Nonlinear Dyn (2014) 76:785–795
DOI 10.1007/s11071-013-1169-0

ORIGINAL PAPER

Lyapunov-based fractional-order controller design to synchronize a class of fractional-order chaotic systems

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Received: 28 April 2013 / Accepted: 27 November 2013 / Published online: 10 December 2013

Abstract In this paper, a novel adaptive fractional-order feedback controller is first developed by extending an adaptive integer-order feedback controller. Then a simple but practical method to synchronize almost all familiar fractional-order chaotic systems has been put forward. Through rigorous theoretical proof by means of the Lyapunov stability theorem and Barbalat lemma, sufficient conditions are derived to guarantee chaos synchronization. A wide range of fractional-order chaotic systems, including the commensurate system and incommensurate case, autonomous system, and nonautonomous case, is just the novelty of this technique. The feasibility and validity of presented scheme have been illustrated by numerical simulations of the fractional-order Chen system, fractional-order hyperchaotic Lü system, and fractional-order Duffing system.

Keywords Fractional-order chaotic system · Adaptive fractional-order feedback · Commensurate and incommensurate system · Barbalat lemma

1 Introduction

The concept of fractional calculus was proposed by Leibniz more than 300 years ago. For a long time, it was unexplored due to its inherent complexity and the fact that it does not have an acceptable geometrical or physical interpretation [1–4]. Only in recent years, it begins to attract more and more attention of physicists and engineers. This is a result of better understanding of the fractional calculus revealed by problems such as viscoelasticity [5, 6], dielectric polarization [7], electrode-electrolyte polarization [8], electromagnetic waves [9], quantitative finance [10], and quantum evolution of complex systems [11]. More recently, many researchers show growing interest in chaotic behavior of fractional-order dynamical systems. Up to now, it has been shown that some fractional-order dynamical systems can display chaotic or hyperchaotic behaviors, such as the fractional-order Lorenz families system [12–14], fractional-order Rössler system [15], fractional-order Liu system [16], fractional-order Chua's system [17], fractional-order Duffing system [18], and so on.

Chaos control and synchronization are two important ways to utilize chaos in practice. The synchronization of chaotic fractional-order system has attracted great attention due to its potential application in secure communication. So far, there exist many methods to realize synchronization for fractional-order chaotic systems, which include the linear feedback method [19–22], nonlinear feedback method [23, 24], unidirectional coupling and bidirectional coupling [25–28], PC method [25, 26, 29], PAD method [25], active control [30, 31], sliding mode control [31, 32] and state observer [27, 30]. However, despite the large

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amount of effort, there are still some problems to be overcome, which can be summarized as the following four points: (1) Firstly, the control methods in most literatures [19–21, 25–31] just focus on a specific dynamical system, not giving a general approach to attain chaos synchronization for fractional-order dynamical systems. (2) Secondly, the control approaches mentioned above can achieve chaos synchronization successfully. But these methods have some defects to a certain degree. For example, the linear feedback method [19–22] is simple and practical, but it is difficult to get the suitable feedback coefficient. Then numerical calculation has to be used, which lead to this technique can only be applied to particular models. There is a similar problem for the unidirectional coupling and bidirectional coupling controller [25–28]. The sliding mode technique [31, 32] is effective in many cases, however, chattering phenomenon is the inherent vice. PC and PAD techniques [25, 26, 29] have high requirements on system structure. (3) Thirdly, the control methods in literatures [23, 24, 30, 31] are none of the above problems, but the structures of controllers are complex. (4) Finally, most synchronization methods [19–31] are applied to autonomous fractional-order chaotic systems but for the nonautonomous case mentioned less. Motivated by the above discussions, in this paper, we will try to find a simple, efficient and practical approach to achieve chaos synchronization for almost all familiar fractional-order chaotic systems.

Adaptive feedback control [33–35] is a mature and effective method to synchronize the integer-order chaotic system. Compared with the linear feedback method, one of the advantages of this technique is that feedback coefficients do not need to know in advance. Hence, we will design a new adaptive feedback controller to realize chaos synchronization for the fractional-order chaotic system. The remainder of this paper is organized as follows. In Sect. 2, some preliminary definitions and numerical computational methods about fractional-order system are introduced. Section 3 presents a new adaptive fractional-order feedback control law to synchronize a commensurate fractional-order chaotic system. Furthermore, Sect. 4 provides numerical simulations for some typical fractional-order chaotic systems. Then the proposed control approach has been generalized for the incommensurate fractional-order chaotic system in Sect. 5. Finally, the concluding remarks are stated in Sect. 6.

2 Preliminaries and notations

2.1 Definitions of fractional derivative

A fractional-order derivative can be considered as a generalization of an integer-order derivative. There are several definitions for fractional derivative of order $\alpha > 0$ [1–4]. The two most commonly used are Riemann–Liouville and Caputo definitions. Each definition employs the Riemann–Liouville fractional integration and derivatives of the whole order. The difference between two definitions is in the order of evaluation. Since the Riemann–Liouville derivative is a continuous operator on fractional order α, it can bridge all the gaps among integer derivatives and integrals [36]. Consequently, we will choose the Riemann–Liouville derivative through this paper. Hereafter, one will use the notation D^α to denote the Riemann–Liouville fractional derivative operator.

