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# Recursive least squares parameter estimation algorithm for dual-rate sampled-data nonlinear systems

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**Abstract** This paper focuses on the identification problem of Hammerstein systems with dual-rate sampling. Using the key-term separation principle, we derive a regression identification model with different input updating and output sampling rates. To solve the identification problem of the dual-rate Hammerstein systems with the unmeasurable variables in the information vector, an auxiliary model-based recursive least squares algorithm is presented by replacing the unmeasurable variables with their corresponding recursive estimates. Convergence properties of the algorithm are analyzed. Simulation results show that the proposed algorithm can estimate the parameters of a class of nonlinear systems.

**Keywords** Hammerstein system · Least squares algorithm · Dual-rate sampling · Key-term separation principle · Auxiliary model identification idea

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## 1 Introduction

Many identification methods focus on discrete-time systems with same sampling rates [1–4]. However, in many industry applications, the inputs and outputs have different updating rates due to the sensor limits. The systems operating at different input and output sampling rates are called multirate systems [5–7]. Dual-rate systems are a class of multirate systems, where the output sampling period is an integer multiple of the input updating period [8,9]. Dual-rate systems are often encountered in control, communication, signal processing. There exist several methods to deal with dual-rate system identification, i.e., the lifting technique [10], the polynomial transformation technique [9]. Ding et al. [11] studied the hierarchical least squares identification for linear SISO systems with dual-rate sampled-data and presented a stochastic gradient identification algorithm for estimate the parameters of the dual-rate systems. Liu et al. presented a least squares estimation algorithm for a class of nonuniformly sampled systems based on the hierarchical identification principle [12–14].

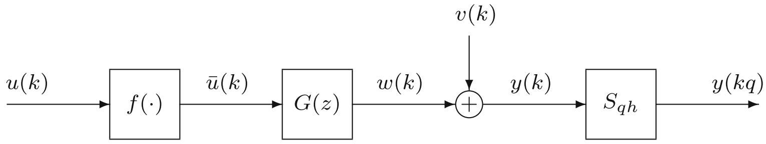


**Fig. 1** The Hammerstein system with dual-rate sampling

Hammerstein systems, which consist of a static nonlinear subsystem followed by a linear dynamic subsystem, can represent many nonlinear dynamic systems [15–22]. There exists a large amount of work on the parametric model identification of Hammerstein systems [23]. Some existing contributions assumed that the nonlinearity is a polynomial combination of a known order in the input [24]. Recently, Li et al. [25] presented a maximum likelihood least squares identification algorithm for input nonlinear finite impulse response moving average systems.

Recursive algorithms are a class of basic parameter estimation approaches which are suitable for online applications [26–29]. For decades, recursive algorithms have been used to estimate the parameters of the linear and nonlinear systems [30–32]. Xiao et al. [33] analyzed the convergence of the recursive least squares (RLS) algorithms for controlled auto-regression models. Wang [34] presented a filtering and auxiliary model-based RLS (AM-RLS) identification algorithm to estimate the parameter of output error moving average systems. This paper extends the identification algorithms from the original single-rate Hammerstein model to the dual-rate sampled-data one. The basic idea is, by means of the key-term separation principle, to present a dual-rate Hammerstein model and then derive the AM-RLS algorithm for the proposed model.

The rest of this paper is organized as follows: Sect. 2 presents the identification model of Hammerstein nonlinear systems with dual-rate sampling. Section 3 derives a AM-RLS algorithm for Hammerstein nonlinear systems with dual-rate sampling. Section 4 proves the convergence of the proposed algorithm. Section 5 provides examples to illustrate the effectiveness of the proposed algorithm. Some conclusions are summarized in Sect. 6.

## 2 Problem formulation

Let us introduce some notation first. The superscript T denotes the matrix transpose; the norm of a matrix $\boldsymbol{X}$ is defined by $\|\boldsymbol{X}\|^2 := \mathrm{tr}[\boldsymbol{X}\boldsymbol{X}^{\mathrm{T}}]$; $\mathbf{1}_n$ being an $n$-dimensional column vector whose elements are all 1; $\lambda_{\min}[\boldsymbol{X}]$ represents the minimum eigenvalue of the symmetric matrix $\boldsymbol{X}$; $f(t) = O(g(t))$ represents that for $g(t) \geqslant 0$, if there exists a positive constant $\delta_1$ such that $\|f(t)\| \geqslant \delta_1 g(t)$.

