}, \mu_{131}, \mu_{132}, \tau_{11}, \tau_{12}, \varsigma_{11}, \varsigma_{12}; \\ \ell_2 = \bar{\delta}_1, \kappa_{21}, \kappa_{22}, \mu_{211}, \mu_{212}, \mu_{231}, \mu_{232}, \xi_{21}, \xi_{22}; \\ \ell_3 = \kappa_{31}, \kappa_{32}, \mu_{311}, \mu_{312}, \mu_{321}, \mu_{322}, \nu_{31}, \nu_{32}, \varsigma_{31}, \varsigma_{32} \end{pmatrix} \end{aligned} \tag{91}$$

The detailed expressions of these coefficients can be found in Appendixes 1 and 4.

6.1 The 1-term truncation

The solvability condition is simplified to

$$\dot{A}_1 + (\kappa_{11} + \alpha\kappa_{12}) A_1^2 \bar{A}_1 = 0 \tag{92}$$

It can be found that Eqs. (92) and (50) are the identical. There is only a trivial solution in the 1-term truncation. The 1-term truncation locates the parametric resonance in the case of ω near ω_3 , which is actually same as the case of ω near ω_1 . On the other hand, the parametric resonance in the case of ω near $3\omega_1 = \omega_1 + \omega_2$ can be regarded as the summation parametric resonance of the first two modes. Thus, as least the 2-term truncation should be used.

6.2 The 2-term truncation

The solvability conditions are simplified to

$$\begin{aligned} & \dot{A}_1 + \gamma_1 \bar{\delta}_2 \bar{A}_2 e^{i\sigma T_1} + (\kappa_{11} + \alpha\kappa_{12}) A_1^2 \bar{A}_1 \\ & + (\mu_{121} + \alpha\mu_{122}) A_1 A_2 \bar{A}_2 = 0, \end{aligned} \tag{93a}$$

$$\begin{aligned} & \dot{A}_2 + \gamma_1 \bar{\delta}_1 \bar{A}_1 e^{i\sigma T_1} + (\kappa_{21} + \alpha\kappa_{22}) A_2^2 \bar{A}_2 \\ & + (\mu_{211} + \alpha\mu_{212}) A_1 A_2 \bar{A}_1 = 0. \end{aligned} \tag{93b}$$

Express the solution to Eqs. (93a) and (93b) in the polar form, substitution of Eq. (55) into them and separation of real and imaginary parts in the resulting equation give

$$\begin{aligned} \dot{\alpha}_1 = & -\alpha\alpha_1 \left(\kappa_{12}\alpha_1^2 + \mu_{122}\alpha_2^2 \right) \\ & - \gamma_1\alpha_2 \left(\bar{\delta}_2^R \cos\theta_1 - \bar{\delta}_2^I \sin\theta_1 \right), \end{aligned} \tag{94a}$$

$$\begin{aligned} \alpha_1\dot{\beta}_1 = & -\alpha_1 \left(\kappa_{11}^I\alpha_1^2 + \mu_{121}^I\alpha_2^2 \right) \\ & - \gamma_1\alpha_2 \left(\bar{\delta}_2^I \cos\theta_1 + \bar{\delta}_2^R \sin\theta_1 \right), \end{aligned} \tag{94b}$$

$$(\alpha_1)_{1,2} = \sqrt{\frac{-H^2 f_1 \sigma \pm (1 + 2H^2) \sqrt{[4H^2 f_1^2 + (1 + 2H^2)^2 f_2^2] \gamma_1^2 |\bar{\delta}_2|^2 - H^2 f_2^2 \sigma^2}}{4H_2^2 f_1^2 + (1 + 2H^2)^2 f_2^2}}, \tag{99}$$

$$\dot{\alpha}_2 = -\alpha \alpha_2 (\mu_{212} \alpha_1^2 + \kappa_{22} \alpha_2^2) - \gamma_1 \alpha_1 (\bar{\delta}_1^R \cos \theta_1 - \bar{\delta}_1^I \sin \theta_1), \tag{94c}$$

$$\alpha_2 \dot{\beta}_2 = -\alpha_2 (\mu_{211}^I \alpha_1^2 + \kappa_{21}^I \alpha_2^2) - \gamma_1 \alpha_1 (\bar{\delta}_1^I \cos \theta_1 + \bar{\delta}_1^R \sin \theta_1), \tag{94d}$$

where $\theta_1 = \sigma T_1 - \beta_1 - \beta_2$. From Eqs. (94b) and (94d), one obtains

$$\begin{aligned} \dot{\theta}_1 = & \sigma + \alpha_1^2 (\kappa_{11}^I + \mu_{211}^I) + \alpha_2^2 (\kappa_{21}^I + \mu_{121}^I) \\ & + \frac{\gamma_1}{\alpha_1 \alpha_2} [(\alpha_1^2 \bar{\delta}_1^R + \alpha_2^2 \bar{\delta}_2^R) \sin \theta_1 \\ & + (\alpha_1^2 \bar{\delta}_1^I + \alpha_2^2 \bar{\delta}_2^I) \cos \theta_1] \end{aligned} \tag{95}$$

Equations (94a)–(94d) possess a zero solution. If it is assumed that there is a nonzero solution, the amplitudes α_1 and α_2 and the phase angle θ_1 should be constant for the steady-state solution. Thus, Eqs. (94a), (94c), and (95) yield

$$-\alpha \alpha_1 (\kappa_{12} \alpha_1^2 + \mu_{122} \alpha_2^2) = \gamma_1 \alpha_2 (\bar{\delta}_2^R \cos \theta_1 - \bar{\delta}_2^I \sin \theta_1), \tag{96a}$$

$$-\alpha \alpha_2 (\mu_{212} \alpha_1^2 + \kappa_{22} \alpha_2^2) = \gamma_1 \alpha_1 (\bar{\delta}_1^R \cos \theta_1 - \bar{\delta}_1^I \sin \theta_1), \tag{96b}$$

$$\begin{aligned} \frac{\alpha_2}{\alpha_1} \sigma + \frac{\alpha_2}{\alpha_1} [(\kappa_{11}^I + \mu_{211}^I) + \frac{\alpha_2^2}{\alpha_1^2} (\kappa_{21}^I + \mu_{121}^I)] \alpha_1^2 \\ + \gamma_1 [(\alpha_1^2 \bar{\delta}_1^R + \frac{\alpha_2^2}{\alpha_1^2} \bar{\delta}_2^R) \sin \theta_1 + (\alpha_1^2 \bar{\delta}_1^I + \frac{\alpha_2^2}{\alpha_1^2} \bar{\delta}_2^I) \cos \theta_1] \\ = 0. \end{aligned} \tag{96c}$$