Definition 1 [1–4] The Riemann–Liouville fractional integral operator with order $\alpha > 0$ for a continuous function $f : R^+ \to R$ is defined as follows:

$$J^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau)\, d\tau, \qquad (1)$$

Definition 2 [1–4] The Riemann–Liouville fractional derivative operator with order $\alpha > 0$ for a continuous function $f : R^+ \to R$ is defined as follows:

$$\begin{aligned} D^\alpha f(t) &= D^m J^{m-\alpha} f(t) \\ &= \frac{1}{\Gamma(m-\alpha)} \frac{d^m}{dt^m} \int_0^t \frac{f(\tau)}{(t-\tau)^{\alpha-m+1}}\, d\tau, \end{aligned} \qquad (2)$$

where $\Gamma(\cdot)$ is the Gamma function, and m is the first integer greater than α, that is, $m-1 < \alpha < m$.

2.2 Approximate calculation method

The numerical calculation of a fractional differential equation is not simple as that of an ordinary differential equation. In literatures of the fractional dynamics research field, two methods have been proposed for a numerical solution of a fractional-order differential equation. One is the frequency-domain approach [37], another is the time-domain approach [38]. The first approximation method is simple and convenient, which has been adopted in [13–15, 18, 19]. However, as the extreme sensitivity to initial conditions of chaotic behavior, this technique is not reliable in the study of a fractional-order chaotic system, which has been verified in [39]. This paper will adopt the generalized Adams–Bashforth–Moulton scheme [40]. It directly derives the analytic expression of fractional differential equation then numerically iterates the formula. Hence, it is superlinearly convergent at least. With good numerical stability, this method has been used on the study of chaotic behavior for fractional-order systems [20, 22, 23, 25, 27–31]. Recently, Deng [41] has proposed an improved predictor-corrector approach in which the numerical approximation is more accurate and the computational cost is largely reduced.

To give the approximate solution of nonlinear fractional-order differential equations by means of this algorithm, consider the following differential equation:

$$D^{\alpha}x(t) = f\big(t, x(t)\big), \quad 0 \le t \le T \tag{3}$$

and

$$x^{(k)}(0) = x_0^{(k)}, \quad k = 0, \dots, m-1. \tag{4}$$

It is equivalent to the Volterra integral equation:

$$x(t) = \sum_{k=0}^{m-1} \frac{t^k}{k!} x_0^{(k)} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f\big(s, x(s)\big)\, ds. \tag{5}$$

Set $h = T/N$, $t_n = nh$, $n = 0, 1, \dots, N \in Z^+$. Equation (5) can be discretized as follows:

$$x_h(t_{n+1}) = \sum_{k=0}^{m-1} \frac{t_{n+1}^k}{k!} x_0^{(k)} + \frac{h^{\alpha}}{\Gamma(\alpha+2)} \Bigg[f\big(t_{n+1}, x_h^p(t_{n+1})\big) + \sum_{j=0}^{n} a_{j,n+1} f\big(t_j, x_h(t_j)\big) \Bigg], \tag{6}$$

where predicted value $x_h^p(t_{n+1})$ is determined by

$$x_h^p(t_{n+1}) = \sum_{k=0}^{m-1} \frac{t_{n+1}^k}{k!} x_0^{(k)} + \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{n} b_{j,n+1} f\big(t_j, x_h(t_j)\big), \tag{7}$$

in which

$$a_{j,n+1} = \begin{cases} n^{\alpha+1} - (n-\alpha)(n+1)^{\alpha}, & j = 0, \\ (n-j+2)^{\alpha+1} + (n-j)^{\alpha+1} - 2(n-j+1)^{\alpha+1}, & 1 \le j \le n, \\ 1, & j = n+1, \end{cases} \tag{8}$$

$$b_{j,n+1} = \frac{h^{\alpha}}{\alpha} \big((n-j+1)^{\alpha} - (n-j)^{\alpha} \big). \tag{9}$$

Therefore, the estimation error of this approximation is

$$\max_{j=0,1,\dots,N} \big| x(t_j) - x_h(t_j) \big| = O\big(h^{\alpha}\big), \quad p = \min(2, 1+\alpha). \tag{10}$$

Remark 1 $\|\cdot\|_{\infty}$, $\|\cdot\|_1$, and $\|\cdot\|_2$, respectively, denote the ∞-norm, 1-norm, and 2-norm of the vector, which is defined as follows:

$$\|x\|_{\infty} = \max_{1 \le i \le n} |x_i|, \qquad \|x\|_1 = \sum_{i=1}^{n} |x_i|, \qquad \|x\|_2 = \left(\sum_{i=1}^{n} x_i^2 \right)^{1/2}, \quad \forall x \in R^n. \tag{11}$$

3 Adaptive synchronization scheme for a commensurate fractional-order chaotic system

Consider a commensurate fractional-order system with a general form, which is usually called a drive system

$$D^{\alpha}x(t) = f(x, t), \tag{12}$$

and a controlled system named response system

$$D^{\alpha}y(t) = f(y, t) + U(x, y), \tag{13}$$

where $0 < \alpha < 1$. $x \in R^n$, $y \in R^n$ are state vectors and Ω is a domain containing the origin. The function $f(\cdot, t) : \Omega \subset R^n \times R^+ \to R^n$ is a nonlinear chaotic vector function and $U(x, y)$ is an unknown vector controller. To study the chaos synchronization, one can define the error signal as $e(t) = y(t) - x(t)$, then error system can be obtained as follows:

$$D^{\alpha}e(t) = f(y, t) - f(x, t) + U(x, y). \tag{14}$$

Assumption 1 For any $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_n)$, nonlinear function $f(\cdot, t)$ satisfies the following inequality:

$$\begin{aligned}&\left|f_i(x,t)-f_i(y,t)\right|\\&\quad\le l\max_{1\le j\le n}|x_j-y_j|,\quad i=1,2,\dots,n,\\&\quad\forall(x,y)\in\Omega,\end{aligned}\tag{15}$$

where $l>0$ is a constant.