Consider a dual-rate Hammerstein output error system shown in Fig. 1, where $y(k)$ is the measured output, $u(k)$ and $\bar{u}(k)$ are the input and output of the nonlinear subsystem, respectively, $S_{qh}$ is a sampler with period $qh$ ($q$ being a positive integer), which yields a discrete-time signal $y(kq)$. The input–output data available are $\{u(k) : k = 0, 1, 2, \ldots\}$ at the fast rate, and $\{y(kq) : k = 0, 1, 2, \ldots\}$ at the slow rate. Thus, the intersample outputs $\{y(kq + i), i = 1, 2, \ldots, q - 1\}$, are not available. Here, we refer to $\{u(k), y(kq)\}$ as the dual-rate measurement data. $\bar{u}(k)$ is assumed to be a linear combination of known nonlinear basis functions $\boldsymbol{f} := (f_1, f_2, \ldots, f_n)$ [35,36]:

$$\begin{aligned}\bar{u}(k) &= f(u(k)) = \gamma_1 f_1(u(k)) + \gamma_2 f_2(u(k)) \\ &\quad + \cdots + \gamma_{n_\gamma} f_{n_\gamma}(u(k)), \\ &= \boldsymbol{f}(u(k))\boldsymbol{\gamma}, \boldsymbol{\gamma} := [\gamma_1, \gamma_2, \ldots, \gamma_{n_\gamma}]^{\mathrm{T}} \in \mathbb{R}^{n_\gamma}.\end{aligned} \tag{1}$$

$G(z) := \frac{B(z)}{A(z)}$ is the transfer function of the linear subsystem, and $A(z)$ and $B(z)$ are the polynomials in $z^{-1}$ ($z^{-1}$ is the unit backward shift operator, i.e., $z^{-1}u(k) = u(k-1)$):

$$A(z) := 1 + \alpha_1 z^{-1} + \alpha_2 z^{-2} + \cdots + \alpha_n z^{-n},$$
$$B(z) := \beta_0 + \beta_1 z^{-1} + \beta_2 z^{-2} + \cdots + \beta_n z^{-n}.$$

From Fig. 1, we have

$$w(k) = \frac{B(z)}{A(z)}\bar{u}(k), \tag{2}$$

$$y(k) = w(k) + v(k). \tag{3}$$

Replacing $k$ in (2) with $kq$ gives

$$w(kq) = [1 - A(z)]\, w(kq) + B(z)\bar{u}(kq). \tag{4}$$

Equation (3) can be expressed as

$$\begin{aligned}y(kq) &= [1 - A(z)]\, w(kq) + B(z)\bar{u}(kq) + v(kq) \\ &= \beta_0\bar{u}(kq) + \beta_1\bar{u}(kq-1) + \beta_2\bar{u}(kq-2) \\ &\quad + \cdots + \beta_n\bar{u}(kq-n) - \alpha_1 w(kq-1) \\ &\quad - \alpha_2 w(kq-2) - \cdots - \alpha_n w(kq-n) + v(kq).\end{aligned} \tag{5}$$

Referring to [37], without loss of generality, we set the coefficient of the key-term $\bar{u}(kq)$ on the right-hand side to be 1 (i.e., $\beta_0 = 1$). Inserting (1) into (5) gives

$$\begin{aligned} y(kq) = &\sum_{i=1}^{n_\gamma} \gamma_i f_i(u(kq)) + \beta_1 \bar{u}(kq-1) \\ &+ \beta_2 \bar{u}(kq-2) + \cdots + \beta_{nq}\bar{u}(kq-n) \\ &- \alpha_1 w(kq-1) - \alpha_2 w(kq-2) \\ &- \cdots - \alpha_n w(kq-n) + v(kq). \end{aligned} \tag{6}$$

The objective of this paper is to present a RLS algorithm to estimate the parameters $\alpha_i$, $\beta_i$ and $\gamma_i$ for the dual-rate Hammerstein output error systems using the key-term separation principle and the auxiliary model identification idea.

## 3 The auxiliary model-based RLS algorithm

This section derives the AM-RLS algorithm for the dual-rate sampled-data Hammerstein output error model.

Define the information vector $\boldsymbol{\varphi}(kq)$ and the parameter vector $\boldsymbol{\theta}$ as

$$\begin{aligned} \boldsymbol{\varphi}(kq) := \big[ &-w(kq-1), -w(kq-2), \ldots, \\ &-w(kq-n), \bar{u}(kq-1), \bar{u}(kq-2)\ldots, \\ &\bar{u}(kq-n), f_1(u(kq)), f_2(u(kq))\ldots, \\ &f_{n_\gamma}(u(kq))\big]^{\mathrm{T}} \in \mathbb{R}^{n_0}, \\ \boldsymbol{\theta} := \big[&\alpha_1, \alpha_2, \ldots, \alpha_n, \beta_1, \beta_2, \ldots, \beta_n, \gamma_1, \\ &\gamma_2, \ldots, \gamma_{n_\gamma}\big]^{\mathrm{T}} \in \mathbb{R}^{n_0}, \quad n_0 := 2n + n_\gamma. \end{aligned}$$

Equation (6) can be written in a regressive form,

$$y(kq) = \boldsymbol{\varphi}^{\mathrm{T}}(kq)\boldsymbol{\theta} + v(kq). \tag{7}$$

Define a quadratic criterion function,

$$J(\boldsymbol{\theta}) := \sum_{i=1}^{k} \left[y(iq) - \boldsymbol{\varphi}^{\mathrm{T}}(iq)\boldsymbol{\theta}\right]^2.$$