Equations (148)–(151) lead to

$$\frac{\bar{\delta}_1^R}{\bar{\delta}_2^R} = \frac{\bar{\delta}_1^I}{\bar{\delta}_2^I} = \frac{1}{2} \tag{97}$$

Substitution of Eqs. (97) into (96a) and (96b) yields

$$\frac{\alpha_2}{\alpha_1} = \sqrt{\frac{\sqrt{128^2 a_1^8 (1 + \cos a_1)^2 + 2 (a_1^2 - 9\pi^2)^4 \cos^2 a_1 \sin^4 a_1 - 128 a_1^4 (1 + \cos a_1)}}{2 (a_1^2 - 9\pi^2)^2 \sin^2 2a_1}}, \tag{98}$$

where Eqs. (107), (109), (117), and (119) are inserted. Substitution of Eqs. (98) into (96a) and (96c) yields

where H denotes the constant in the right hand of equation (98) and

$$\begin{aligned} f_1 = & H^2 (\kappa_{21}^I + \mu_{121}^I) + \kappa_{11}^I + \mu_{211}^I, \\ f_2 = & \alpha (\kappa_{12} + H^2 \mu_{122}) / H. \end{aligned} \tag{100}$$

The existence conditions of periodic responses are derived from Eq. (99) as

$$\sigma \leq \frac{\sqrt{4H^2 f_1^2 + (1 + 2H^2)^2 f_2^2}}{2H f_2} \gamma_1 |\bar{\delta}_2|, \tag{101a}$$

$$\sigma \geq \pm \frac{1 + 2H^2}{2H} \gamma_1 |\bar{\delta}_2|. \tag{101b}$$

The Jacobian matrix of the right hand functions of Eqs. (94a) and (95) calculated at the nontrivial solutions is

$$\mathbf{J} = \begin{pmatrix} -2H f_2 (\alpha_1)_{1,2}^2 - \frac{2H^2 (\alpha_1)_{1,2} [\sigma + f_1 (\alpha_1)_{1,2}^2]}{1 + 2H^2} \\ 2f_1 (\alpha_1)_{1,2} \quad (\frac{1}{2H} + 2H) f_2 (\alpha_1)_{1,2}^2 \end{pmatrix} \tag{102}$$

The characteristic equation of the Jacobian matrix is

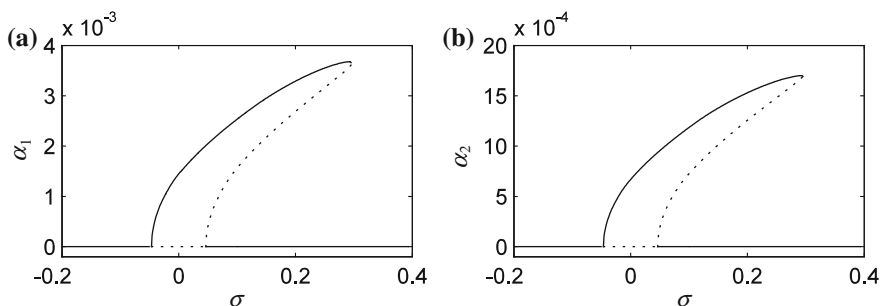
$$\begin{aligned} \lambda^2 + \frac{(2H^2 - 1) f_2 (\alpha_1)_{1,2}^2}{2H} \lambda \\ + \frac{4H^2 f_1 (\alpha_1)_{1,2}^2 [\sigma + f_1 (\alpha_1)_{1,2}^2]}{1 + 2H^2} \\ - (1 + H^2) f_2^2 (\alpha_1)_{1,2}^4 = 0 \end{aligned} \tag{103}$$

The Routh–Hurwitz criterion guarantees that the first nontrivial solution is always unstable and the second nontrivial solution is always stable.

Figure 7 depicts the relationship between the amplitude and the detuning parameter for the responses based on the 2-term truncation for the parameters used in 5.2, namely $\alpha = 0.1, \gamma_0 = 0.2, \gamma_1 = 0.01, k_1 = 20,$

and $\eta = 0.5$. In Fig. 7, the solid lines denote the stable response and the dot lines denote the unstable

Fig. 7 The steady-state responses in parametric resonance ($\omega \approx 3\omega_1$): **a** the first mode and **b** the second mode



response. The responses in the first two modes α_1 and α_2 have stable branches and unstable branches from the points at $\sigma = -0.0464$ and $\sigma = 0.0464$, respectively. The amplitudes of both the stable and unstable nonzero responses increase with the increasing detuning parameters. The amplitude of the response in the first mode is larger than that in the second mode.

The effects of the viscosity, the mean axial speed, the axial speed fluctuation amplitude, and the axial support rigidity are illustrated in Fig. 8. In the summation parametric resonance, the response amplitude and the instability range increase with the axial speed fluctuation amplitude, but they are not sensitive to the mean axial speed, the axial speed fluctuation amplitude, and the axial support rigidity.

