Nonlinear Dyn (2014) 76:1041–1050
DOI 10.1007/s11071-013-1187-y

ORIGINAL PAPER

Hybrid control of Hopf bifurcation in a dual model of Internet congestion control system

Da-Wei Ding · Xue-Mei Qin · Nian Wang · Ting-Ting Wu · Dong Liang

Received: 28 April 2013 / Accepted: 13 December 2013 / Published online: 28 December 2013

Abstract In this paper, a hybrid control strategy using both state feedback and parameter perturbation is applied to control the Hopf bifurcation in a dual model of Internet congestion control system. By choosing communication delay as a bifurcation parameter, it is proved that when it passes through a critical value, a Hopf bifurcation occurs. However, by adjusting the control parameters of the hybrid control strategy, the Hopf bifurcation has been delayed without changing the original equilibrium point of the system. Theoretical analysis and numerical results show that this method can delay the onset of bifurcation effectively. Therefore, it can extend the stable range in parameter space and improve the performance of congestion control system.

Keywords Internet congestion control · Dual model · Hopf bifurcation · Bifurcation control · Hybrid control

1 Introduction

With the rapid development of the Internet, the Internet congestion control algorithm has become a subject of intense research activities [1,2]. One of the important properties of congestion control algorithm is stability [3]. In particular, the local asymptotic stability with communication delays is widely studied by linearizing the model around the equilibrium point [1,2,4,5]. When the congestion control system loses its local stability, it causes some nonlinear dynamical behaviors such as chaos and bifurcation [1,2,4–11]. Such a kind of unstable phenomenon degrades the performance of network [6]. For example, in [12], the period-doubling bifurcation leads to the chaos state. In [1,2,4–11], the Hopf bifurcations change the stability of Internet congestion control model. Therefore, it is very significant to study the problem of bifurcation and chaos in the Internet congestion control system.

In reality, these complex dynamic behaviors mean that the system changes from a stable state to an unstable one, which may be harmful to the system [2]. So, in order to delay the onset of such an unstable phenomenon and enlarge the stable range of the Internet congestion control system, many effective control methods have been proposed, such as delay feedback control method [13,14], heterogeneous delay approach [11,15,16], dynamic delayed feedback control [3,17], and hybrid control strategy [12,18–21]. Especially, the hybrid control has also been widely used in recent years. In [12,18,19], a hybrid control method has been used to control bifurcation and chaos in discrete nonlinear dynamical systems, and in [18], the authors have applied hybrid control to a continuous nonlinear dynamical system. In [20], hybrid control of bifurcation is considered in a predator–prey system with three

D.-W. Ding (✉) · X.-M. Qin · N. Wang · T.-T. Wu · D. Liang
Department of Electronics and Information Engineering, Anhui University, Hefei 230039, China
e-mail:dwding@ahu.edu.cn

delays. In [21], a parameter perturbation control and a hybrid control are proposed.

The main contribution of this paper is that a hybrid control strategy using both state feedback and parameter perturbation is applied to control the Hopf bifurcation in a first-order dual model of congestion control system. Here we choose the communication delay as a bifurcation parameter. Because the round-trip delay varies depending on the network's congestion status, the system may exhibit complex behaviors in practice [4]. By using this strategy, first, we increase the critical value of communication delay and delay the onset of undesirable Hopf bifurcation. Second, we extend the stable range and improve the performance of the Internet congestion control system. Therefore, this control strategy is applicable in practice.

The rest of this paper is organized as follows. In Sect. 2, we summarize some properties of the original uncontrolled Internet congestion control system. In Sect. 3, the proposed hybrid control strategy is applied to the dual model, and the Hopf bifurcation of the controlled system is studied. In Sect. 4, by applying the center manifold theorem and the normal form theory, the stability of bifurcating solutions and the direction of the Hopf bifurcation are analyzed. Finally, numerical examples and conclusion remarks are given in Sects. 5 and 6, respectively.

2 Hopf bifurcation in uncontrolled model

In this section, we consider a first-order dual model of the Internet congestion control systems, the model can be formulated as follows [2]:

$$\dot{p}(t) = kp(t)(x(t-\tau) - c), \tag{1}$$

where $p(t)$ is the price at the link (packets), and $x(t) = f(p(t)) = 1/p(t)$ is a nonnegative continuous, strictly decreasing demand function and has at least third-order continuous derivatives (1/packets). The scalar c is the capacity of the bottleneck link (packets/s), and k is a gain parameter. τ is the round-trip time which consists of the propagation delay and queuing delay (s).

Let p^* be the nonzero equilibrium point of system (1). It then satisfies the following equation:

$$f(p^*) = c.$$

For convenience, the results of system (1) are summarized as follows. The analysis of specific details for the system (1) has been described in [2].

Theorem 1 *For the system (1), when $\tau_0 = -\frac{\pi}{2b_2}$ and $\omega_0 = -b_2$, we can get the following results in [2]:*

(i) When $\tau < \tau_0$, the equilibrium point of the system (1) is locally asymptotically stable;
(ii) When $\tau = \tau_0$, the system (1) exists a Hopf bifurcation;
(iii) When $\tau > \tau_0$ the equilibrium point of the system (1) is unstable and a limit cycle exists.

3 Hybrid control of bifurcation

Equation (1) is donated as

$$\dot{p}(t) = g(p(t), \tau). \tag{2}$$

Now, the hybrid control strategy is added to the model (2), and then we can obtain the following controlled system:

$$\begin{aligned}\dot{p}(t) &= \alpha g(p(t), \tau) + (1-\alpha)(p(t) - p^*),\\ &= \alpha kp(t)(x(t-\tau) - c) + (1-\alpha)(p(t) - p^*),\end{aligned} \tag{3}$$

where α is a control parameter. The system (3) has the same fixed points as the original system (1) [21]. So, we know

$$f(p^*) = c. \tag{4}$$

Set $u(t) = p(t) - p^*$, and then Eq. (3) is expanded by a Taylor expansion around the equilibrium point p^*, we can get

$$\begin{aligned}\dot{u}(t) = {} & a_1u(t) + a_2u(t-\tau) + a_4u(t)u(t-\tau)\\ & + a_5u^2(t-\tau) + a_8u(t)u^2(t-\tau)\\ & + a_9u^3(t-\tau) + O(u^4),\end{aligned} \tag{5}$$