Let $\hat{\boldsymbol{\theta}}(kq)$ be the estimate of $\boldsymbol{\theta}$ at time $kq$. Minimizing $J(\boldsymbol{\theta})$, gives the following RLS algorithm:

$$\begin{aligned} \hat{\boldsymbol{\theta}}(kq) = &\hat{\boldsymbol{\theta}}(kq-q) + \boldsymbol{P}(kq)\boldsymbol{\varphi}(kq) \\ &\times \left[y(kq) - \boldsymbol{\varphi}^{\mathrm{T}}(kq)\hat{\boldsymbol{\theta}}(kq-q)\right], \end{aligned} \tag{8}$$

$$\hat{\boldsymbol{\theta}}(kq+i) = \hat{\boldsymbol{\theta}}(kq), \quad i = 0, 1, \ldots, q-1, \tag{9}$$

$$\boldsymbol{P}(kq) = \boldsymbol{P}(kq-q) - \frac{\boldsymbol{P}(kq-q)\boldsymbol{\varphi}(kq)\boldsymbol{\varphi}^{\mathrm{T}}(kq)\boldsymbol{P}(kq-q)}{1 + \boldsymbol{\varphi}^{\mathrm{T}}(kq)\boldsymbol{P}(kq-q)\boldsymbol{\varphi}(kq)}. \tag{10}$$

Because the information vector $\boldsymbol{\varphi}(kq)$ in (8) contains unknown inner variables $w(kq-j)$ and $\bar{u}(kq-j)$, the standard least squares method fails to give the estimate of the parameter vector $\boldsymbol{\theta}$. The solution is based on the auxiliary model identification idea [38–40]: to replace the unmeasurable term $w(kq+i)$ in $\boldsymbol{\varphi}(kq)$ with its estimate,

$$\hat{w}(kq+i) = \hat{\boldsymbol{\varphi}}^{\mathrm{T}}(kq+i)\hat{\boldsymbol{\theta}}(kq), \; i = 1, 2, \ldots, q-1. \tag{11}$$

Replacing $\gamma_i$ in (2) with its estimate $\hat{\gamma}_i(kq)$, we can get the estimate $\hat{u}(kq+i)$ of $\bar{u}(kq+i)$:

$$\begin{aligned} \hat{u}(kq+i) = &\hat{\gamma}_1(kq) f_1(u(kq+i)) + \hat{\gamma}_2(kq) \\ &\times f_2(u(kq+i)) + \cdots + \hat{\gamma}_{n_\gamma}(kq) \\ &\times f_{n_\gamma}(u(kq+i)), \quad i = 1, 2, \ldots, q-1. \end{aligned} \tag{12}$$

Define the estimate of $\boldsymbol{\varphi}(kq)$:

$$\begin{aligned} \hat{\boldsymbol{\varphi}}(kq) := \Big[ &-\hat{w}(kq-1), -\hat{w}(kq-2), \ldots, \\ &-\hat{w}(kq-n), \hat{u}(kq-1), \hat{u}(kq-2), \ldots, \\ &\hat{u}(kq-n), f_1(u(kq)), f_2(u(kq))\ldots, \\ &f_{n_\gamma}(u(kq))\Big]^{\mathrm{T}}. \end{aligned} \tag{13}$$

Using $\hat{\boldsymbol{\varphi}}(kq)$ in place of $\boldsymbol{\varphi}(kq)$ in (8) and (10), we have

$$\begin{aligned} \hat{\boldsymbol{\theta}}(kq) = &\hat{\boldsymbol{\theta}}(kq-q) + \boldsymbol{P}(kq)\hat{\boldsymbol{\varphi}}(kq) \\ &\times \left[y(kq) - \hat{\boldsymbol{\varphi}}^{\mathrm{T}}(kq)\hat{\boldsymbol{\theta}}(kq-q)\right], \end{aligned} \tag{14}$$

$$\boldsymbol{P}(kq) = \boldsymbol{P}(kq-q) - \frac{\boldsymbol{P}(kq-q)\hat{\boldsymbol{\varphi}}(kq)\hat{\boldsymbol{\varphi}}^{\mathrm{T}}(kq)\boldsymbol{P}(kq-q)}{1+\hat{\boldsymbol{\varphi}}^{\mathrm{T}}(kq)\boldsymbol{P}(kq-q)\hat{\boldsymbol{\varphi}}(kq)}. \tag{15}$$