6.3 The 3-term truncation

Express the solution to Eqs. (90a–c) in the polar form. Substitution of Eq. (62) into them and separation of real and imaginary parts in the resulting equation give

$$\begin{aligned} \dot{\alpha}_1 = & -\alpha\alpha_1 \left(\kappa_{12}\alpha_1^2 + \mu_{122}\alpha_2^2 + \mu_{132}\alpha_3^2 \right) \\ & -\gamma_1\alpha_2 \left(\bar{\delta}_2^R \cos \theta_1 - \bar{\delta}_2^I \sin \theta_1 \right) \\ & -\alpha_2^2\alpha_3 \left(\varsigma_{11}^I \sin \theta_2 + \alpha\varsigma_{12} \cos \theta_2 \right) \\ & -\alpha_1^2\alpha_3 \left(\tau_{11}^I \sin \theta_3 + \alpha\tau_{12} \cos \theta_3 \right), \end{aligned} \tag{104a}$$

$$\begin{aligned} \alpha_1\dot{\beta}_1 = & -\alpha_1 \left(\kappa_{11}^I\alpha_1^2 + \mu_{121}^I\alpha_2^2 + \mu_{131}^I\alpha_3^2 \right) \\ & -\gamma_1\alpha_2 \left(\bar{\delta}_2^I \cos \theta_1 + \bar{\delta}_2^R \sin \theta_1 \right) \\ & -\alpha_2^2\alpha_3 \left(\varsigma_{11}^I \cos \theta_2 - \alpha\varsigma_{12} \sin \theta_2 \right) \\ & -\alpha_1^2\alpha_3 \left(\tau_{11}^I \cos \theta_3 - \alpha\tau_{12} \sin \theta_3 \right), \end{aligned} \tag{104b}$$

$$\begin{aligned} \dot{\alpha}_2 = & -\alpha\alpha_2 \left(\mu_{212}\alpha_1^2 + \kappa_{22}\alpha_2^2 + \mu_{232}\alpha_3^2 \right) \\ & -\gamma_1\alpha_1 \left(\bar{\delta}_1^R \cos \theta_1 - \bar{\delta}_1^I \sin \theta_1 \right) \\ & +\alpha_1\alpha_2\alpha_3 \left(\xi_{21}^I \sin \theta_2 - \alpha\xi_{22} \cos \theta_2 \right), \end{aligned} \tag{104c}$$

$$\begin{aligned} \alpha_2\dot{\beta}_2 = & -\alpha_2 \left(\mu_{211}^I\alpha_1^2 + \kappa_{21}^I\alpha_2^2 + \mu_{231}^I\alpha_3^2 \right) \\ & -\gamma_1\alpha_1 \left(\bar{\delta}_1^I \cos \theta_1 + \bar{\delta}_1^R \sin \theta_1 \right) \\ & -\alpha_1\alpha_2\alpha_3 \left(\xi_{21}^I \cos \theta_2 + \alpha\xi_{22} \sin \theta_2 \right), \end{aligned} \tag{104d}$$

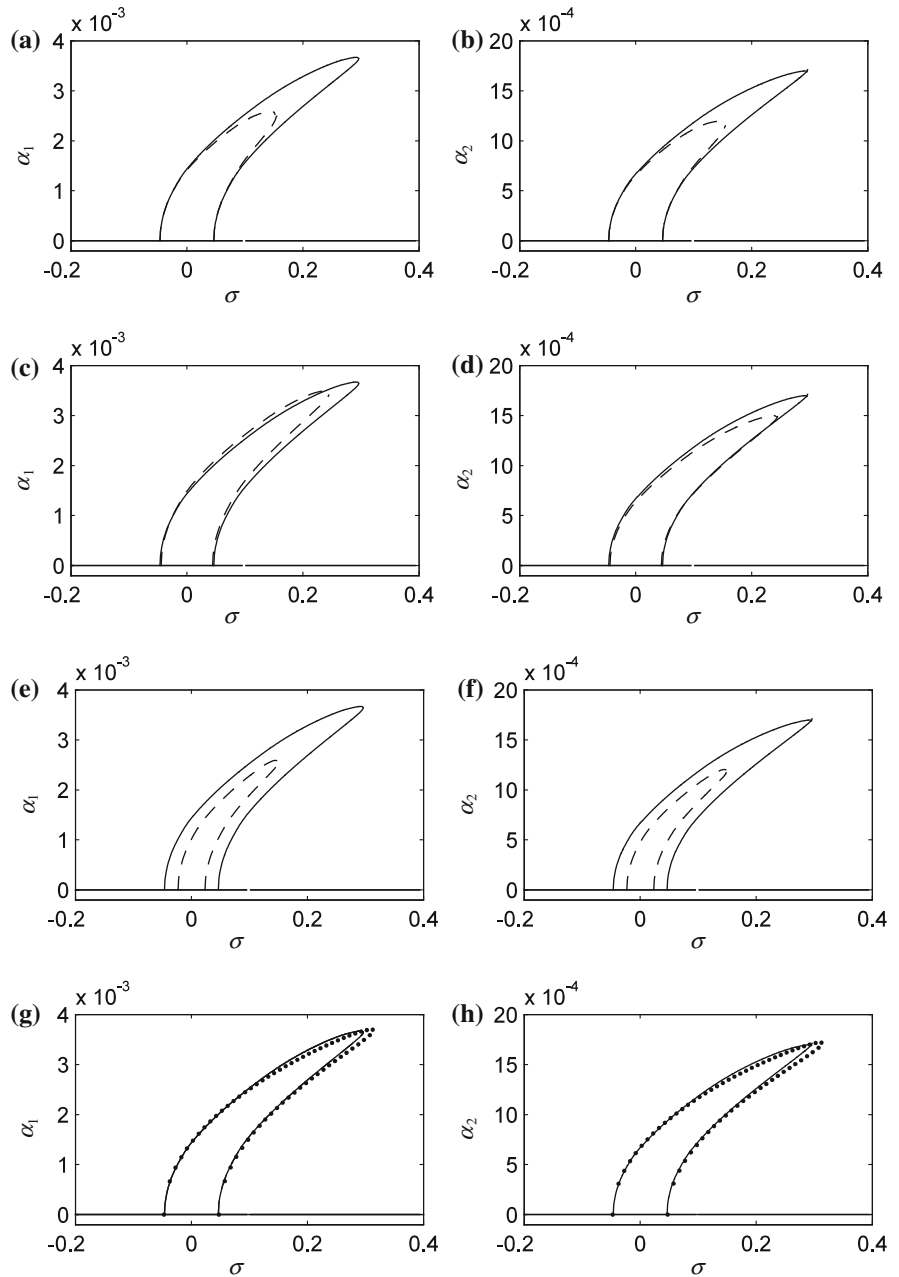
$$\begin{aligned} \dot{\alpha}_3 = & -\alpha\alpha_3 \left(\mu_{312}\alpha_1^2 + \mu_{322}\alpha_2^2 + \kappa_{32}\alpha_3^2 \right) \\ & -\alpha_1\alpha_2^2 \left(\varsigma_{31}^R \sin \theta_2 + \alpha\varsigma_{32}^I \cos \theta_2 \right) \\ & +\alpha_1^3 \left(\nu_{31}^I \sin \theta_3 - \alpha\nu_{32} \cos \theta_3 \right), \end{aligned} \tag{104e}$$