Remark 2 Such condition may be called local Lipschitz condition, which can ensure the existence and uniqueness of fractional-order dynamical system [42]. Most fractional-order chaotic systems [12–18] satisfy the above condition.

Theorem 1 *For the drive system (12) and response system (13), if the controller is given by*

$$u_i(t)=k_ie_i=k_i(y_i-x_i),\tag{16}$$

and adaptive control parameters update according to the following laws:

$$D^\alpha k_i(t)=-\gamma_ie_i^2=-\gamma_i(y_i-x_i)^2,\tag{17}$$

where $\gamma_i>0$, $(i=1,\dots,n)$ are arbitrary constants. Then the chaos synchronization for two commensurate fractional-order chaotic systems can be achieved.

Assumption 2 Suppose that vector function $k(e,t)$ meets the following condition:

$$m_i\le k_i(e,t)\le M_i,\quad i=1,2,\dots,n.\tag{18}$$

Remark 3 As the state variables of chaotic system are bounded, the assumption is reasonable.

Lemma 1 (Barbalat lemma [43]) *If $\omega(t):R\to R^+$ is a uniformly positive function for $t\ge 0$and if the integral $\lim_{t\to\infty}\int_0^t\omega(\tau)\,d\tau$ exists and is finite, then* $\lim_{t\to\infty}\omega(t)=0$.

Proof Applying the controller (16) to error system (14) results in the following new system:

$$D^\alpha e_i=D^\alpha y_i-D^\alpha x_i=f_i(y,t)-f_i(x,t)+k_ie_i,\tag{19}$$

Now, we introduce the following Lyapunov function:

$$V(t,e)=\frac{1}{2}\sum_{i=1}^n\left(J_0^{1-\alpha}e_i\right)^2+\frac{L_1}{2}\sum_{i=1}^n\frac{1}{\gamma_i}\left(J_0^{1-\alpha}k_i\right)^2,\tag{20}$$

where $L_1>\frac{(M+l)n}{m}$, $M=\max_{1\le i\le n}\{|M_i|\}$, $m=\min_{1\le i\le n}\{|m_i|\}$. The derivative of Valong the trajectories of error dynamic (19) is

$$\begin{aligned}\dot V&=\sum_{i=1}^n\left(J_0^{1-\alpha}e_i\right)\left(D_0^\alpha e_i\right)+L_1\sum_{i=1}^n\frac{1}{\gamma_i}\left(J_0^{1-\alpha}k_i\right)\left(D_0^\alpha k_i\right)\\
&=\frac{1}{\Gamma(1-\alpha)}\left\{\sum_{i=1}^n\left(\int_0^t(t-\tau)^{-\alpha}e_i(\tau)\,d\tau\right)\right.\\
&\quad\times\left(f_i(y)-f_i(x)+k_ie_i\right)\\
&\quad\left.-L_1\sum_{i=1}^n\left(\int_0^t(t-\tau)^{-\alpha}k_i(\tau)\,d\tau\right)e_i^2\right\}\\
&\le\frac{1}{\Gamma(1-\alpha)}\left\{\sum_{i=1}^n\left(\int_0^t(t-\tau)^{-\alpha}\left|e_i(\tau)\right|d\tau\right)\right.\\
&\quad\times\left|\left(f_i(y)-f_i(x)\right)+k_ie_i\right|\\
&\quad\left.-L_1\sum_{i=1}^n\left(\int_0^t(t-\tau)^{-\alpha}k_i(\tau)\,d\tau\right)e_i^2\right\}\\
&\le\frac{1}{\Gamma(1-\alpha)}\left\{\sum_{i=1}^n\|e\|_\infty\left(\int_0^t(t-\tau)^{-\alpha}\,d\tau\right)\right.\\
&\quad\times\left|\left(f_i(y)-f_i(x)\right)+k_ie_i\right|\\
&\quad\left.-L_1\sum_{i=1}^n m_i\left(\int_0^t(t-\tau)^{-\alpha}\,d\tau\right)e_i^2\right\}\\
&\le\frac{t^{1-\alpha}}{\Gamma(2-\alpha)}\left[\sum_{i=1}^n\|e\|_\infty\left|f_i(y)-f_i(x)\right|\right.\\
&\quad\left.+\sum_{i=1}^n\|e\|_\infty|k_ie_i|-L_1\sum_{i=1}^n m_ie_i^2\right]\\
&\le\frac{t^{1-\alpha}}{\Gamma(2-\alpha)}\left[l\|e\|_\infty\sum_{i=1}^n\max_{1\le j\le n}|y_j-x_j|\right.\\
&\quad\left.+M\|e\|_\infty\sum_{i=1}^n|e_i|-L_1m\sum_{i=1}^n e_i^2\right]\\
&=\frac{t^{1-\alpha}}{\Gamma(2-\alpha)}\left[nl\|e\|_\infty^2+M\|e\|_\infty\|e\|_1-L_1m\|e\|_2^2\right]\\
&\le\frac{t^{1-\alpha}}{\Gamma(2-\alpha)}\left[nl\|e\|_\infty^2+Mn\|e\|_\infty^2-L_1m\|e\|_\infty^2\right]\\
&=\frac{t^{1-\alpha}}{\Gamma(2-\alpha)}\left[(M+l)n-L_1m\right]\|e\|_\infty^2\le 0,\\
&\le\frac{t^{1-\alpha}}{\Gamma(2-\alpha)}\left[(M+l)n-L_1m\right]\|e\|_2^2\\
&=-e^TP(t)e=-\omega(t)\le 0,\end{aligned}\tag{21}$$