Equations (9), (11)–(15) form the AM-RLS algorithm for system by using the key-term separation principle, which can be summarized as

$$\begin{aligned} \hat{\boldsymbol{\theta}}(kq) = &\hat{\boldsymbol{\theta}}(kq-q) \\ &+ \boldsymbol{P}(kq)\hat{\boldsymbol{\varphi}}(kq)\left[y(kq) - \hat{\boldsymbol{\varphi}}^{\mathrm{T}}(kq)\hat{\boldsymbol{\theta}}(kq-q)\right], \end{aligned} \tag{16}$$

$$\hat{\boldsymbol{\theta}}(kq+i) = \hat{\boldsymbol{\theta}}(kq), \quad i = 1, 2, \ldots, q-1, \tag{17}$$

$$\boldsymbol{P}(kq) = \boldsymbol{P}(kq-q) - \frac{\boldsymbol{P}(kq-q)\hat{\boldsymbol{\varphi}}(kq)\hat{\boldsymbol{\varphi}}^{\mathrm{T}}(kq)\boldsymbol{P}(kq-q)}{1 + \hat{\boldsymbol{\varphi}}^{\mathrm{T}}(kq)\boldsymbol{P}(kq-q)\hat{\boldsymbol{\varphi}}(kq)}, \tag{18}$$

$$\hat{\boldsymbol{\varphi}}(kq) = \Big[-\hat{w}(kq-1), -\hat{w}(kq-2), \ldots, -\hat{w}(kq-n), \hat{u}(kq-1), \hat{u}(kq-2), \ldots, \hat{u}(kq-n), f_1(u(kq)), f_2(u(kq)), \ldots, f_{n_\gamma}(u(kq))\Big]^{\mathrm{T}}, \tag{19}$$

$$\hat{u}(kq+i) = \hat{\gamma}_1(kq) f_1(u(kq+i)) + \hat{\gamma}_2(kq) f_2(u(kq+i)) + \cdots + \hat{\gamma}_{n_\gamma}(kq) f_{n_\gamma}(u(kq+i)), \quad i = 1, 2, \ldots, q-1, \tag{20}$$

$$\hat{w}(kq+i) = \hat{\boldsymbol{\varphi}}^{\mathrm{T}}(kq+i)\hat{\boldsymbol{\theta}}(kq) \quad i = 1, 2, \ldots, q-1. \tag{21}$$

To initialize the algorithm, we take $\hat{\boldsymbol{\theta}}(0)$ to be a small real vector, e.g., $\hat{\boldsymbol{\theta}}(0) = \mathbf{1}_{n_0}/p_0$ and with $p_0$ normally a large positive number (e.g., $p_0 = 10^6$), and $\boldsymbol{P}(0) = p_0\boldsymbol{I}$ with $\boldsymbol{I}$ representing an identity matrix of appropriate dimension.

## 4 Convergence of parameter estimation

In this section, we focus on analyzing the convergence properties of the proposed RLS algorithm which is under weak conditions. Assume that $\{v(kq), \mathcal{F}_{kq}\}$ is a martingale difference sequence defined on a probability space $\{\Omega, \mathcal{F}, P\}$, where $\{\mathcal{F}_{kq}\}$ is the $\sigma$ algebra sequence generated by $\{v(kq)\}$. The noise sequence $\{v(kq)\}$ satisfies the following assumptions [41,42]:

(A1) $E[v(kq)|\mathcal{F}_{kq-q}] = 0$, a.s.,

(A2) $E[\|v(kq)\|^2|\mathcal{F}_{kq-q}] \leqslant \sigma_v^2 < \infty$.

Defining $r(kq) = \mathrm{tr}[\boldsymbol{P}^{-1}(kq)]$, it follows that

$$\ln|\boldsymbol{P}^{-1}(kq)| = O(\ln r(kq)). \tag{22}$$

**Theorem 1** *For the systems in (7) and the algorithm (16)–(21), if assumptions* (A1) and (A2) *hold, for any $c > 1$, the parameter estimation error associated with the AM-LS algorithm for the dual-rate sampled-data Hammerstein output error model satisfies:*

$$\|\tilde{\boldsymbol{\theta}}(kq)\|^2 = O\left(\frac{\ln\|r(kq)\|]^c}{\lambda_{\min}[\boldsymbol{P}^{-1}(kq)]}\right). \tag{23}$$

*Proof* Define the parameter estimation error vector

$$\begin{aligned}\tilde{\boldsymbol{\theta}}(kq) &:= \hat{\boldsymbol{\theta}}(kq) - \boldsymbol{\theta} \\ &= \tilde{\boldsymbol{\theta}}(kq-q) + \boldsymbol{P}(kq)\hat{\boldsymbol{\varphi}}(kq) \Big[\hat{\boldsymbol{\varphi}}^{\mathrm{T}}(kq)\boldsymbol{\theta}(kq) + v(kq) - \hat{\boldsymbol{\varphi}}^{\mathrm{T}}(kq)\hat{\boldsymbol{\theta}}(kq-q)\Big] \\ &=: \tilde{\boldsymbol{\theta}}(kq-q) + \boldsymbol{P}(kq)\hat{\boldsymbol{\varphi}}(kq)\left[-\tilde{y}(kq) + v(kq)\right],\end{aligned} \tag{24}$$

where

$$\begin{aligned}\tilde{y}(kq) &:= \hat{\boldsymbol{\varphi}}^{\mathrm{T}}(kq)\hat{\boldsymbol{\theta}}(kq-q) - \hat{\boldsymbol{\varphi}}^{\mathrm{T}}(kq)\boldsymbol{\theta}(kq) \\ &= \hat{\boldsymbol{\varphi}}^{\mathrm{T}}(kq)\tilde{\boldsymbol{\theta}}(kq-q).\end{aligned} \tag{25}$$