$$\begin{aligned} \alpha_3\dot{\beta}_3 = & -\alpha_3 \left(\mu_{311}^I\alpha_1^2 + \mu_{321}^I\alpha_2^2 + \kappa_{31}^I\alpha_3^2 \right) \\ & -\alpha_1\alpha_2^2 \left(\varsigma_{31}^R \cos \theta_2 - \alpha\varsigma_{32}^I \sin \theta_2 \right) \\ & -\alpha_1^3 \left(\nu_{31}^I \cos \theta_3 + \alpha\nu_{32} \sin \theta_3 \right), \end{aligned} \tag{104f}$$

where $\theta_1 = \sigma T_1 - \beta_1 - \beta_2$, $\theta_2 = \beta_1 - 2\beta_2 + \beta_3$, and $\theta_3 = 3\beta_1 - \beta_3$. Equations (104a–f) possess a zero solution. It can be derived from Eqs. (104b), (104d), and (104f)

$$\begin{aligned} \alpha_1\alpha_2\dot{\theta}_1 = & \alpha_1\alpha_2 \left[\sigma + \alpha_1^2 \left(\kappa_{11}^I + \mu_{211}^I \right) \right. \\ & +\alpha_2^2 \left(\kappa_{21}^I + \mu_{121}^I \right) + \alpha_3^2 \left(\mu_{131}^I + \mu_{231}^I \right) \\ & +\gamma_1 \left[\left(\alpha_1^2\bar{\delta}_1^I + \alpha_2^2\bar{\delta}_2^I \right) \cos \theta_1 \right. \\ & + \left. \left(\alpha_1^2\bar{\delta}_1^R + \alpha_2^2\bar{\delta}_2^R \right) \sin \theta_1 \right] \\ & +\alpha_2\alpha_3 \left[\left(\alpha_1^2\xi_{21}^I + \alpha_2^2\varsigma_{11}^I \right) \cos \theta_2 \right. \end{aligned}$$

Fig. 8 The effects of parameters on the steady-state responses in parametric resonance ($\omega \approx 3\omega_1$): **a** the viscosity coefficients (solid $\alpha = 0.1$, dashed $\alpha = 0.2$); the first mode, **b** the viscosity coefficients (solid $\alpha = 0.1$, dashed $\alpha = 0.2$); the second mode, **c** the mean axial speeds (solid $\gamma_0 = 0.2$; dashed $\gamma_0 = 0.1$); the first mode, **d** the mean axial speeds (solid $\gamma_0 = 0.2$; dashed $\gamma_0 = 0.1$); the second mode, **e** the axial speed fluctuation amplitudes (solid $\gamma_1 = 0.01$; dashed $\gamma_1 = 0.005$); the first mode, **f** the axial speed fluctuation amplitudes (solid $\gamma_1 = 0.01$; dashed $\gamma_1 = 0.005$); the second mode, **g** the axial support rigidity parameter (dot $\eta = 0.0$; solid $\eta = 0.5$); the first mode, **h** the axial support rigidity parameter (dot $\eta = 0.0$; solid $\eta = 0.5$); the second mode



$$\begin{aligned}
 & + \alpha \left(\alpha_1^2 \xi_{22} - \alpha_2^2 \zeta_{12} \right) \sin \theta_2 \Big] \\
 & + \alpha_1^2 \alpha_2 \alpha_3 \left(\tau_{11}^I \cos \theta_3 - \alpha \tau_{12} \sin \theta_3 \right), \tag{105a}
 \end{aligned}$$

$$\begin{aligned}
 & \alpha_1 \alpha_2 \alpha_3 \theta_2 \\
 & = -\alpha_1 \alpha_2 \alpha_3 \left[\alpha_1^2 \left(\kappa_{11}^I - 2\mu_{211}^I + \mu_{311}^I \right) \right. \\
 & \quad \left. - \alpha_2^2 \left(\mu_{121}^I - 2\kappa_{21}^I + \mu_{321}^I \right) \right.
 \end{aligned}$$

$$\begin{aligned}
 & \left. - \alpha_3^2 \left(\kappa_{31}^I - 2\mu_{231}^I + \mu_{131}^I \right) \right] \\
 & - \gamma_1 \alpha_3 \left[\left(2\alpha_1^2 \delta_1^I - \alpha_2^2 \delta_2^I \right) \cos \theta_1 \right. \\
 & \quad \left. + \left(2\alpha_1^2 \bar{\delta}_1^R - \alpha_2^2 \bar{\delta}_2^R \right) \sin \theta_1 \right] \\
 & - \alpha_2 \cdot \left\{ \left[\alpha_2^2 \left(\alpha_1^2 \zeta_{31}^I + \alpha_3^2 \zeta_{11}^I \right) - 2\alpha_1^2 \alpha_3^2 \xi_{21}^I \right] \cos \theta_2 \right. \\
 & \quad \left. - \alpha \left[\alpha_2^2 \left(\alpha_1^2 \zeta_{32}^I + \alpha_3^2 \zeta_{12}^I \right) + 2\alpha_1^2 \alpha_3^2 \xi_{22}^I \right] \sin \theta_2 \right\}
 \end{aligned}$$

