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Finite-time optimal formation tracking control of vehicles in horizontal plane

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Abstract This paper studies the problem of finite-time optimal formation tracking for planar vehicles which are considered as rigid bodies, under the condition that the tracking time is given according to task requirements in advance. By using Pontryagin's maximum principle (PMP) on a Lie group, an optimal control law is designed for vehicles with holonomic dynamics to track a desired reference trajectory at the given tracking time in the manner of rigid formation which is also specified by task requirements. Simultaneously, a corresponding cost function is considered and guaranteed to be optimal. Then, the above mentioned result of tracking is extended to the case of multi-vehicle systems with a directed-tree communication topology. Furthermore, some conditions are proposed to ensure the adjoint orbits of vehicles to be non-holonomic. Finally, the numerical simulations are provided to illustrate the effectiveness of the theoretical results.

Keywords Finite-time · Optimal control · Formation tracking · Lie group

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1 Introduction

In the past few years, control and coordination of multiple autonomous vehicles have been considerably studied for the potential both civilian and military applications. Compared with the traditional monolithic systems, vehicle team can perform tasks that are difficult for one single vehicle, such as, formation flying of unmanned air vehicles, large area exploration, surveillance, and spacecraft interferometry tasks.

Vehicle formation control has attracted much attention in multi-vehicle coordination, since vehicles moving in formation can reduce the system cost, increase the robustness and efficiency of the system [1]. Generally, the formation control approaches can be roughly categorized into three cases [1–3]: leader-follower, behavior-based, and virtual structure methods. Specially, for the case of leader-follower formation, one vehicle is designated as the leader, which tracks predefined trajectories, and the other vehicles are controlled to follow their respective leaders with given separations. Formation and coordination control of vehicles with various dynamics have been well studied in recent research. In many works, the formation and coordination control of vehicles is studied on Euclidean space. Nevertheless, the configuration space of vehicles is a nonlinear space in many practical applications. For instance, the attitude of a satellite is defined on the Lie group SO(3) and planar robots travel on Lie group SE(2). Differing from solving these problems on vector space, the approaches developed on Lie group make use of the intrinsic geometric properties of the mechanical control systems, and the obtained results are independent of the choice of coordinates [4].

Based on the nonlinear space, the control problem is inherently more difficult than on Euclidean space. It should be noted that linear operation is no longer effective in such a nonlinear manifold. Euclidean space is a linear space and has closure for addition. Thus, for systems evolving on Euclidean space, the state space and velocity space are on the same space and the velocity error can be directly derived by subtraction. However, for the system evolving on nonlinear manifold, such as SE(2), the state space and velocity space are different, and every state has the velocity space of its own. The velocity error is obtained by using coordinate transformation from one velocity space to another, which is provided by the algebraic structure of Lie group and the symmetry of the vector field on Lie group. Furthermore, considering that SE(2) is a matrix Lie group, the ordinary operations, such as derivation, integral, become complicated. So, the research for vehicles on nonlinear space is significative and important [5].

Motivated by the above analysis, we focus on the problem of finite-time optimal formation tracking control for vehicles with kinematics evolving on Lie groups SE(2), where vehicles are not modeled as particles. The translational and rotational formation tracking of vehicles are considered simultaneously. Considering the holonomic systems, an optimal control law is proposed for vehicles to track a desired reference trajectory at the given tracking time in the manner of rigid formation, where the tracking time and the rigid formation are specified by the task requirements in advance. Besides, the minimization of the given performance index is guaranteed. The tracking control law is designed based on Pontryagin's minimum principle and is given by explicit formulation, instead of by numerical algorithms. This is crucial for autonomous control design in practical applications. Furthermore, this paper extends the optimal control results in [6] to formation tracking problem. Comparing with [6], the communication topology is extended from complete graph to a directed tree and the position of the formation can be specified by the leader vehicle. Finally, some conditions are proposed to ensure the adjoint orbits of vehicles to be non-holonomic.

The remainder of this paper is organized as follows. The related work and preliminaries are given in Sects. 2 and 3, respectively. Main theoretical results are provided in Sect. 4. In Sect. 5, some numerical simulations are reported to illustrate the theoretical results. Concluding remarks are finally given in Sect. 6.