Define a non-negative definite function

$$T(kq) = \tilde{\boldsymbol{\theta}}^{\mathrm{T}}(kq)\boldsymbol{P}^{-1}(kq)\tilde{\boldsymbol{\theta}}(kq). \tag{26}$$

Using (7), (24), and (25), we have

$$\begin{aligned}T(kq) &= \Big\{\tilde{\boldsymbol{\theta}}(kq-q) + \boldsymbol{P}(kq)\hat{\boldsymbol{\varphi}}(kq)\big[-\tilde{y}(kq) + v(kq)\big]\Big\}^{\mathrm{T}}\boldsymbol{P}^{-1}(kq)\Big\{\tilde{\boldsymbol{\theta}}(kq-q) + \boldsymbol{P}(kq)\hat{\boldsymbol{\varphi}}(kq)\left[-\tilde{y}(kq) + v(kq)\right]\Big\} \\ &= \tilde{\boldsymbol{\theta}}^{\mathrm{T}}(kq-q)\boldsymbol{P}^{-1}(kq)\tilde{\boldsymbol{\theta}}(kq-q) + 2\tilde{\boldsymbol{\theta}}^{\mathrm{T}}(kq-q)\hat{\boldsymbol{\varphi}}(kq)\left[-\hat{y}(kq) + v(kq)\right] \\ &\quad + \hat{\boldsymbol{\varphi}}^{\mathrm{T}}(kq)\boldsymbol{P}(kq)\hat{\boldsymbol{\varphi}}(kq)[-\hat{y}(kq) + v(kq)]^2 \\ &= T(kq-q) - \left[1 - \hat{\boldsymbol{\varphi}}^{\mathrm{T}}(kq)\boldsymbol{P}(kq)\hat{\boldsymbol{\varphi}}(kq)\right]\tilde{y}^2(kq) + \hat{\boldsymbol{\varphi}}^{\mathrm{T}}(kq)\boldsymbol{P}(kq)\hat{\boldsymbol{\varphi}}(kq)v^2(kq) \\ &\quad + 2\left[1 - \hat{\boldsymbol{\varphi}}^{\mathrm{T}}(kq)\boldsymbol{P}(kq)\hat{\boldsymbol{\varphi}}(kq)\right]\tilde{y}(kq)v(kq) \\ &\leqslant T(kq-q) + \hat{\boldsymbol{\varphi}}^{\mathrm{T}}(kq)\boldsymbol{P}(kq)\hat{\boldsymbol{\varphi}}(kq)v^2(kq) + 2\left[1 - \hat{\boldsymbol{\varphi}}^{\mathrm{T}}(kq)\boldsymbol{P}(kq)\hat{\boldsymbol{\varphi}}(kq)\right]\tilde{y}(kq)v(kq).\end{aligned}$$

Here, we have used the relation

$$\begin{aligned}&\left[1 - \hat{\boldsymbol{\varphi}}^{\mathrm{T}}(kq)\boldsymbol{P}(kq)\hat{\boldsymbol{\varphi}}(kq)\right] \\ &\quad = \left[1+\hat{\boldsymbol{\varphi}}^{\mathrm{T}}(kq)\boldsymbol{P}(kq)\hat{\boldsymbol{\varphi}}(kq)\right]^{-1} > 0.\end{aligned} \tag{27}$$

Since $\tilde{y}(kq)$, $\hat{\boldsymbol{\varphi}}^{\mathrm{T}}(kq)\boldsymbol{P}(kq)\hat{\boldsymbol{\varphi}}(kq)$, and $T(kq-q)$ are uncorrelated with $v(kq)$ and $\mathcal{F}_{kq-q}$ are measurable, taking the conditional expectation on both sides with respect to $\mathcal{F}_{kq-q}$ and using (A1) and (A2) give

$$\begin{aligned}&E\left[T(kq)|\mathcal{F}_{kq-q}\right] \\ &\quad \leqslant T(kq-q) + 2\hat{\boldsymbol{\varphi}}^{\mathrm{T}}(kq)\boldsymbol{P}(kq)\hat{\boldsymbol{\varphi}}(kq)\sigma_v^2, \text{ a.s.}\end{aligned} \tag{28}$$

Let

$$V(kq) = \frac{T(kq)}{\left[\ln\|\boldsymbol{P}^{-1}(kq)\|\right]^c}, \quad c > 1.$$