$$\begin{aligned}
 & -\alpha_1^2 \alpha_2 \left[(\alpha_1^2 \nu_{31}^I + \alpha_3^2 \tau_{11}^I) \cos \theta_3 \right. \\
 & \left. + \alpha \left(\alpha_1^2 \nu_{32} - \alpha_3^2 \tau_{12} \right) \sin \theta_3 \right], \tag{105b}
 \end{aligned}$$

$$\begin{aligned}
 \alpha_1 \alpha_3 \dot{\theta}_3 = & \alpha_1 \alpha_3 \left[\alpha_1^2 \left(\mu_{311}^I - 3\kappa_{11}^I \right) \right. \\
 & \left. + \alpha_2^2 \left(\mu_{321}^I - 3\mu_{121}^I \right) + \alpha_3^2 \left(\kappa_{31}^I - 3\mu_{131}^I \right) \right] \\
 & - 3\gamma_1 \alpha_2 \alpha_3 \left(\delta_2^I \cos \theta_1 + \delta_2^R \sin \theta_1 \right) \\
 & + \alpha_2^2 \left[\left(\alpha_1^2 \varsigma_{31}^I - 3\alpha_3^2 \varsigma_{11}^I \right) \cos \theta_2 \right. \\
 & \left. - \alpha \left(\alpha_1^2 \varsigma_{32} - 3\alpha_3^2 \varsigma_{12} \right) \sin \theta_3 \right] \\
 & + \alpha_1^2 \left[\left(\alpha_1^2 \nu_{31}^I - 3\alpha_3^2 \tau_{11}^I \right) \cos \theta_3 \right. \\
 & \left. + \alpha \left(\alpha_1^2 \nu_{32} + 3\alpha_3^2 \tau_{12} \right) \sin \theta_3 \right]. \tag{105c}
 \end{aligned}$$

If we assume that there is a nonzero steady-state response, the amplitudes α_1, α_2 , and α_3 and the new phase angles θ_1, θ_2 , and θ_3 should be constant for the response. Numerical tests strongly indicate that α_3 is equal to zero. Therefore, the third mode has actually no effect on the first two modes in the 3-term truncation. The 3-term truncation yields the same results as the 2-term truncation.

7 Conclusions

The present work treats nonlinear parametric vibration of axially accelerating viscoelastic strings. The longitudinally varying tensions are recognized in the modeling, and the exact internal resonances are accounted in the analysis. An integro-partial-differential governing equation with time/space-dependent coefficients is derived to governing the transverse vibration of the string. The method of multiple scales is applied to the integro-partial-differential governing to seek the approximate analytical solution. Based on 1–3 term truncation, the parametric resonances in the cases of ω

there is only zero response in the $\omega \approx \omega_1$ parametric resonance. The amplitude, the existence conditions, and the stability are determined, and the effects of the viscosity, the mean axial speed, the axial speed fluctuation amplitude, and the axial support rigidity on the amplitude and the existence are examined. It is found that the 2-term and the 3-term truncation yield the qualitatively same and the quantitatively close results in the $\omega \approx 2\omega_1$ parametric resonance and the exact same results in the $\omega \approx 3\omega_1$ parametric resonance. It is suggested that the effects of infinite exact internal resonances may be neglectable.

Acknowledgments This work was supported by the National Outstanding Young Scientists Foundation of China (No. 10725209), the State Key Program of National Natural Science of China (Nos. 10932006 and 11232009), the National Natural Science Foundation of China (No. 11202135) and Shanghai Leading Academic Discipline Project (No. S30106).

Appendix A

$$\kappa_{11} = -\frac{2a_1^2 (a_1^2 + \pi^2)^2 + \pi^4 \sin^2 a_1}{8a_1^2 (\omega_1 + a_1\gamma_0)} k_1^2 i \tag{106}$$

$$\kappa_{12} = \frac{\omega_1 \pi^4 \sin^2 a_1}{4a_1^2 (\omega_1 + a_1\gamma_0)} k_1^2 \tag{107}$$

$$\begin{aligned}
 \mu_{121} = & -\left\{ 9(9a_1^6 - 73a_1^4\pi^2 - 73a_1^2\pi^4 + 9\pi^6)^2 \right. \\
 & + 1280a_1^2\pi^4(73a_1^4 - 18a_1^2\pi^2 + 9\pi^4) \\
 & + 128a_1^2\pi^4 \left[9(a_1^2 - \pi^2)^2 \cos a_1 \right. \\
 & \left. \left. + (a_1^2 - 9\pi^2)^2 \cos 3a_1 \right] \right\} \\
 & / \left[9(9a_1^4 - 82a_1^2\pi^2 + 9\pi^4)^2 (\omega_1 + a_1\gamma_0) \right] k_1^2 i \tag{108}
 \end{aligned}$$

$$\mu_{122} = \frac{512a_1^2\pi^4 \omega_1 \cos^2 0.5a_1 \left[120(\pi^4 - a_1^4) + (a_1^2 - 9\pi^2)^2 (\cos 2a_1 - 2\cos a_1) \right]}{3(9a_1^4 - 82a_1^2\pi^2 + 9\pi^4)^2 (\omega_1 + a_1\gamma_0)} k_1^2 \tag{109}$$

near $\omega_1, 2\omega_1$, and $3\omega_1$ are investigated with the presence of the exact internal resonances. In all truncations, there are steady-state nonzero periodic responses in the $\omega \approx 2\omega_1$ and $\omega \approx 3\omega_1$ parametric resonances, while