2 Related work

Early seminal works for formation and coordination control of vehicles with integrator-type dynamics have been launched by Olfati-Saber and Murray [7] and Ren and Beard [8]. However, most actual multi-vehicle systems have very complex physical dynamics. Motivated by this observation, the works by Tuna [9], Seo, Shim, and Back [10], and Qu, Wang, and Hull [11] have further extended the dynamics of vehicles into the linear system and the nonlinear system, respectively. Formation and coordination control of rigid bodies and non-holonomic systems (the unicycle model) have also been extensively studied in [12–15]. Additionally, authors considered the finite-time tracking problem of rigid bodies in [16–18], where results of the finite-time tracking are verified by theoretical derivations. And, the tracking time is estimated by the initial conditions, the designed Lyapunov function and control laws, cannot given by task requirements in advance. In all above works, a common feature is to study the formation and coordination control of vehicles on Euclidean space.

As mentioned in above section, in many practical applications, the configuration space of vehicles is a nonlinear space. For vehicles evolving on nonlinear space, the formation and coordination control of vehicles have been studied in many literatures such as Justh and Krishnaprasad [19], Nair and Leonard [20], Sarlette [21], Sarlette, Bonnabel, and Sepulchre [22], Dong and Geng [5]. In these papers, the authors took into account the geometry structure of the nonlinear space, i.e. symmetries. Considering the kinematics model for vehicles evolving on SE(3), the authors presented a Lie group setting for the formation problem in [19], and achieved the formation for the two-agent case by stabilizing relative equilibria, which is determined by the kinematics model under consideration. In [22], the authors gave a general problem formulation, analyzed ensuing conditions and proposed the control laws for the coordinated motion. However, the derived result guarantees that the relative configuration between vehicles are constants, which are determined by initial conditions and cannot specified by task requirements in advance. Furthermore, a formation algorithm is proposed in [5], which can apply to arbitrary formation requirements. For the dynamics model with multiple agents, the stable synchronization on Lie groups is considered in [20]. The asymptotical control laws are proposed to stabilize the desired relative equilibrium. Besides, the tracking problem for vehicles has been considered in [4, 23–25], where the exponential tracking control was designed and the obtained results were coordinate-free. Obviously, the aforementioned methods can only achieve asymptotic formation or tracking. In many practical applications, the formation algorithms that achieve the formation in finite time are more desirable, especially when the multiple maneuvers are needed and a high precision control is required. The problem of finite-time formation for systems evolving on nonlinear space has been studied in [6], where the desired formation is achieved in finite time, but the position of the formation cannot been specified. In this paper, the optimal control results in [6] are extended to formation tracking problem and the formation position can be determined by the leader. Besides, Pontryagin's minimum principle is also used to derive the optimal control for system evolving on nonlinear space in [26–28], where the control laws are given by numerical algorithms. By contrast, the control law is given by explicit formulation in this paper, which is crucial for autonomous control design in practical applications.

3 Preliminaries

This section introduces the elements used to formulate the optimal formation tracking control of vehicles.

3.1 Lie group SE(2)

For vehicles such as aerial and underwater autonomous vehicles, robotics, and spacecraft, one of the distinct feature is that their motions include translation and rotation, which are represented by changes in position and attitude, i.e., the changes in configuration. For a rigid body, the configuration is described by the position of center of mass, and the body-fixed frame of its own, respectively. The orthogonal matrices are used to describe the basis vectors of the body-fixed frame. Therefore, the rotation of vehicle is represented by a matrix $R \in \mathrm{SO}(3)$, where the special orthogonal group SO(3) is the matrix Lie group of 3×3 orthogonal matrices with determinant of one, i.e., $\mathrm{SO}(3) = \{R \in \mathbb{R}^{3\times 3} | R^T R = I, \det R = 1\}$. Together with the position vector of center of mass, the configuration of the vehicle is denoted by a matrix

$$g = \begin{bmatrix} R & d \\ 0_{1\times 3} & 1 \end{bmatrix},$$

where $d \in \mathbb{R}^3$ is the position vector. All the configurations constitute the matrix Lie group SE(3), i.e.,

$$\mathrm{SE}(3) = \left\{ \begin{bmatrix} R & d \\ 0_{1\times 3} & 1 \end{bmatrix} \in \mathbb{R}^{4\times 4} \middle| R \in \mathrm{SO}(3), d \in \mathbb{R}^3 \right\}.$$