Since $\boldsymbol{P}^{-1}(kq)$ is non-decreasing, we have

$$\begin{aligned}E[T(kq)|\mathcal{F}_{kq-q}] &\leqslant \frac{T(kq-q)}{\left[\ln\|\boldsymbol{P}^{-1}(kq)\|\right]^c} + \frac{2\hat{\boldsymbol{\varphi}}^{\mathrm{T}}(kq)\boldsymbol{P}(kq)\hat{\boldsymbol{\varphi}}(kq)}{\left[\ln\|\boldsymbol{P}^{-1}(kq)\|\right]^c}\sigma_v^2 \\ &= V(kq-q) + \frac{2\hat{\boldsymbol{\varphi}}^{\mathrm{T}}(kq)\boldsymbol{P}(kq)\hat{\boldsymbol{\varphi}}(kq)}{\left[\ln\|\boldsymbol{P}^{-1}(kq)\|\right]^c}\sigma_v^2, \text{ a.s.}\end{aligned}$$

Applying the martingale convergence theorem to the above inequality, we conclude that $V(kq)$ converges a.s. to a finite random variable $V_0$, i.e.,

$$V(kq) = \frac{T(kq)}{[\ln\|\boldsymbol{P}^{-1}(kq)\|]^c} \to V_0 < \infty, \text{ a.s., or}$$

$$T(kq) = O\left(\left[\ln\|\boldsymbol{P}^{-1}(kq)\|\right]^c\right), \text{ a.s} \tag{29}$$

**Table 1** The parameter estimates and their errors ($\sigma^2 = 2.00^2$) for Example 1

| $k$ | $\alpha_1$ | $\alpha_2$ | $\beta_1$ | $\beta_2$ | $\gamma_1$ | $\gamma_2$ | $\gamma_3$ | $\delta$ (%) |
|---|---|---|---|---|---|---|---|---|
| 100 | −0.08957 | 0.28833 | 0.69233 | 0.48838 | 0.92168 | 0.56569 | 0.18008 | 22.79018 |
| 500 | −0.28598 | 0.36002 | 0.77980 | 0.19288 | 1.27630 | 0.52086 | 0.16452 | 20.20585 |
| 1,000 | −0.29097 | 0.37232 | 0.76827 | 0.14979 | 1.04526 | 0.48545 | 0.23760 | 7.87752 |
| 2,000 | −0.27804 | 0.40340 | 0.77800 | 0.18180 | 1.07561 | 0.51477 | 0.23658 | 7.69908 |
| 3,000 | −0.25412 | 0.38338 | 0.77716 | 0.22105 | 0.99830 | 0.52817 | 0.27124 | 3.30136 |
| 4,000 | −0.25887 | 0.36744 | 0.77806 | 0.22566 | 0.98179 | 0.51115 | 0.26994 | 2.60922 |
| 5,000 | −0.24646 | 0.37291 | 0.78578 | 0.24913 | 1.00930 | 0.51781 | 0.25039 | 1.83474 |
| 6,000 | −0.24375 | 0.37378 | 0.78031 | 0.24489 | 0.98735 | 0.52491 | 0.25895 | 2.24174 |
| True values | −0.24000 | 0.36000 | 0.78000 | 0.24000 | 1.00000 | 0.50000 | 0.25000 | |

**Table 2** The parameter estimates and their errors ($\sigma^2 = 0.50^2$) for Example 1

| $k$ | $\alpha_1$ | $\alpha_2$ | $\beta_1$ | $\beta_2$ | $\gamma_1$ | $\gamma_2$ | $\gamma_3$ | $\delta$ (%) |
|---|---|---|---|---|---|---|---|---|
| 100 | −0.29748 | 0.35725 | 0.66015 | 0.20367 | 0.96348 | 0.57311 | 0.29819 | 11.36996 |
| 500 | −0.28523 | 0.36869 | 0.73395 | 0.17640 | 1.07809 | 0.51902 | 0.23762 | 8.29469 |
| 1,000 | −0.27272 | 0.36994 | 0.74840 | 0.18517 | 1.01127 | 0.50468 | 0.25392 | 4.96478 |
| 2,000 | −0.26083 | 0.37572 | 0.76209 | 0.20688 | 1.01878 | 0.50787 | 0.25066 | 3.40692 |
| 3,000 | −0.25161 | 0.36961 | 0.76637 | 0.22162 | 0.99872 | 0.50959 | 0.25838 | 2.05509 |
| 4,000 | −0.25119 | 0.36491 | 0.76906 | 0.22534 | 0.99501 | 0.50471 | 0.25718 | 1.64000 |
| 5,000 | −0.24702 | 0.36590 | 0.77285 | 0.23308 | 1.00210 | 0.50596 | 0.25179 | 1.02232 |
| 6,000 | −0.24561 | 0.36587 | 0.77274 | 0.23323 | 0.99668 | 0.50742 | 0.25364 | 1.06076 |
| True values | −0.24000 | 0.36000 | 0.78000 | 0.24000 | 1.00000 | 0.50000 | 0.25000 | |