where $P(t)=\frac{t^{1-\alpha}}{\Gamma(2-\alpha)}[L_1 m-(M+l)n]I$ is a positive definite matrix. Now integrating (21) from zero to t yields

$$V(t)+\int_0^t \omega(\tau)\,d\tau=V(0) \tag{22}$$

and the above equation means that

$$\int_0^t \omega(\tau)\,d\tau\le V(0) \tag{23}$$

since $V\ge 0$. As t approach to infinite, the above integral is always less than or equal to $V(0)$, so $\lim_{t\to\infty}\int_0^t \omega(\tau)\,d\tau$ exists and is finite. Based on Lemma 1, one has

$$\lim_{t\to\infty}\omega(t)=\lim_{t\to\infty}e^T P(t)e=0 \quad\Leftrightarrow\quad \lim_{t\to\infty}e(t)=0. \tag{24}$$

Then the state trajectories of error system (19) will converge to zero point. That is, the chaos synchronization between drive system (12) and response system (13) has been realized. □

4 Numerical simulations

In this section, the performance of the above proposed scheme will be studied through three typical fractional-order chaotic systems. As mentioned in Sect. 2.2, one has implemented the improved Adams–Bashforth–Moulton algorithm for numerical simulation in FORTRAN, in which the step-size is fixed as $h=0.01$.

4.1 Fractional-order Chen system

The fractional-order Chen system [26] is described by

$$\begin{aligned}\frac{d^\alpha x_1}{dt^\alpha}&=a(x_2-x_1),\\ \frac{d^\alpha x_2}{dt^\alpha}&=(c-a)x_1+cx_2-x_1x_3,\\ \frac{d^\alpha x_3}{dt^\alpha}&=-bx_3+x_1x_2,\end{aligned} \tag{25}$$

where system parameters $a,b,c>0$. Set $a=35$, $b=3$, $c=28$, $\alpha=0.96$ fractional-order Chen system (25) has a chaotic attractor as depicted in Fig. 1, where final time is $t=100$ s. The initial values of the drive system, response system, and adaptive control parameters are selected as $(-1,5,2,6,-4,7,1,2,3)$ and $\gamma_i=-1, (i=1,2,3)$. The synchronization errors between the drive system and response system are displayed in Fig. 2, which show the error vector (e_1,e_2,e_3) converges to zero quickly with the control law (16).

4.2 Fractional-order hyperchaotic Lü system

The fractional-order hyperchaotic Lü system [44] is described by

$$\begin{aligned}\frac{d^\alpha x_1}{dt^\alpha}&=a(x_2-x_1)+x_4,\\ \frac{d^\alpha x_2}{dt^\alpha}&=-x_1x_3+bx_2,\end{aligned} \tag{26}$$

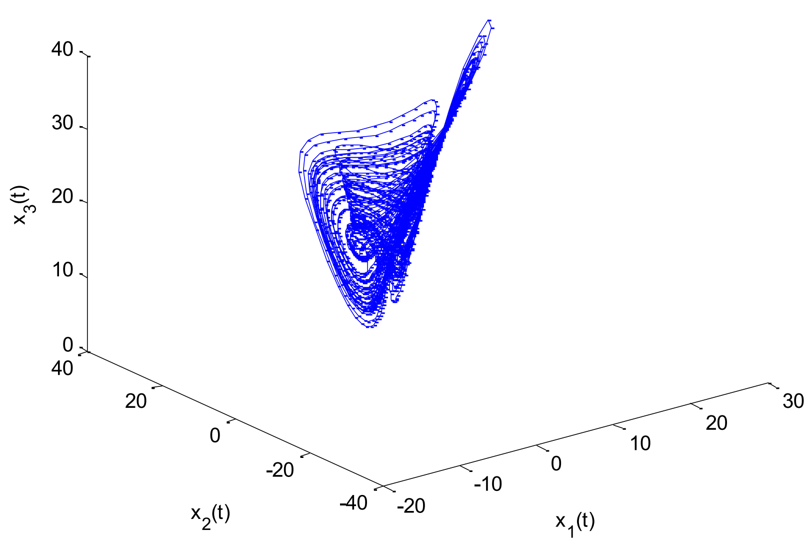


Fig. 1 Chaotic attractor of the commensurate fractional-order Chen system with order $\alpha=0.96$ and $a=35$, $b=3$, $c=28$