where

$$\begin{aligned}a_1 &= \frac{\partial}{\partial p(t)}[\dot{p}(t)]|_{p^*} = \alpha k(x(t-\tau) - c)\\ &\quad + (1-\alpha)|_{p^*} = 1-\alpha,\\ a_2 &= \frac{\partial}{\partial x(t-\tau)}[\dot{p}(t)]|_{p^*} = \alpha kp(t)x'(t-\tau)|_{p^*}\\ &= \alpha kp^*x'(p^*),\\ a_4 &= \frac{\partial^2}{2!\partial x(t-\tau)\partial p(t)}[\dot{p}(t)]|_{p^*} = \frac{1}{2}\alpha kx'(t-\tau)|_{p^*}\\ &= \frac{1}{2}\alpha kx'(p^*),\end{aligned}$$

$$a_5 = \frac{\partial^2}{2!\partial^2 x(t-\tau)}[\dot{p}(t)]|_{p^*} = \frac{1}{2}\alpha k p(t) x''(t-\tau)|_{p^*}$$
$$= \frac{1}{2}\alpha k p^* x''(p^*),$$
$$a_8 = \frac{\partial^3}{3!\partial^2 x(t-\tau)\partial p(t)}[\dot{p}(t)]|_{p^*} = \frac{1}{6}\alpha k x''(t-\tau)|_{p^*}$$
$$= \frac{1}{6}\alpha k x''(p^*),$$
$$a_9 = \frac{\partial^3}{3!\partial^3 x(t-\tau)}[\dot{p}(t)]|_{p^*} = \frac{1}{6}\alpha k p(t) x'''(t-\tau)|_{p^*}$$
$$= \frac{1}{6}\alpha k p^* x'''(p^*).$$

Consider the linear part of Eq. (5)

$$\dot{u}(t) = (1-\alpha)u(t) + a_2 u(t-\tau) \tag{6}$$

The characteristic equation of Eq. (6) is

$$\lambda - (1-\alpha) - a_2 e^{-\lambda\tau} = 0. \tag{7}$$

Lemma 1 *When $\tau = \tau_0$, $\lambda = \pm i\omega_0$, $\omega_0 > 0$ are pure imaginary roots of Eq. (7).*

Proof Assume that Eq. (7) has pure imaginary roots, i.e., $\lambda = \pm i\omega$, $\omega > 0$. Then substituting them into Eq. (7) and separating the real and imaginary parts, it is straightforward to get

$$\begin{cases} (1-\alpha) + a_2\cos(\omega\tau) = 0, \\ \omega + a_2\sin(\omega\tau) = 0. \end{cases} \tag{8}$$

From Eq. (8), we obtain

$$\omega_0 = \sqrt{a_2^2 - (1-\alpha)^2}, \tag{9}$$

$$\tau_0 = \frac{1}{\omega_0}\arccos\left(-\frac{1-\alpha}{a_2}\right). \tag{10}$$

□

Lemma 2 *Equation (7) has roots with positive real parts except for $\tau = \tau_0$.*

Proof Let $\lambda = \beta + i\omega$ be a root of Eq. (7) with $\beta > 0$ and $\omega > 0$, then

$$\begin{cases} \beta - (1-\alpha) - a_2 e^{-\beta\tau}\cos(\omega\tau) = 0, \\ \omega + a_2 e^{-\beta\tau}\sin(\omega\tau) = 0. \end{cases} \tag{11}$$

From the first equation of Eq. (11), we get

$$\frac{(2n+1)\pi}{2} < \omega\tau < \frac{(2n+3)\pi}{2}, \quad n = 0, 2, 4, \ldots$$

and from the second equation of Eq. (11), we get

$$\omega\tau < \frac{(2n+1)\pi}{2}.$$

Therefore, Eq. (7) may have positive real parts except for $\tau = \tau_0$. Finally, we will show that the transversality condition of the Hopf bifurcation is also satisfied. □

Lemma 3 *Let $\lambda = \beta + i\omega$ be the root of Eq. (7), the following transversality condition holds:*

$$\mathrm{Re}\left(\frac{d\lambda}{d\tau}\right)_{\tau=\tau_0} \neq 0.$$

Proof Therefore, evaluating

$$\frac{d\lambda}{d\tau} = -\frac{a_2\lambda e^{-\lambda\tau}}{1 + a_2\tau e^{-\lambda\tau}}.$$

Hence, let $\lambda = \beta + i\omega$ be the root of Eq. (7), we have

$$\begin{aligned} \frac{d\lambda}{d\tau} &= -\frac{a_2(\beta+i\omega)e^{-(\beta+i\omega)\tau}}{1 + a_2\tau e^{-(\beta+i\omega)\tau}} \\ &= -\frac{a_2(\beta+i\omega)e^{-\beta\tau}(\cos(\omega\tau) - i\sin(\omega\tau))}{1 + a_2\tau e^{-\beta\tau}(\cos(\omega\tau) - i\sin(\omega\tau))} \end{aligned} \tag{12}$$

From Eq. (12), we can get

$$\mathrm{Re}\left(\frac{d\lambda}{d\tau}\right) = -\frac{a_2 e^{-\beta\tau}(\beta\cos(\omega\tau) + \omega\sin(\omega\tau) + a_2\beta\tau e^{-\beta\tau})}{(1 + a_2\tau e^{-\beta\tau}\cos(\omega\tau))^2 + (a_2\tau e^{-\beta\tau}\sin(\omega\tau))^2}$$

and

$$\mathrm{Im}\left(\frac{d\lambda}{d\tau}\right) = -\frac{a_2 e^{-\beta\tau}(\omega\cos(\omega\tau) - \beta\sin(\omega\tau) + a_2\omega\tau e^{-\beta\tau})}{(1 + a_2\tau e^{-\beta\tau}\cos(\omega\tau))^2 + (a_2\tau e^{-\beta\tau}\sin(\omega\tau))^2}.$$

When $\tau = \tau_0$, $\beta = 0$, and $\omega_0\tau_0 = -\omega_0/a_2$, then we obtain the following formula:

$$\mathrm{Re}\left(\frac{d\lambda}{d\tau}\right)_{\tau=\tau_0} = -\frac{\omega_0^2}{(1-(1-\alpha)\tau_0)^2 + (\omega_0\tau_0)^2} > 0 \tag{13}$$

and

$$\mathrm{Im}\left(\frac{d\lambda}{d\tau}\right)_{\tau=\tau_0} = \frac{\omega_0((1-\alpha) - a_2^2\tau_0)}{(1-(1-\alpha)\tau_0)^2 + (\omega_0\tau_0)^2}. \tag{14}$$

It is well known that when $\tau > \tau_0$, the characteristic equation of Eq. (6) has at least one root with positive real parts. At the same time, the equilibrium point of the system (3) is unstable and a limit cycle bifurcates out from the equilibrium point. □

Based on above analysis, we can get the following theorem. When $\alpha = 1$, the controlled system (3) becomes the original uncontrolled system (1). The conclusion is as Theorem 1.