$$\begin{aligned}
 \mu_{131} = & -9 \left\{ 8(4a_1^6 - 13a_1^4\pi^2 - 13a_1^2\pi^4 + 4\pi^6)^2 \right. \\
 & + 45a_1^2\pi^4(13a_1^4 - 8a_1^2\pi^2 + 4\pi^4) \\
 & \left. - 9a_1^2\pi^4 \left[4(4a_1^2 - \pi^2)^2 \cos 2a_1 \right. \right.
 \end{aligned}$$

$$\begin{aligned} & + (a_1^2 - 4\pi^2)^2 \cos 4a_1 \Big] \\ & \Big/ \left[32 (4a_1^4 - 17a_1^2\pi^2 + 4\pi^4)^2 (\omega_1 + a_1\gamma_0) \right] k_1^2 i \end{aligned} \quad \begin{aligned} & \Big/ \left[(-4a_1^6 + 137a_1^4\pi^2 - 934a_1^2\pi^4 + 225\pi^6) \right. \\ & \left. \times (\omega_1 + a_1\gamma_0) \right] k_1^2 \end{aligned} \quad (115)$$

$$\begin{aligned} \mu_{132} &= \frac{81a_1^2\pi^4\omega_1 \sin^2 a_1 \left[15(\pi^4 - a_1^4) + (a_1^2 - 4\pi^2)^2 \cos 2a_1 \right]}{2(4a_1^4 - 17a_1^2\pi^2 + 4\pi^4)^2 (\omega_1 + a_1\gamma_0)} k_1^2 \quad (111) \\ \kappa_{21} &= -\frac{8a_1^2(a_1^2 + \pi^2)^2 + \pi^4 \sin^2 2a_1}{4a_1^2(\omega_1 + a_1\gamma_0)} k_1^2 i \quad (116) \\ \kappa_{22} &= \frac{\omega_1\pi^4 \sin^2 2a_1}{a_1^2(\omega_1 + a_1\gamma_0)} k_1^2 \quad (117) \end{aligned}$$

$$\tau_{11} = -\frac{27\pi^4 \sin^2 a_1}{8(4\pi^2 - a_1^2)(\omega_1 + a_1\gamma_0)} k_1^2 i \quad (112)$$

$$\tau_{12} = \frac{9\pi^4\omega_1 \sin^2 a_1}{4(4\pi^2 - a_1^2)(\omega_1 + a_1\gamma_0)} k_1^2 \quad (113)$$

$$\begin{aligned} \zeta_{11} &= 9\pi^4 \left[3(2731a_1^4 - 694a_1^2\pi^2 + 75\pi^4) \right. \\ & \left. - (a_1^4 - 34a_1^2\pi^2 + 225\pi^4) \cos 4a_1 \right. \\ & \left. \Big/ \left[18(9a_1^4 - 82a_1^2\pi^2 + 9\pi^4)^2 (\omega_1 + a_1\gamma_0) \right] \right] k_1^2 i \end{aligned} \quad (118)$$

$$\mu_{212} = \frac{256a_1^2\pi^4\omega_1 \cos^2 0.5a_1 \left[(123a_1^4 - 54a_1^2\pi^2 + 123\pi^4) + (a_1^2 - 9\pi^2)^2 (\cos 2a_1 - 2 \cos a_1) \right]}{3(9a_1^4 - 82a_1^2\pi^2 + 9\pi^4)^2 (\omega_1 + a_1\gamma_0)} k_1^2 \quad (119)$$

$$\begin{aligned} & + 2048a_1^2(4a_1^2 - \pi^2) \cos a_1 \Big] \\ & \Big/ \left[16(-4a_1^6 + 137a_1^4\pi^2 - 934a_1^2\pi^4 + 225\pi^6) \right. \\ & \left. \times (\omega_1 + a_1\gamma_0) \right] k_1^2 i \end{aligned} \quad (114)$$

$$\begin{aligned} \zeta_{12} &= -9\pi^4\omega_1 \cos^2 0.5a_1 \left[(a_1^4 - 34a_1^2\pi^2 + 225\pi^4) \right. \\ & \left. \times (2 \cos 2a_1 - 3 \cos a_1 - \cos 3a_1) \right. \\ & \left. + 2(225\pi^4 + 94a_1^2\pi^2 - 511a_1^4) \right] \end{aligned}$$

$$\begin{aligned} \mu_{231} &= -9i \left\{ 25(25a_1^6 - 601a_1^4\pi^2 - 601a_1^2\pi^4 + 25\pi^6)^2 \right. \\ & \left. + 29952a_1^2\pi^4(601a_1^4 - 50a_1^2\pi^2 + 25\pi^4) \right. \\ & \left. + 1152a_1^2\pi^4 \left[25(25a_1^2 - \pi^2)^2 \cos a_1 \right. \right. \\ & \left. \left. + (a_1^2 - 25\pi^2)^2 \cos 5a_1 \right] \right\} \\ & \Big/ \left[50(25a_1^4 - 626a_1^2\pi^2 + 25\pi^4)^2 \right. \\ & \left. \times (\omega_1 + a_1\gamma_0) \right] k_1^2 \end{aligned} \quad (120)$$

$$\mu_{232} = \frac{5184a_1^2\pi^4\omega_1 \left[4(155\pi^4 + 50a_1^2\pi^2 - 781a_1^4) - 5(25a_1^2 - \pi^2)^2 \cos a_1 + (a_1^2 - 25\pi^2)^2 \cos 5a_1 \right]}{5(25a_1^4 - 626a_1^2\pi^2 + 25\pi^4)^2 (\omega_1 + a_1\gamma_0)} k_1^2 \quad (121)$$