From the definition of $V(kq)$, we have

$$\begin{aligned}\|\tilde{\boldsymbol{\theta}}(kq)\|^2 &\leqslant \frac{\mathrm{tr}\left[\boldsymbol{\theta}^{\mathrm{T}}(kq)\boldsymbol{P}^{-1}(kq)\boldsymbol{\theta}(kq)\right]}{\lambda_{\min}\left[\boldsymbol{P}^{-1}(kq)\right]} \\ &= \frac{T(kq)}{\lambda_{\min}\left[\boldsymbol{P}^{-1}(kq)\right]}. \end{aligned} \tag{30}$$

Using (29) and (22) gives

$$\begin{aligned}\|\tilde{\boldsymbol{\theta}}(kq)\|^2 &= O\left(\frac{\ln\|\boldsymbol{P}^{-1}(kq)\|]^c}{\lambda_{\min}\left[\boldsymbol{P}^{-1}(kq)\right]}\right) \\ &= O\left(\frac{\ln\|r(kq)\|]^c}{\lambda_{\min}\left[\boldsymbol{P}^{-1}(kq)\right]}\right).\end{aligned}$$

This gives the conclusion of Theorem 1. □

*Remark* Theorem 1 shows that if the noise has a bounded variance, then the parameter estimates converge to the true values.

## 5 Numerical examples

In this section, three examples are given to illustrate effectiveness of the AM-RLS algorithm.

*Example 1* Consider the following Hammerstein system with dual-rate sampling,

$$\begin{aligned} y(kq) &= [1 - A(z)]\, w(kq) + B(z)\bar{u}(kq) + v(kq), \\ A(z) &= 1 - 0.24z^{-1} + 0.36z^{-2}, \\ B(z) &= 1 + 0.78z^{-1} + 0.24z^{-2}, \\ \bar{u}(k) &= \gamma_1 u(k) + \gamma_2 u^2(k) + \gamma_3 u^3(k) \\ &= u(k) + 0.5u^2(k) + 0.25u^3(k). \end{aligned}$$

In simulation, $\{u(k)\}$ is taken as a persistent excitation signal sequence with zero mean and unit variance, and $\{v(k)\}$ as a white noise sequence with zero mean and variance $\sigma^2 = 0.50^2$ and $\sigma^2 = 2.00^2$, respectively. Applying the proposed algorithm in (16)–(21) to estimate the parameters $(\alpha_i, \beta_i, \gamma_i)$ of this system, the parameter estimates and their errors with different noise variances are shown in Tables 1 and 2. The parameter estimation errors $\delta := \|\hat{\boldsymbol{\theta}}(kq) - \boldsymbol{\theta}\|/\|\boldsymbol{\theta}\|$ versus $k$ are shown in Fig. 2.