Theorem 2 *For the controlled system (3), we can easily obtain*

(i) *When $\tau \in [0, \tau_0)$, the equilibrium point p^* of the controlled system (3) is locally asymptotically stable;*
(ii) *When $\tau = \tau_0$, the controlled system (3) exists a Hopf bifurcation at equilibrium point p^*;*
(iii) *When $\tau \in (\tau_0, +\infty)$, the equilibrium point p^* of the controlled system (3) is unstable.*

4 Stability and direction of bifurcating periodic solutions

In this section, the normal form theory and the center manifold theorem are used to analyze the direction of the bifurcation and the stability of bifurcating periodic solutions of the controlled system (3). The analysis process is as follows.

In order to simplify calculation, let $\tau = \tau_0 + \mu$. So $\mu = 0$ is the value of Hopf bifurcation for Eq. (6). Let

$$L_\mu \varphi = (1-\alpha)\phi(0) + a_2\phi(-\tau)$$

and

$$\begin{aligned} &F(\varphi, \mu) \\ &\quad = a_4\varphi(0)\varphi(-\tau) + a_5\varphi^2(-\tau) + a_8\varphi(0)\varphi^2(-\tau) \\ &\qquad + a_9\varphi^3(-\tau) + O(|\varphi|^4). \end{aligned}$$

Therefore, the system (5) can be written in the following form:

$$\dot{u}(t) = L_\mu u_t + F(u_t, \mu). \tag{15}$$

According to the Riesz representation theorem, there is a function of bounded variation $\eta(\theta, \mu)$ with $\theta \in [-\tau, 0]$ such that

$$L_\mu \phi = \int_{-\tau}^{0} d\eta(\theta, \phi)\phi(\theta)$$

which can be satisfied by choosing

$$d_\eta(\theta, \varphi) = (1-\alpha)\delta(0) + a_2\delta(-\tau),$$

where δ is the Dirac delta function.

For arbitrary $\phi \in C^1([-\tau, 0], \mathrm{R})$, define

$$A(\mu)\phi = \begin{cases} \frac{d\phi}{d\theta}, & \theta \in [-\tau, 0) \\ \int_{-\tau}^{0} d\eta(\theta, \mu)\phi(\theta), & \theta = 0 \end{cases} \tag{16}$$

and

$$R(\mu)\phi = \begin{cases} 0, & \theta \in [-\tau, 0) \\ F(\mu, \phi), & \theta = 0 \end{cases} \tag{17}$$

Hence, we can rewrite Eq. (5) as

$$\dot{u}(t) = A(\mu)u_t + R(\mu)u_t, \tag{18}$$

where $u_t = u(t+\theta)$, $\theta \in [-\tau, 0]$.

For arbitrary $\psi \in C^1([0, \tau], \mathrm{R})$, the adjoint operator A^* of A is defined as

$$A^*(\mu)\psi(s) = \begin{cases} \frac{d\phi(s)}{d\theta}, & s \in (0, \tau] \\ \int_{-\tau}^{0} d\eta(s, \mu)\psi(-s), & s = 0 \end{cases}$$

For arbitrary $\phi \in C^1([-\tau, 0], \mathrm{R})$ and $\psi \in C^1([0, \tau], \mathrm{R})$, an inner product is defined as follows:

$$\langle \psi, \varphi \rangle = \bar{\psi}(0)\varphi(0) - \int_{\theta=-\tau}^{0} \int_{s=0}^{\theta} \bar{\psi}(s-\theta)d\eta(\theta)\varphi(s)ds, \tag{19}$$

where $d\eta(\theta) = d\eta(\theta, 0)$.

In order to determine the Poincare normal form of the operator $A(0)$, we need to compute the eigenvector $q(\theta)$ of $A(0)$ associated with the eigenvalue $i\omega_0$ and the eigenvector $q^*(s)$ of $A^*(0)$ associated with the eigenvalue $-i\omega_0$. We can easily get

$$q(\theta) = \exp(i\omega_0\theta), \quad \theta \in [-\tau, 0)$$

and

$$q(s) = B\exp(-iw_0 s), \quad s \in (0, \tau].$$

Now, we can prove that $\langle q^*, \bar{q} \rangle = 1$ and $\langle q^*, q \rangle = 0$. From Eq. (19), we have

$$\begin{aligned} \langle q^*, q \rangle &= \bar{q}^*(0)q(0) - \int_{\theta=-\tau}^{0} \int_{s=0}^{\theta} \bar{q}^*(s-\theta)d\eta(\theta)q(s)ds, \\ &= \bar{B} - \int_{\theta=-\tau}^{0} \int_{s=0}^{\theta} \bar{B}\exp(-i\omega_0(s-\theta))d\eta(\theta)\exp(i\omega_0\tau)ds, \\ &= \bar{B} - \bar{B}\int_{\theta=-\tau}^{0} \theta\exp(-i\omega_0\tau)d\eta(\theta). \\ &= \bar{B}(1 + a_2\tau\exp(-i\omega_0\tau)) \end{aligned}$$

So, let $B = \frac{1}{1+a_2\tau\exp(i\omega_0\tau)}$, we can obtain $\langle q^*, q \rangle = 1$.