$$\xi_{21} = -\frac{18\pi^4 \left[(a_1^4 - 34a_1^2\pi^2 + 225\pi^4) \cos 4a_1 - 2048a_1^2(4a_1^2 - \pi^2) \cos a_1 - 3(2731a_1^4 - 694a_1^2\pi^2 + 75\pi^4) \right]}{32(-4a_1^6 + 137a_1^4\pi^2 - 934a_1^2\pi^4 + 225\pi^6) (\omega_1 + a_1\gamma_0)} k_1^2 i \quad (122)$$

$$\xi_{22} = \frac{9\pi^4 \omega_1 \sin^2 2a_1}{2(4a_1^2 - \pi^2)(\omega_1 + a_1\gamma_0)} k_1^2 \tag{123}$$

$$\begin{aligned} & \left/ \left[25(25a_1^4 - 626a_1^2\pi^2 + 25\pi^4) \right. \right. \\ & \left. \left. \times (\omega_1 + a_1\gamma_0) \right] k_1^2 i \right. \end{aligned} \tag{128}$$

$$\mu_{322} = \frac{3456a_1^2\pi^4 \omega_1 \left[6(521a_1^4 - 50a_1^2\pi^2 + 105\pi^4) + 5(25a_1^2 - \pi^2)^2 \cos a_1 + (a_1^2 - 25\pi^2)^2 \cos 5a_1 \right]}{5(25a_1^4 - 626a_1^2\pi^2 + 25\pi^4)^2 (\omega_1 + a_1\gamma_0)} k_1^2 \tag{129}$$

$$\kappa_{31} = -\frac{3 \left[18a_1^2(a_1^2 + \pi^2)^2 + \pi^4 \sin^2 3a_1 \right]}{8a_1^2(\omega_1 + a_1\gamma_0)} k_1^2 i \tag{124}$$

$$\nu_{31} = -\frac{3\pi^4 \sin^2 a_1}{8(4\pi^2 - a_1^2)(\omega_1 + a_1\gamma_0)} k_1^2 i \tag{130}$$

$$\nu_{32} = \frac{3\pi^4 \omega_1 \sin^2 a_1}{4(4\pi^2 - a_1^2)(\omega_1 + a_1\gamma_0)} k_1^2 \tag{131}$$

$$\kappa_{32} = \frac{9\omega_1\pi^4 \sin^2 3a_1}{4a_1^2(\omega_1 + a_1\gamma_0)} k_1^2 \tag{125}$$

$$\zeta_{31} = \frac{6\pi^4 \left[3(2731a_1^4 - 694a_1^2\pi^2 + 75\pi^4) + 2048a_1^2(4a_1^2 - \pi^2) \cos a_1 - (a_1^4 - 34a_1^2\pi^2 + 225\pi^4) \cos 4a_1 \right]}{32(-4a_1^6 + 137a_1^4\pi^2 - 934a_1^2\pi^4 + 225\pi^6)(\omega_1 + a_1\gamma_0)} k_1^2 i \tag{132}$$

$$\begin{aligned} \mu_{311} = & -3 \left\{ 8(4a_1^6 - 13a_1^4\pi^2 - 13a_1^2\pi^4 + 4\pi^6)^2 \right. \\ & + 45a_1^2\pi^4(13a_1^4 - 8a_1^2\pi^2 + 4\pi^4) \\ & - 9a_1^2\pi^4 \left[4(4a_1^2 - \pi^2)^2 \cos 2a_1 \right. \\ & \left. \left. + (a_1^2 - 4\pi^2)^2 \cos 4a_1 \right] \right\} \\ & \left/ \left[32(4a_1^4 - 17a_1^2\pi^2 + 4\pi^4)^2 \right. \right. \\ & \left. \left. \times (\omega_1 + a_1\gamma_0) \right] k_1^2 i \right. \end{aligned} \tag{126}$$

$$\begin{aligned} \zeta_{32} = & -3\pi^4 \omega_1 \cos^2 0.5a_1 \left[- (a_1^4 - 34a_1^2\pi^2 + 225\pi^4) \right. \\ & \left. \times (3 \cos a_1 - 2 \cos 2a_1 + \cos 3a_1) \right. \\ & \left. + 18(57a_1^4 - 18a_1^2\pi^2 + 25\pi^4) \right] \\ & \left/ \left[(-4a_1^6 + 137a_1^4\pi^2 - 934a_1^2\pi^4 + 225\pi^6) \right. \right. \\ & \left. \left. \times (\omega_1 + a_1\gamma_0) \right] k_1^2 \right. \end{aligned} \tag{133}$$

$$\mu_{312} = \frac{27a_1^2\pi^4 \omega_1 \sin^2 a_1 \left[17a_1^4 - 16a_1^2\pi^2 + 17\pi^4 + (a_1^2 - 4\pi^2)^2 \cos 2a_1 \right]}{2(4a_1^4 - 17a_1^2\pi^2 + 4\pi^4)^2 (\omega_1 + a_1\gamma_0)} k_1^2 \tag{127}$$

$$\begin{aligned} \mu_{321} = & 2\mu_{231}/3 \\ = & -3 \left\{ 25(25a_1^6 - 601a_1^4\pi^2 - 601a_1^2\pi^4 + 25\pi^6)^2 \right. \\ & + 29952a_1^2\pi^4(601a_1^4 - 50a_1^2\pi^2 + 25\pi^4) \\ & + 1152a_1^2\pi^4 \left[25(25a_1^2 - \pi^2)^2 \cos a_1 \right. \\ & \left. \left. + (a_1^2 - 25\pi^2)^2 \cos 5a_1 \right] \right\} \end{aligned}$$