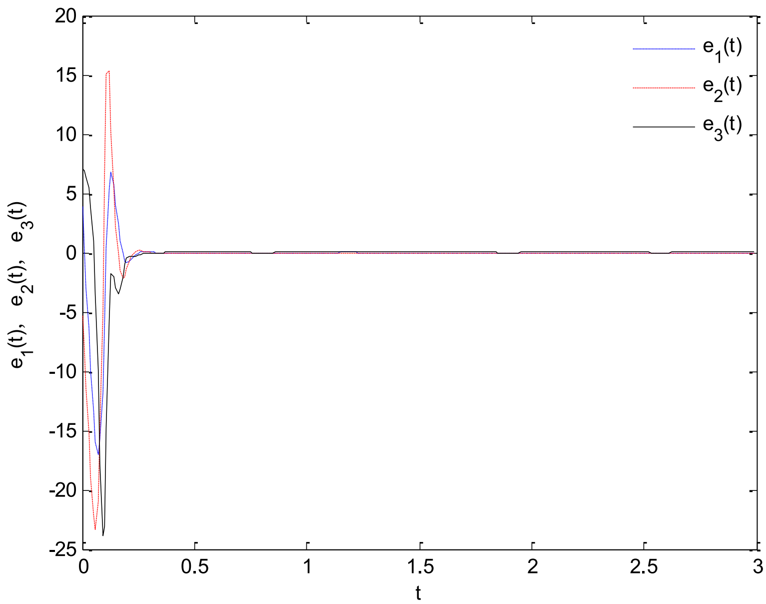


Fig. 2 Time history of synchronization error for the commensurate fractional-order chaotic Chen system with order $\alpha = 0.96$

$$\frac{d^\alpha x_3}{dt^\alpha} = x_1 x_2 - c x_3,$$
$$\frac{d^\alpha x_4}{dt^\alpha} = x_1 x_3 - d x_2,$$

where system parameters $a, b, c, d > 0$. Set $a = 36$, $b = 20$, $c = 3$, $d = 1$, $\alpha = 0.95$, fractional-order Lü system (26) has a hyperchaotic attractor as depicted in Fig. 3, where final time is $t = 60$ s. The initial values of the drive system, response system, and adaptive parameters are selected as $(-15, -18, 9, 10, -13, -21, 14, 18, -5, 7, -6, 2)$ and $\gamma_i = -1, (i = 1, \ldots, 4)$. The synchronization errors between drive system and response system are displayed in Fig. 4, which show the error vector (e_1, e_2, e_3, e_4) converge to zero rapidly with the control law (16).

4.3 Fractional-order Duffing system

The fractional-order Duffing system [45] is described by

$$\frac{d^\alpha x_1}{dt^\alpha} = x_2,$$
$$\frac{d^\alpha x_2}{dt^\alpha} = x_1 - \beta x_1^3 - \delta x_2 + f\cos(\omega t), \tag{27}$$

where system parameters $\beta, \delta, f, \omega > 0$. Set $\beta = 1$, $\delta = 0.15$, $f = 0.3$, $\omega = 1.0$, $\alpha = 0.97$, fractional-order Duffing system (27) has a chaotic attractor as depicted in Fig. 5, where final time is $t = 300$ s. The initial values of drive system, response system and adaptive parameters are selected as $(0.5, -0.5, -2, 3.5, 2.5, -4)$ and $\gamma_i = -1, (i = 1, 2)$. The synchronization errors between drive system and response system are displayed in Fig. 6, which show the error vector (e_1, e_2) converge to zero fleetly with the control law (16).

5 Adaptive synchronization scheme for incommensurate fractional-order chaotic system

Since modeling the actual dynamical system with incommensurate fractional-order system is more reasonable than the commensurate case, we will discuss chaos synchronization for the incommensurate fractional-order chaotic system in this section.

Consider the incommensurate fractional-order system with a general form, which is called a drive system

$$D^{\bar{\alpha}} x(t) = f(x, t), \tag{28}$$

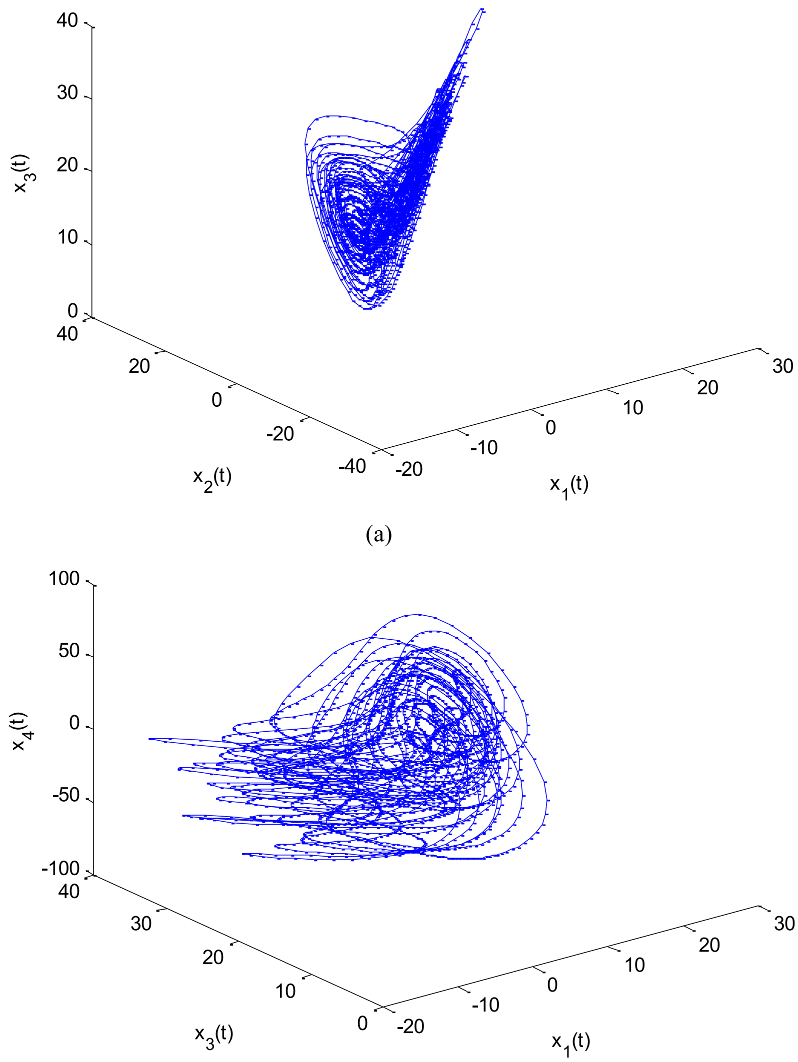
and a controlled system named the response system