Similarly, we need to prove that $\langle q^*, \bar{q}\rangle = 0$. Also by using Eq. (19), we get

$$\begin{aligned}\langle q^*, \bar{q}\rangle &= \bar{q}^*(0)\bar{q}(0) - \int_{\theta=-\tau}^{0}\int_{s=0}^{\theta} \bar{q}^*(s-\theta)d\eta(\theta)\bar{q}(s)ds \\ &= \bar{B} - \int_{\theta=-\tau}^{0}\int_{s=0}^{\theta} \bar{B}\exp(-i\omega_0(s-\theta))d\eta(\theta)\exp(-i\omega_0\tau)ds \\ &= \bar{B} + \frac{\bar{B}}{i2\omega_0}\int_{\theta=-\tau}^{0}(\exp(-i\omega_0\tau)-\exp(i\omega_0\tau))d\eta(\theta) \\ &= \bar{B}\left[1+\frac{a_2(\exp(-i\omega_0\tau)-\exp(i\omega_0\tau))}{i2\omega_0}\right].\end{aligned}$$

Since $A(0)q(0) = i\omega_0 q(0)$ and $A^*(0)q(0) = -i\omega_0 q^*(0)$, we get

$$(1-\alpha) + a_2\exp(-i\omega_0\tau) = i\omega_0$$

and

$$(1-\alpha) + a_2\exp(i\omega_0\tau) = -i\omega_0.$$

Hence

$$a_2(\exp(-i\omega_0\tau) - \exp(i\omega_0\tau)) = -i2\omega_0.$$

Therefore $\langle q^*, \bar{q}\rangle = 0$. This completes the proof.

In the following, let u_t be the solution of Eq. (18) at $\mu = 0$, we define

$$z(t) = \langle q^*, u_t\rangle$$

and

$$W(t,\theta) = u_t - zq - \overline{zq} = u_t - 2Re\{z(t)q(\theta)\}.$$

Then, on the manifold C_0, we have

$$W(t,\theta) = W(z(t), \bar{z}(t), \theta),$$

where

$$W(z,\bar{z},\theta) = W_{20}(\theta)\frac{z^2}{2} + W_{11}(\theta)z\bar{z} + W_{02}(\theta)\frac{\bar{z}^2}{2} + \cdots \tag{20}$$

Here z and $\bar{z}$ are local coordinates for C_0 and C in the directions of q and $\overline{q^*}$, respectively. Note that W is real if u_t is real; we deal with real solutions only. At $\mu = 0$, it is easy to get

$$\begin{aligned}\dot{z}(t) &= \langle q^*, u_t\rangle, \\ &= \langle q^*, A(0)u_t + R(0)u_t\rangle, \\ &= i\omega_0 z(t) + \bar{q}^*(0)F_0(z,\bar{z}).\end{aligned} \tag{21}$$

Equation (21) is simply written as

$$\dot{z}(t) = i\omega_0 z(t) + g(z,\bar{z}), \tag{22}$$

where

$$\begin{aligned}g(z,\bar{z}) &= \bar{q}^*(0)F_0(z,\bar{z}) \\ &= g_{20}\tfrac{z^2}{2} + g_{11}z\bar{z} + g_{02}\tfrac{\bar{z}^2}{2} + g_{21}\tfrac{z^2\bar{z}}{2} + \cdots\end{aligned} \tag{23}$$

Following the algorithms in [22], we have

$$\dot{W} = \dot{u}_t - \dot{z}q - \dot{\bar{z}}\bar{q}. \tag{24}$$

By using Eqs. (18) and (22), we obtain

$$\dot{W} = \begin{cases} AW - 2Re\{\bar{q}^*(0)F_0 q(\theta)\}, & \theta \in [-\tau, 0) \\ AW - 2Re\{\bar{q}^*(0)F_0 q(0)\} + F_0, & \theta = 0\end{cases}$$

which is rewritten as

$$\dot{W} = AW + H(z,\bar{z},\theta), \tag{25}$$

where

$$H(z,\bar{z},\theta) = H_{20}(\theta)\frac{z^2}{2} + H_{11}(\theta)z\bar{z} + H_{02}(\theta)\frac{\bar{z}^2}{2} + \cdots \tag{26}$$

On the other hand, on C_0

$$\dot{W} = W_z\dot{z} + W_{\bar{z}}\dot{\bar{z}}. \tag{27}$$

Using Eqs. (20) and (22) to replace W_z, $\dot{z}$, and their conjugates by their power series expansions, we get a second expression for $\dot{W}$:

$$\dot{W} = i\omega_0 W_{20}(\theta)z^2 - i\omega_0 W_{02}(\theta)\bar{z}^2 + \cdots \tag{28}$$

Comparing the coefficients of the above equation with those of Eq. (25), we get

$$\begin{cases}(A - i2\omega_0)W_{20}(\theta) = -H_{20}(\theta) \\ W_{11}(\theta) = -H_{11}(\theta) \\ (A + i2\omega_0)W_{02}(\theta) = -H_{02}(\theta)\end{cases} \tag{29}$$

Observing

$$\begin{aligned}u_t(\theta) &= W(z,\bar{z},\theta) + zq(\theta) + \bar{z}\cdot\bar{q}(\theta) \\ &= W_{20}(\theta)\cdot\tfrac{z^2}{2} + W_{11}(\theta)z\bar{z} + W_{02}(\theta)\cdot\tfrac{\bar{z}^2}{2} \\ &\quad + z\exp(i\omega_0\theta) + \bar{z}\exp(-i\omega_0\theta) + \cdots\end{aligned}$$

which we obtain $u_t(0)$ and $u_t(-\tau)$

$$u_t(0) = W(z,\bar{z},0) + z + \bar{z},$$

$$u_t(-\tau) = W(z,\bar{z},-\tau) + z\exp(-i\omega_0\tau) + \bar{z}\exp(i\omega_0\tau).$$