*Example 2* Consider a Hammerstein system in Example 1 with

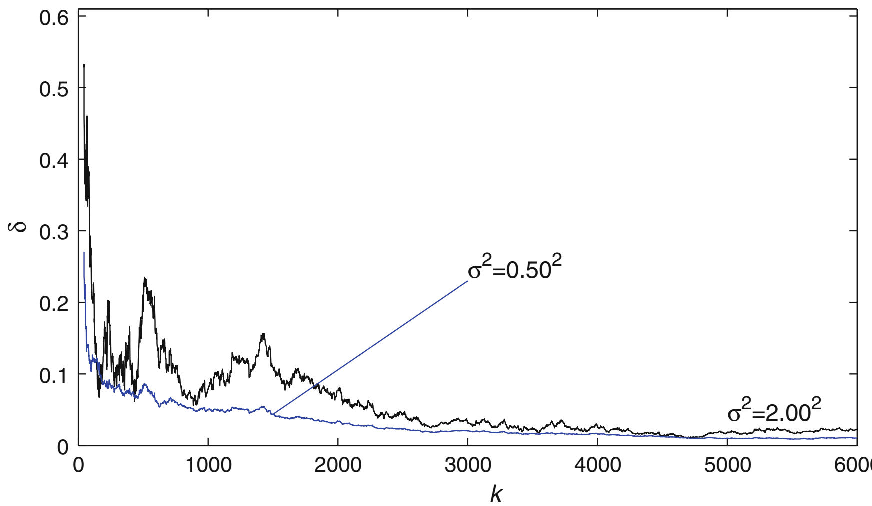


**Fig. 2** The estimation errors $\delta$ versus $k$ with $\sigma^2 = 0.50^2$ and $\sigma^2 = 2.00^2$ in Example 1

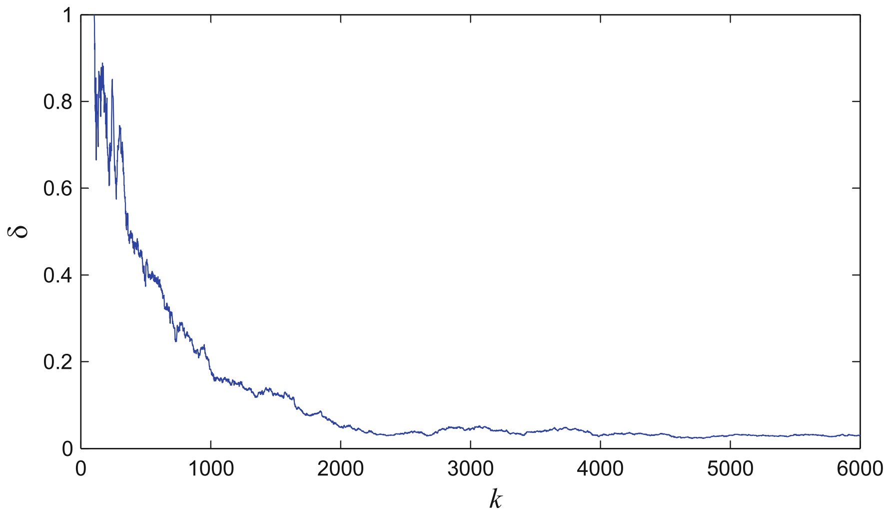


**Fig. 3** The estimation errors $\delta$ versus $k$ with $\sigma^2 = 0.50^2$ for Example 2

$$\begin{cases} A(z) = 1 + 0.44z^{-1} + 0.36z^{-2} + 0.14z^{-3} \\ \qquad\quad + 0.21z^{-4} + 0.12z^{-5}, \\ B(z) = 1 + 0.58z^{-1} - 0.54z^{-2} + 0.34z^{-3} \\ \qquad\quad + 0.25z^{-4} + 0.13z^{-5}, \\ \bar{u}(k) = u(k) + 0.5u^2(k) + 0.25u^3(k). \end{cases}$$

*Example 3* Consider a nonlinear subsystem in Example 1 with

$$\begin{cases} A(z) = 1 + 0.54z^{-1} - 0.36z^{-2}, \\ B(z) = 1 + 0.48z^{-1} + 0.14z^{-2}, \\ \bar{u}(k) = u(k) + 0.70u^2(k) + 0.65u^3(k) \\ \qquad\quad + 0.55u^4(k) + 0.45u^5(k). \end{cases}$$

The simulation conditions of Examples 2 and 3 are the same as in Example 1 with the noise variance $\sigma^2 = 0.50^2$. The parameter estimation errors $\delta$ versus $k$ are shown in Figs. 3 and 4.

From Tables 1 and 2 and Figs. 2, 3, and 4, we can draw the following conclusions:

- It is clear that the estimation errors become smaller (in general) as the recursive step $k$ increases—see Tables 1 and 2.
- A higher noise level results in a slower convergence rate of the parameter estimates; after about $k = 5000$, the parameter estimates converge to their true values—see the error curves in Fig. 2 and the estimation errors of the last columns of Tables 1 and 2.
- Under the same noise level, a complex model structure results in a slower convergence rate—see Figs. 3 and 4.
- Increasing the complexity of the nonlinear subsystem causes a slower convergence rate than increasing the complexity of the linear subsystem—see Figs. 3 and 4.

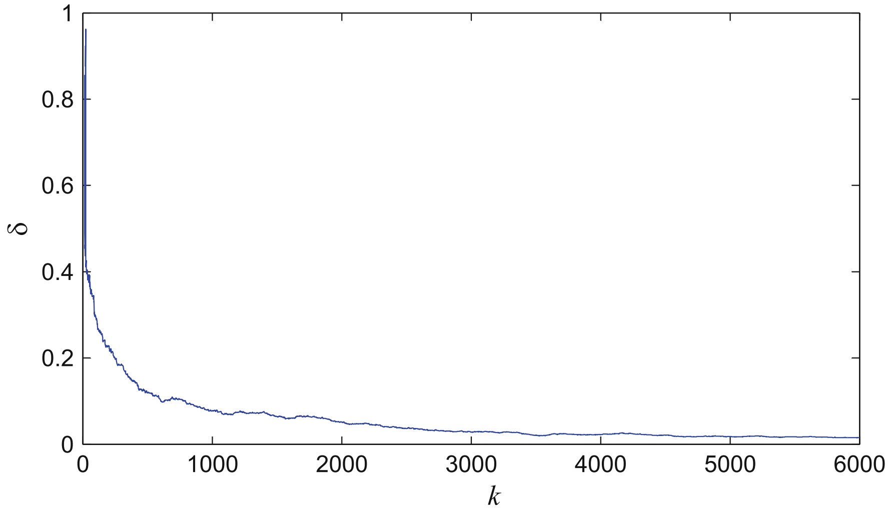


**Fig. 4** The estimation errors $\delta$ versus $k$ with $\sigma^2 = 0.50^2$ for Example 3

## 6 Conclusions

In this paper, we present a RLS identification algorithm for dual-rate sampled-data Hammerstein nonlinear systems. Using the key-term separation principle, we construct the identification model of Hammerstein nonlinear systems with dual-rate sampling. All the model parameters can be estimated by the AM-RLS algorithm. The simulation results verified the effectiveness of the proposed algorithm. The method in the paper can combine iterative methods [43–47] to study identification problems for other linear or nonlinear systems [48–52].

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