$$D^{\bar{\alpha}} y(t) = f(y, t) + U(x, y), \tag{29}$$

where $\bar{\alpha} = [\alpha_1, \alpha_2, \ldots, \alpha_n]$, $0 < \alpha_i < 1 (i = 1, \ldots, n)$.

Theorem 2 *For the drive system* (28) *and response system* (29), *if the controller is given by*

$$u_i = k_i e_i = k_i (y_i - x_i), \tag{30}$$



(b)

Fig. 3 Hyper-chaotic attractor of the commensurate fractional-order Lü system in different projection coordinate surface with order $\alpha = 0.95$ and $a = 36$, $b = 20$, $c = 3$, $d = 1$. (**a**) $x_1 - x_2 - x_3$; (**b**) $x_1 - x_3 - x_4$

and adaptive control parameters update according to the following laws:

$$D^{\alpha_i} k_i(t) = -\gamma_i e_i^2 = -\gamma_i (y_i - x_i)^2, \tag{31}$$

where $\gamma_i > 0$ $(i = 1, \dots, n)$ *are arbitrary constants. Then the chaos synchronization for two incommensurate fractional-order chaotic systems can be achieved.*

Similarly, we introduce the following Lyapunov function:

$$V(t, e) = \frac{1}{2} \sum_{i=1}^{n} \left(J_0^{1-\alpha_i} e_i\right)^2 + \frac{L_1}{2} \sum_{i=1}^{n} \frac{1}{\gamma_i} \left(J_0^{1-\alpha_i} k_i\right)^2, \tag{32}$$

where L_1, M, m are the same as in Sect. 3. Taking the time derivative of V, one has

$$\begin{aligned}
\dot{V} &= \sum_{i=1}^{n} \left(J_0^{1-\alpha_i} e_i\right)\left(D_0^{\alpha_i} e_i\right) \\
&\quad + L_1 \sum_{i=1}^{n} \frac{1}{\gamma_i} \left(J_0^{1-\alpha_i} k_i\right)\left(D_0^{\alpha_i} k_i\right) \\
&= \sum_{i=1}^{n} \frac{1}{\Gamma(1-\alpha_i)} \int_0^t (t-\tau)^{-\alpha_i} e_i(\tau)\, d\tau \\
&\quad \times \left(f_i(y) - f_i(x) + k_i e_i\right) \\
&\quad - \sum_{i=1}^{n} \frac{L_1 e_i^2}{\Gamma(1-\alpha_i)} \int_0^t (t-\tau)^{-\alpha_i} k_i(\tau)\, d\tau
\end{aligned}$$

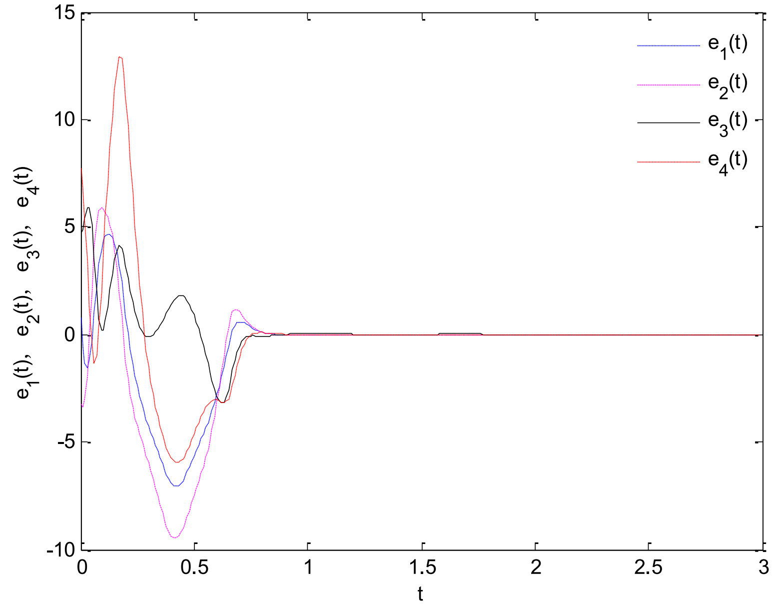


Fig. 4 Time history of synchronization error for the commensurate fractional-order hyperchaotic Lü system with order $\alpha = 0.95$