As we only need the coefficients of z^2, $z\bar{z}$, $\bar{z}^2$, and $z^2\bar{z}$, we keep these relevant terms in the following expansions:

$$\begin{aligned}u_t(0)u_t(-\tau) &= z^2\exp(-i\omega_0\tau) + z\bar{z}(\exp(i\omega_0\tau) \\ &\quad + \exp(-i\omega_0\tau)) + \bar{z}^2\exp(i\omega_0\tau) \\ &\quad + z^2\bar{z}\left[W_{11}(0)\exp(-i\omega_0\tau) + \frac{W_{20}(0)}{2}\exp(i\omega_0\tau)\right. \\ &\qquad \left. + W_{11}(-\tau) + \frac{W_{20}(-\tau)}{2}\right] + \cdots\end{aligned}$$

$$u_t^2(-\tau) = z^2\exp(-i2\omega_0\tau) + \bar{z}^2\exp(i2\omega_0\tau) + 2z\bar{z} + z^2\bar{z}\left[2\exp(-i\omega_0\tau)W_{11}(-\tau) + \exp(i\omega_0\tau)W_{20}(-\tau)\right] + \cdots$$

$$u_t(0)u_t^2(-\tau) = z^2\bar{z}(\exp(-i2\omega_0\tau) + 2) + \cdots$$

$$u_t^3(-\tau) = 3z^2\bar{z}\exp(-i\omega_0\tau) + \cdots$$

Therefore, we have

$$\begin{aligned} g(z,\bar{z}) &= z^2\bar{B}(a_4\exp(-i\omega_0\tau) + a_5\exp(-i2\omega_0\tau)) \\ &+ z\bar{z}\bar{B}\left[a_4(\exp(i\omega_0\tau) + \exp(-i\omega_0\tau)) + 2a_5\right] \\ &+ \bar{z}^2\bar{B}(a_4\exp(i\omega_0\tau) + a_5\exp(i2\omega_0\tau)) \\ &+ z^2\bar{z}\bar{B}\left[a_4W_{11}(0)\exp(-i\omega_0\tau) + \tfrac{W_{20}(0)}{2}\exp(i\omega_0\tau) \right. \\ &+ W_{11}(-\tau) + \tfrac{W_{20}(-\tau)}{2} \\ &+ a_5(2W_{11}(-\tau)\exp(-i\omega_0\tau) \\ &+ W_{20}(-\tau)\exp(i\omega_0\tau)) + a_8(\exp(-i2\omega_0\tau) + 2) \\ &\left. + 3a_9\exp(-i\omega_0\tau)\right] \end{aligned}$$

Comparing above coefficients with those in Eq. (23), we get

$$g_{20} = 2\bar{B}(a_4\exp(-i\omega_0\tau) + a_5\exp(-i2\omega_0\tau)) \tag{30}$$

$$g_{11} = \bar{B}\left[a_4(\exp(i\omega_0\tau) + \exp(-i\omega_0\tau)) + 2a_5\right] \tag{31}$$

$$g_{02} = 2\bar{B}(a_4\exp(i\omega_0\tau) + a_5\exp(i2\omega_0\tau)) \tag{32}$$

$$\begin{aligned} g_{21} &= 2\bar{B}[a_4W_{11}(0)\exp(-i\omega_0\tau) + \tfrac{W_{20}(0)}{2}\exp(i\omega_0\tau) \\ &+ W_{11}(-\tau) + \tfrac{W_{20}(-\tau)}{2} + a_5(2W_{11}(-\tau)\exp(-i\omega_0\tau) \\ &+ W_{20}(-\tau)\exp(i\omega_0\tau)) \\ &+ a_8(\exp(-i2\omega_0\tau) + 2) + 3a_9\exp(-i\omega_0\tau)] \end{aligned} \tag{33}$$

We still need to compute $W_{20}(0)$, $W_{20}(-\tau)$, $W_{11}(0)$ and $W_{11}(-\tau)$ for the expression of g_{21}. For $\theta \in [-\tau, 0)$,

$$\begin{aligned} H(z,\bar{z},\theta) &= -2Re\{\bar{q}^*(0)F_0q(\theta)\} \\ &= -2Re\{g(z,\bar{z})q(\theta)\} \\ &= -g(z,\bar{z})q(\theta) - \bar{g}(z,\bar{z})\bar{q}(\theta) \\ &= -\left(g_{20}\frac{z^2}{2} + g_{11}z\bar{z} + g_{02}\frac{\bar{z}^2}{2} + \cdots\right)q(\theta) \\ &\quad - \left(\bar{g}_{20}\frac{\bar{z}^2}{2} + \bar{g}_{11}z\bar{z} + \bar{g}_{02}\frac{z^2}{2} + \cdots\right)\bar{q}(\theta) \end{aligned}$$

Comparing the coefficients of above equation with those of Eq. (26), we obtain

$$\begin{cases} H_{20}(\theta) = -g_{20}q(\theta) - \bar{g}_{02}\bar{q}(\theta). \\ H_{11}(\theta) = -g_{11}q(\theta) - \bar{g}_{11}\bar{q}(\theta). \end{cases}$$

From Eqs. (16) and (29), we get

$$\dot{W}_{20}(\theta) = i2\omega_0W_{20}(\theta) + g_{20}q(\theta) + \bar{g}_{02}\bar{q}(\theta) \tag{34}$$

and

$$\dot{W}_{11}(\theta) = g_{11}q(\theta) + \bar{g}_{11}\bar{q}(\theta) \tag{35}$$

Solving the Eqs. (34) and (35), we have

$$\begin{aligned} W_{20}(\theta) &= -\frac{g_{20}}{i\omega_0}q(0)\exp(i\omega_0\theta) - \frac{\bar{g}_{02}}{i3\omega_0}\bar{q}(0) \\ &\quad \exp(-i\omega_0\theta) + E_1\exp(i2\omega_0\theta) \end{aligned} \tag{36}$$

and

$$\begin{aligned} W_{11}(\theta) &= \frac{g_{11}}{i\omega_0}q(0)\exp(i\omega_0\theta) \\ &\quad -\frac{\bar{g}_{11}}{i\omega_0}\bar{q}(0)\exp(-i\omega_0\theta) + E_2, \end{aligned} \tag{37}$$

where E_1 and E_2 are both constants and can be determined by setting $\theta = 0$ in $H(z,\bar{z},\theta)$. It is evident that

$$H(z,\bar{z},0) = -2Re\left\{\bar{q}^*(0)F_0q(0)\right\} + F_0$$

Thus,

$$\begin{aligned} H_{20}(0) &= -g_{20}q(0) - \bar{g}_{20}\bar{q}(0) + 2(a_4\exp(-i\omega_0\tau) \\ &\quad + a_5\exp(-i2\omega_0\tau)) \end{aligned} \tag{38}$$

$$\begin{aligned} H_{11}(0) &= -g_{11}q(0) - \bar{g}_{11}\bar{q}(0) \\ &\quad + \left[a_4(\exp(i\omega_0\tau) + \exp(-i\omega_0\tau)) + 2a_5\right] \end{aligned} \tag{39}$$