Appendix B

$$\begin{aligned} \delta_2^R = & 4\pi^2 \\ & \times \frac{a_1\omega_1(a_1^2 - 9\pi^2) + \sin a_1[\omega_1(a_1^2 + 9\pi^2) - 2a_1\kappa\gamma_0(a_1^2 - 9\pi^2)]}{(a_1^2 - 9\pi^2)^2(\omega_1 + a_1\gamma_0)} \end{aligned} \tag{134}$$

$$\delta_2^I = -\frac{4\pi^2(1 + \cos a_1)[\omega_1(a_1^2 + 9\pi^2) - 2a_1\kappa\gamma_0(a_1^2 - 9\pi^2)]}{(a_1^2 - 9\pi^2)^2(\omega_1 + a_1\gamma_0)} \tag{135}$$

$$\chi_1^R = -2\pi^2 \times \frac{a_1\omega_1(a_1^2 - 9\pi^2) + \sin a_1[\omega_1(a_1^2 + 9\pi^2) - 2a_1\kappa\gamma_0(a_1^2 - 9\pi^2)]}{(a_1^2 - 9\pi^2)^2(\omega_1 + a_1\gamma_0)} \tag{136}$$

$$\chi_1^I = -\frac{2\pi^2(1 + \cos a_1)[\omega_1(a_1^2 + 9\pi^2) - 2a_1\kappa\gamma_0(a_1^2 - 9\pi^2)]}{(a_1^2 - 9\pi^2)^2(\omega_1 + a_1\gamma_0)} \tag{137}$$

$$\delta_3^R = 18\pi^2 \times \frac{a_1\omega_1(a_1^2 - 25\pi^2) + \sin a_1[\omega_1(a_1^2 + 25\pi^2) - 2a_1\kappa\gamma_0(a_1^2 - 25\pi^2)]}{(a_1^2 - 25\pi^2)^2(\omega_1 + a_1\gamma_0)} \tag{138}$$

$$\delta_3^I = -\frac{18\pi^2(1 + \cos a_1)[\omega_1(a_1^2 + 25\pi^2) - 2a_1\kappa\gamma_0(a_1^2 - 25\pi^2)]}{(a_1^2 - 25\pi^2)^2(\omega_1 + a_1\gamma_0)} \tag{139}$$

$$\chi_2^R = -12\pi^2 \times \frac{a_1\omega_1(a_1^2 - 25\pi^2) + \sin a_1[\omega_1(a_1^2 + 25\pi^2) - 2a_1\kappa\gamma_0(a_1^2 - 25\pi^2)]}{(a_1^2 - 25\pi^2)^2(\omega_1 + a_1\gamma_0)} \tag{140}$$

$$\chi_2^I = -\frac{24\pi^2 \cos^2 0.5a_1[\omega_1(a_1^2 + 25\pi^2) - 2a_1\kappa\gamma_0(a_1^2 - 25\pi^2)]}{(a_1^2 - 25\pi^2)^2(\omega_1 + a_1\gamma_0)} \tag{141}$$

Appendix C

$$\delta_3^R = 9\pi^2 \times \frac{2a_1\omega_1(a_1^2 - 4\pi^2) - \sin 2a_1[\omega_1(a_1^2 + 4\pi^2) - 2a_1\kappa\gamma_0(a_1^2 - 4\pi^2)]}{8(a_1^2 - 4\pi^2)^2(\omega_1 + a_1\gamma_0)} \tag{142}$$

$$\delta_3^I = -\frac{9\pi^2 \sin^2 a_1[\omega_1(a_1^2 + 4\pi^2) - 2a_1\kappa\gamma_0(a_1^2 - 4\pi^2)]}{4(a_1^2 - 4\pi^2)^2(\omega_1 + a_1\gamma_0)} \tag{143}$$

$$\delta_1^R = \pi^2 \frac{2a_1\omega_1 - \sin 2a_1(\omega_1 - 2a_1\kappa\gamma_0)}{8a_1^2(\omega_1 + a_1\gamma_0)} \tag{144}$$

$$\delta_1^I = \frac{\pi^2 \sin^2 a_1(\omega_1 - 2a_1\kappa\gamma_0)}{4a_1^2(\omega_1 + a_1\gamma_0)} \tag{145}$$

$$\chi_1^R = -3\pi^2 \times \frac{2a_1\omega_1(a_1^2 - 4\pi^2) - \sin 2a_1[\omega_1(a_1^2 + 4\pi^2) - 2a_1\kappa\gamma_0(a_1^2 - 4\pi^2)]}{8(a_1^2 - 4\pi^2)^2(\omega_1 + a_1\gamma_0)} \tag{146}$$

$$\chi_1^I = -\frac{3\pi^2 \sin^2 a_1[\omega_1(a_1^2 + 4\pi^2) - 2a_1\kappa\gamma_0(a_1^2 - 4\pi^2)]}{4(a_1^2 - 4\pi^2)^2(\omega_1 + a_1\gamma_0)} \tag{147}$$

Appendix D

$$\delta_2^R = 4\pi^2 \times \frac{3a_1\omega_1(9a_1^2 - \pi^2) + \sin 3a_1[\omega_1(9a_1^2 + \pi^2) - 2a_1\kappa\gamma_0(9a_1^2 - \pi^2)]}{3(9a_1^2 - \pi^2)^2(\omega_1 + a_1\gamma_0)}$$

$$\delta_2^I = \frac{8\pi^2 \cos^2(3a_1/2)[\omega_1(9a_1^2 + \pi^2) - 2a_1\kappa\gamma_0(9a_1^2 - \pi^2)]}{3(9a_1^2 - \pi^2)^2(\omega_1 + a_1\gamma_0)} \tag{148}$$

$$\delta_1^R = 2\pi^2 \times \frac{3a_1\omega_1(9a_1^2 - \pi^2) + \sin 3a_1[\omega_1(9a_1^2 + \pi^2) - 2a_1\kappa\gamma_0(9a_1^2 - \pi^2)]}{3(9a_1^2 - \pi^2)^2(\omega_1 + a_1\gamma_0)} \tag{150}$$

$$\delta_1^I = \frac{4\pi^2 \cos^2(3a_1/2)[\omega_1(9a_1^2 + \pi^2) - 2a_1\kappa\gamma_0(9a_1^2 - \pi^2)]}{3(9a_1^2 - \pi^2)^2(\omega_1 + a_1\gamma_0)} \tag{151}$$

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