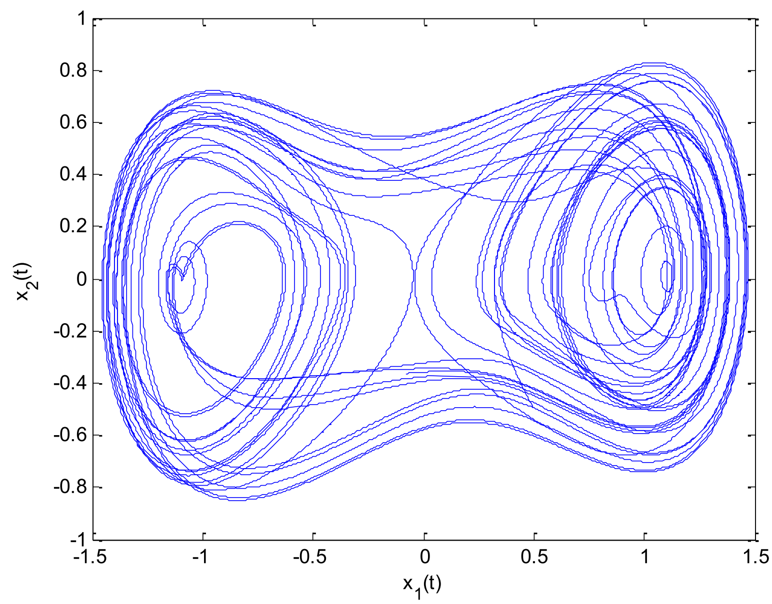


Fig. 5 Chaotic attractor of the commensurate fractional-order Duffing system with order $\alpha = 0.97$ and $\beta = 1.0$, $\delta = 0.15$, $f = 0.3$, $\omega = 1.0$

$$
\begin{aligned}
&\le \sum_{i=1}^{n} \frac{1}{\Gamma(1-\alpha_i)} \int_0^t (t-\tau)^{-\alpha_i} \left| e_i(\tau) \right| d\tau \\
&\quad \times \left| \big(f_i(y) - f_i(x)\big) + k_i e_i \right| \\
&\quad - \sum_{i=1}^{n} \frac{L_1 e_i^2}{\Gamma(1-\alpha_i)} \int_0^t (t-\tau)^{-\alpha_i} k_i(\tau)\, d\tau \\
&\le \sum_{i=1}^{n} \frac{\|e\|_\infty}{\Gamma(1-\alpha_i)} \int_0^t (t-\tau)^{-\alpha_i}\, d\tau \\
&\quad \times \left| \big(f_i(y) - f_i(x)\big) + k_i e_i \right| \\
&\quad - L_1 \sum_{i=1}^{n} \frac{m_i e_i^2}{\Gamma(1-\alpha_i)} \int_0^t (t-\tau)^{-\alpha_i}\, d\tau \\
&= \sum_{i=1}^{n} \frac{t^{1-\alpha_i} \|e\|_\infty}{\Gamma(2-\alpha_i)} \left| \big(f_i(y) - f_i(x)\big) + k_i e_i \right| \\
&\quad - L_1 \sum_{i=1}^{n} \frac{m_i t^{1-\alpha_i}}{\Gamma(1-\alpha_i)} e_i^2.
\end{aligned}
\tag{33}
$$

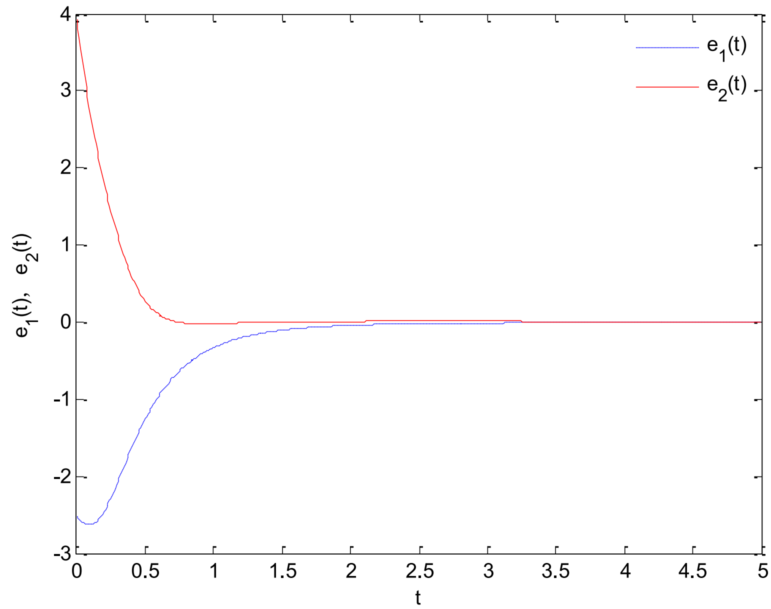


Fig. 6 Time history of synchronization error for the commensurate fractional-order chaotic Duffing system with order $\alpha = 0.97$