From Eq. (29) and recall that

$$AW_{20}(0) = (1-\alpha)W_{20}(0) + a_2W_{20}(-\tau),$$
$$AW_{11}(0) = (1-\alpha)W_{11}(0) + a_2W_{11}(-\tau).$$

We get

$$\begin{aligned} &(1-\alpha)W_{20}(0) + a_2W_{20}(-\tau) - i2\omega_0W_{20}(0) \\ &\quad = g_{20}q(0) + \bar{g}_{20}\bar{q}(0) - 2(a_4\exp(-i\omega_0\tau) \\ &\quad\quad + a_5\exp(-i2\omega_0\tau)) \end{aligned} \tag{40}$$

and

$$\begin{aligned} &(1-\alpha)W_{11}(0) + a_2W_{11}(-\tau) \\ &\quad = g_{11}q(0) + \bar{g}_{11}\bar{q}(0) - [a_4(\exp(i\omega_0\tau) \\ &\quad\quad + \exp(-i\omega_0\tau)) + 2a_5] \end{aligned} \tag{41}$$

Substituting Eq. (36) into Eq. (40), we obtain

$$E_1 = \frac{\Phi_1}{(1-\alpha) + a_2\exp(-i2\omega_0\tau) - i2\omega_0},$$

where

$$\begin{aligned} \Phi_1 &= ((1-\alpha) - i2\omega_0)\left(\tfrac{g_{20}}{i\omega_0} + \tfrac{\bar{g}_{02}}{i3\omega_0}\right) \\ &+ a_2(\tfrac{g_{20}}{i\omega_0}\exp(-i\omega_0\tau) + \tfrac{\bar{g}_{02}}{i3\omega_0}\exp(i\omega_0\tau)) \\ &+ g_{20} + \bar{g}_{02} - 2(a_4\exp(-i\omega_0\tau) + a_5\exp(-i2\omega_0\tau)) \end{aligned}$$

Similarly, substituting Eq. (37) into Eq. (41), weget

$$E_2 = \frac{\Phi_2}{(1-\alpha) + a_2},$$

where

$$\Phi_2 = -(1-\alpha)\left(\frac{g_{11}}{i\omega_0} - \frac{\bar{g}_{11}}{i\omega_0}\right) - a_2\left(\frac{g_{11}}{i\omega_0}\exp(i\omega_0\tau) - \frac{\bar{g}_{11}}{i\omega_0}\exp(-i\omega_0\tau)\right) + g_{11} + \bar{g}_{11} - [a_4(\exp(i\omega_0\tau) + \exp(-i\omega_0\tau)) + 2a_5].$$

Therefore, we have formulas to compute the following parameters:

$$\begin{aligned} C_1(0) &= \frac{i}{2\omega_0}\left(g_{20}g_{11} - 2|g_{11}|^2 - \frac{1}{3}|g_{02}|^2\right) + \frac{g_{21}}{2}, \\ \mu_2 &= -\frac{Re\{C_1(0)\}}{Re\lambda'(0)}, \\ T_2 &= -\frac{Im\{C_1(0)\} + \mu_2 Im\lambda'(0)}{\omega_0}, \\ \beta_2 &= 2Re\{C_1(0)\} \end{aligned} \tag{42}$$

where $\mu_2 > 0 (< 0)$ is the Lyapunov coefficient. Now we give the main results of this section.

Theorem 3 *For the controlled system (3), we summarize the above analysis:*

(i) *If $\mu_2 > 0(<0)$, the Hopf bifurcation is supercritical (subcritical), and the bifurcating periodic solutions exist for $\tau > \tau_0 (\tau < \tau_0)$, i.e., μ_2 determines the direction of the Hopf bifurcation.*
(ii) *If $\beta_2 < 0(>0)$, the bifurcating periodic solutions are stable (unstable), i.e., β_2 determines the stability of the bifurcating periodic solution.*
(iii) *If $T_2 > 0(<0)$, the period increases (decreases), i.e., T_2 determines the period of the bifurcating periodic solution.*

5 Numerical simulations

In this section, we illustrate the effectiveness of the hybrid control by numerical simulation. For comparison, we choose the same parameter as [2], i.e., $x(t) = 1/p(t)$, $c = 50$, $k = 0.01$.

First we choose $\alpha = 1$, therefore, the system is the uncontrolled model. By direct calculation we can get

$$p^* = 0.02, \quad \omega_0 = 0.5, \quad \tau_0 = 3.1416$$
$$\mu_2 = 5259.2, \quad T_2 = 2125, \quad \beta_2 = -758.38.$$

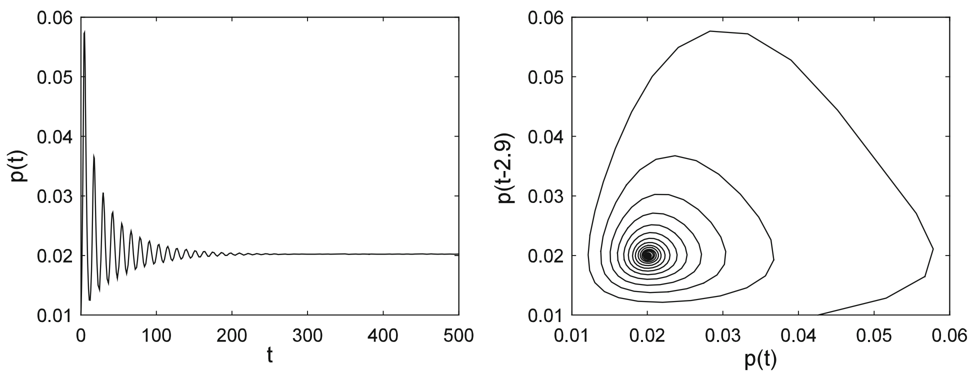

Fig. 1 Waveform plot and phase portrait of uncontrolled system (1) with $\tau = 2.9$