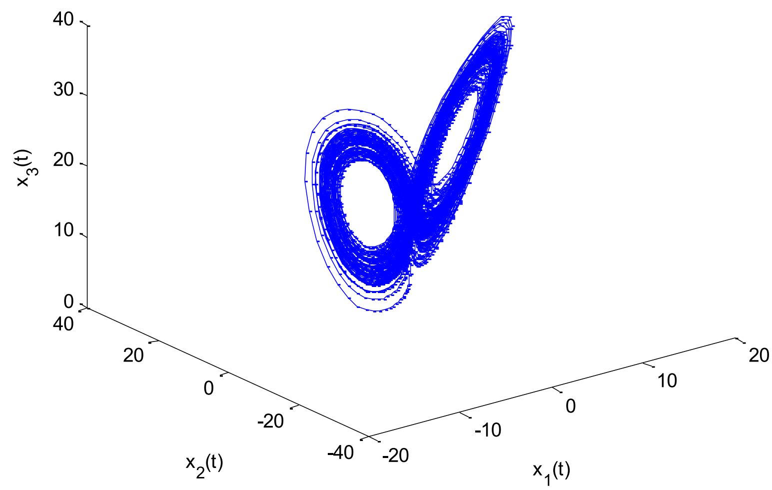


Fig. 7 Chaotic attractor of the incommensurate fractional-order Chen system with order $(\alpha_1, \alpha_2, \alpha_3) = (0.8, 1.0, 0.9)$ and $(a, b, c) = (35, 3, 28)$

For sufficiently large t, $\frac{t^{1-\alpha_i}}{\Gamma(2-\alpha_i)} \leq \frac{t^{1-\underline{\alpha}}}{\Gamma(2-\bar{\alpha})}$, where $\underline{\alpha} = \min_{1\leq i\leq n}\{\alpha_i\}$, $\bar{\alpha} = \max_{1\leq i\leq n}\{\alpha_i\}$. Thus, the following inequality can be obtained:

$$\dot{V} \leq \frac{t^{1-\underline{\alpha}}}{\Gamma(2-\bar{\alpha})}\big[(M+l)n - L_1 m\big]\|e\|_\infty^2 \leq 0. \qquad (34)$$

Then, by derivation resembling in Sect. 3, we can deduce that the state trajectories of the error system will converge to the zero point. That is, chaos synchronization between the drive system (28) and response system (29) has been realized. Below the incommensurate fractional-order Chen system will be taken as an example to show the effectiveness of the proposed method.

Consider the incommensurate fractional-order Chen system [46] as follows:

$$\begin{aligned} \frac{d^{\alpha_1}x_1}{dt^{\alpha_1}} &= a(x_2 - x_1), \\ \frac{d^{\alpha_2}x_2}{dt^{\alpha_2}} &= (c-a)x_1 + cx_2 - x_1x_3, \\ \frac{d^{\alpha_3}x_3}{dt^{\alpha_3}} &= -bx_3 + x_1x_2, \end{aligned} \qquad (35)$$

where system parameters values are the same as Sect. 4.1. and $(\alpha_1, \alpha_2, \alpha_3) = (0.8, 1.0, 0.9)$, then the incommensurate fractional-order Chen system (35) also has a chaotic attractor, which is depicted in Fig. 7,

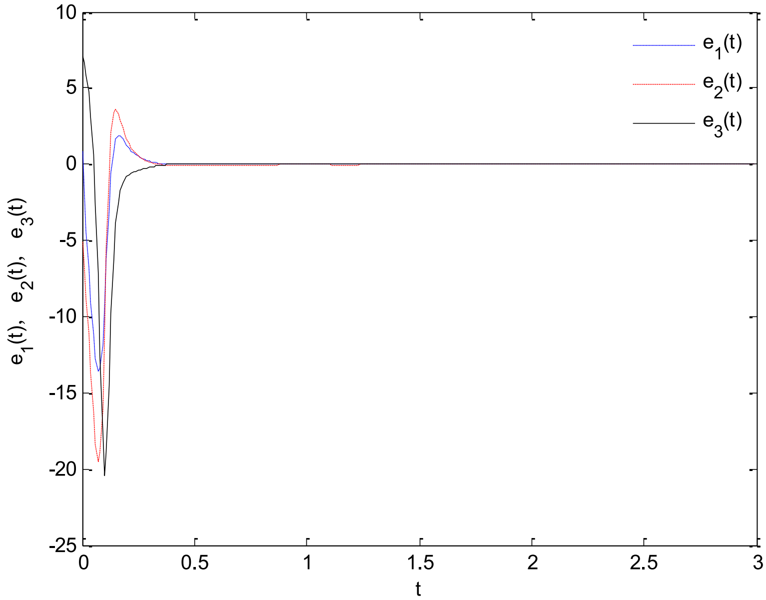


Fig. 8 Time history of synchronization error for the incommensurate fractional-order chaotic Chen system with order $(\alpha_1, \alpha_2, \alpha_3) = (0.8, 1.0, 0.9)$

in which final time is $t = 100$ s. And the synchronization errors between the drive system and response system are displayed in Fig. 8, which show the error vector (e_1, e_2, e_3) also can converge to zero quickly with the control law (30).

6 Conclusions

This work is concerned with chaos synchronization of two fractional-order chaotic systems. According to the Lyapunov stability theorem and Barbalat lemma, a novel adaptive fractional-order feedback control law is given to achieve chaos synchronization. In comparison with existing methods, the proposed scheme supplies a simple and uniform controller to synchronize almost all familiar fractional-order chaotic systems with a very loose condition. Three groups of numerical examples are provided to show the effectiveness of developed methods. The technique is easy to implement in practice, so we have reason to believe that such a simple synchronization method will be very beneficial for applications of chaos synchronization. In the future, this new approach will be extended to achieve other types of chaos synchronization for a fractional-order chaotic system, such as projective synchronization, lag synchronization, antisynchronization, and so on.

Acknowledgements The present work is supported by National Natural Science Foundation of China (No. 11202155). And the authors would like to thank the editor and the reviewers for their constructive comments and suggestions.

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