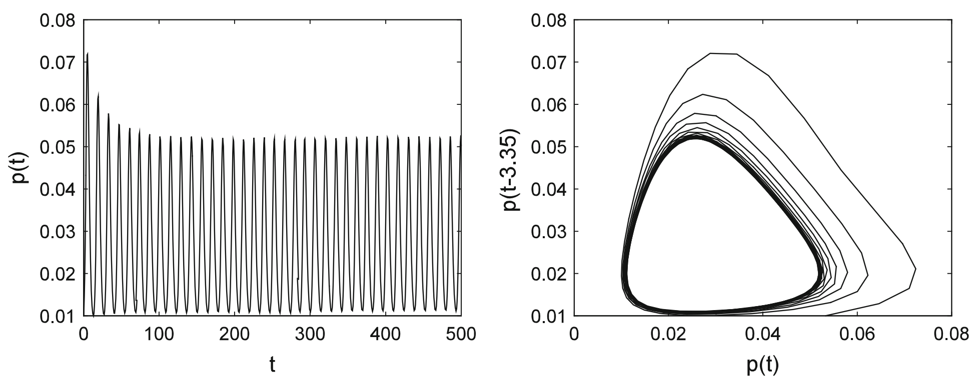

Fig. 2 Waveform plot and phase portrait of uncontrolled system (1) with $\tau = 3.35$, the equilibrium loses its stability

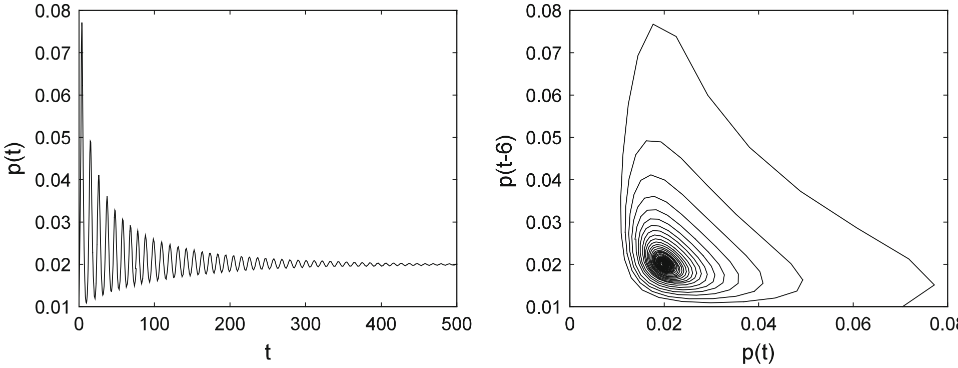

Fig. 3 Waveform plot and phase portrait of controlled system (3) with $\alpha = 1.3$ and $\tau = 3.35$, the equilibrium is asymptotically stable

Figures 1 and 2 illustrate the dynamic behaviors of the uncontrolled model (1). The simulation results are as follows:

(i) When $\tau < \tau_0$, the equilibrium point is asymptotically stable (see Fig. 1);
(ii) When τ passes through τ_0, the equilibrium point p^* loses its stability, and a Hopf bifurcation occurs (see Fig. 2);
(iii) Since $\mu_2 > 0$ and $\beta_2 < 0$, the Hopf bifurcation is supercritical, and the bifurcating periodic solutions are stable.

Now we consider the influence of the hybrid control for the Hopf bifurcation. By choosing an appropriate control parameter, we can delay the onset of Hopf bifurcation without changing the original equilibrium point. For example, when choosing $\alpha = 1.3$, the system becomes the controlled model, the critical value τ_0 increases from 3.1416 to 3.5561. By adjusting the control parameter, limit cycle in the uncontrolled system (see Fig. 2) becomes an asymptotically stable equilibrium point in the controlled system (see Fig. 3). Moreover, the bifurcation diagram of uncontrolled system and controlled system are obtained in Figs. 4 and 5, respectively. It is shown that the onset of the Hopf bifurcation is delayed, and the stable range in parameter space is extended.

When choosing $\alpha = 1.5$, for the controlled model, we obtain

$$p^* = 0.02, \quad \omega_0 = 0.5590, \quad \tau_0 = 4.1153,$$
$$\mu_2 = 11167, \quad T_2 = 2410.7, \quad \beta_2 = -476.6734.$$

It is seen from Figs. 6 and 7 that when $\tau = 3.9$, the equilibrium point of the controlled system (3) is asymptotically stable. When τ passes the critical value $\tau_0 = 4.1153$, a Hopf bifurcation occurs.

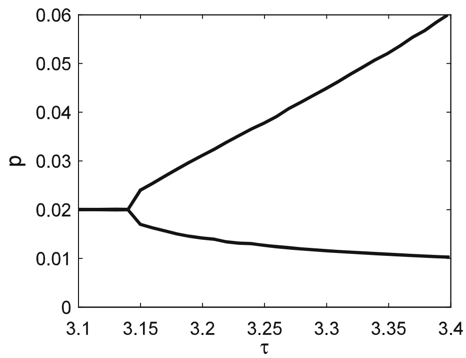

Fig. 4 Bifurcation diagram of uncontrolled system (1) with $\tau = \tau_0 = 3.14$

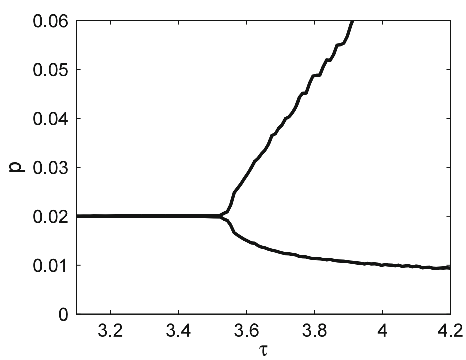

Fig. 5 Bifurcation diagram of controlled system (3) with $\alpha = 1.3$ and $\tau = \tau_0 = 3.5561$

The relationship between the critical value τ_0 and the control parameter α is shown in Fig. 8. From this Fig. 8 we know that when increasing α, the critical value increases. For example, by choosing $\alpha = 1.85$, the critical value is $\tau_0 = 7.4991$. In detail, we choose $\tau = 6$ and $\alpha = 1.85$, we can see the Hopf bifurcation is delayed from Fig. 9.

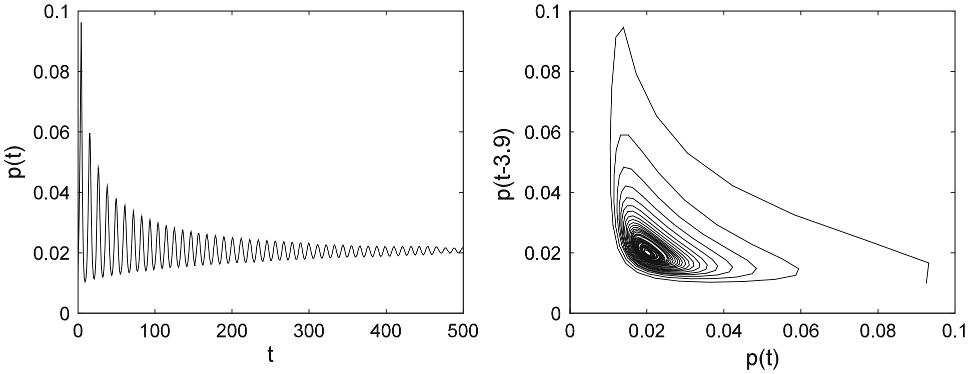

Fig. 6 Waveform plot and phase portrait of controlled system (3) with $\alpha = 1.5$ and $\tau = 3.9$

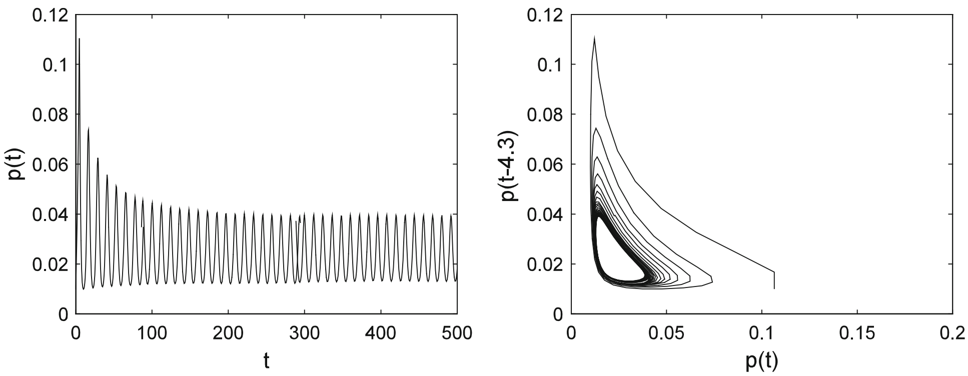

Fig. 7 Waveform plot and phase portrait of controlled system (3) with $\alpha = 1.5$ and $\tau = 4.3$

Finally we can draw the conclusion that the controlled system (3) is more stable than uncontrolled system (1). Limit cycle in the uncontrolled system becomes an asymptotically stable equilibrium point in the controlled system.

Thus, by the control strategy, first, we can increase the critical value of communication delay and delay the onset of undesirable Hopf bifurcation. Second, we extend the stable range in parameter and guarantee a stable sending rate for a larger delay. It is suitable for future high bandwidth-delay-product networks which may have a large communication delay in the networks. The proposed control method can also be used to study the higher dimensional nonlinear time-delay systems.

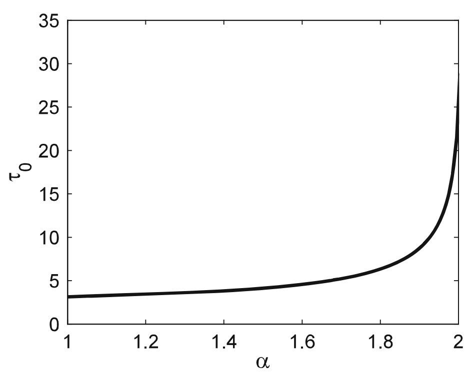

Fig. 8 The fluctuation of τ_0 depending on α

6 Conclusions

In this paper, a hybrid control strategy is used to control the Hopf bifurcation in a dual model of Internet congestion control system. By choosing an appropriate control parameter, we can delay the onset of Hopf bifurcation. It has been shown that the hybrid control can effectively control Hopf bifurcation. Furthermore, by the normal form theory and the center manifold theorem, we analyze the stability and direction of periodic solutions bifurcating. Finally, numerical simulations have demonstrated the correctness of theoretical analysis.

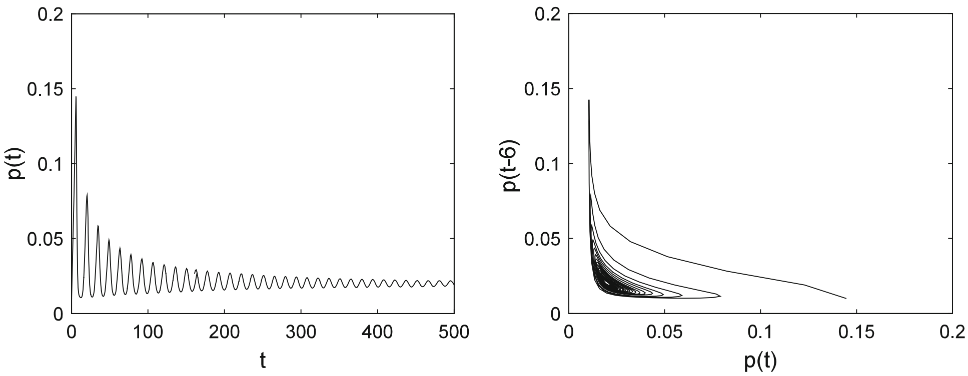

Fig. 9 Waveform plot and phase portrait of controlled system (3) with $\alpha = 1.85$ and $\tau = 6$

Acknowledgments This paper was supported by NSFC-Guangdong Joint Fund (No:U1201255), the National Natural Science Foundation of China (No:61201227 and No:61172127), the Natural Science Foundation of Anhui (No:1208085MF93) (No:1208085MF93), 211 Innovation Team of Anhui University (No: KJTD007A and No:KJTD001B), Young-backbone Teacher Project of Anhui University, The Doctoral Scientific Research Foundation of Anhui University.

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