Similarly, for vehicle in horizontal plane, we have the matrix Lie group SE(2) and SO(2). The element of SE(2) is denoted by

$$g = \begin{bmatrix} R & d \\ 0_{1\times 2} & 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & x \\ \sin\theta & \cos\theta & y \\ 0 & 0 & 1 \end{bmatrix},$$

where $d \in \mathbb{R}^2$ is the position vector, and θ is the attitude angle. Let g^{-1} denote the group inverse of $g \in \mathrm{SE}(2)$. $\mathrm{T}_g\mathrm{SE}(2)$ is the tangent space to SE(2) at the base element g, and for $g = I$ (identity element), define the following Lie bracket in $\mathrm{T}_I\mathrm{SE}(2)$:

$$[\hat{X}, \hat{Y}] = \hat{X}\hat{Y} - \hat{Y}\hat{X}, \quad \hat{X}, \hat{Y} \in \mathrm{T}_I\mathrm{SE}(2).$$

Then, $\mathrm{T}_I\mathrm{SE}(2)$ is denoted by $\mathfrak{se}(2)$ and is called Lie algebra of the Lie group SE(2). We identify $\mathfrak{se}(2)$ with $\mathbb{R}^3$ by the following isomorphic mapping $\wedge$: $\mathbb{R}^3 \to \mathfrak{se}(2)$:

$$\wedge : \begin{pmatrix} v_x \\ v_y \\ \omega \end{pmatrix} \longrightarrow \begin{pmatrix} 0 & -\omega & v_x \\ \omega & 0 & v_y \\ 0 & 0 & 0 \end{pmatrix}$$

According to the isomorphic mapping, the basis of $\mathfrak{se}(2)$ are given by

$$\hat{i}_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad \hat{i}_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$

$$\hat{i}_3 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

For the formation tracking of vehicles, it is inevitable to compare the velocities of different vehicles, which cannot be considered locally in a neighborhood. Every configuration have its own velocity space, i.e. the tangent space of the corresponding configuration. Thus, for the velocity comparison between different vehicles, the coordinate transformations from one tangent space to another are essential. The algebraic structure of Lie group and the symmetry of the vector field on Lie group provide such coordinate transformations. For the control of vehicles, especially for attitude control, most of prior work is based on three-parameter representation, such as Euler angle, modified Rodriguez parameters, or unit quaternions (four-parameter). Parameterization methods convert the configuration space from nonlinear space to normal Euclidean space by identifying the different velocities as the same Euclidean space essentially. For the stability problems or tracking problems in a neighborhood, these methods are good approximations. However, for the systems with rigid bodies which cannot be considered locally in a neighborhood, first, this can present difficulties for different vehicles to keep rigid formation when the nonlinear trajectory tracking problem is considered. For example, when the relative positions and velocity of center of mass for two vehicles are equal, it is impossible to keep the rigid rotation for these two vehicles. As we all know, keeping the rigid rotation is important for the problem of multiple vehicle formation. Second, these parameterization methods cause singularities or ambiguities. In addition, the control inputs of vehicle should be represented in the own body-fixed frame. The coordinate transformations are also needed for the control input comparison of different vehicles. So, ignoring the Lie group structure for vehicles is equivalent to putting the control inputs of different vehicles in same coordinate frame, obviously which does not agree with reality. Thus, in this paper, we consider the tracking problem of vehicles in the Lie group frame.

Remark 1 For parameterization methods, it is well known that any three-parameter representations are defined only locally, and they exhibit singularities for larger angle rotational maneuvers. Quaternions do not have singularities, but they have ambiguities in representing an attitude, as the three-sphere S^3 double covers SO(3). Thus, the tracking problems of vehicles on Lie group defined globally, without singularities or ambiguities, is very significative.

3.2 Model of vehicles

In this paper, vehicles are considered to evolve on Lie group SE(2). Suppose that the model of a vehicle in horizontal plane is given by

$$\dot{g} = g\hat{\xi} = g\hat{i}_1 v_x + g\hat{i}_2 v_y + g\hat{i}_3\omega, \tag{1}$$

where $\hat{\xi} \in \mathfrak{se}(2)$ is called a twist which can be written as a linear combination of the basis of $\mathfrak{se}(2)$, i.e. $\hat{i}_k$ $(k = 1, 2, 3)$, and $\xi = [v_x, v_y, \omega]^T \in \mathbb{R}^3$ is considered as the control input. For the case that the number of independent control inputs is equal to the dimension of SE(2), the system is called a holonomic system. More challenges arise when dealing with nonholonomic systems. In this paper, the following nonholonomic model is considered:

$$\dot{g} = g\hat{\xi} = g\hat{i}_1 v_x + g\hat{i}_3\omega, \tag{2}$$

that is to say, the vehicle cannot move laterally. Generally, this model is used to describe the kinematic model of aircraft and underwater vehicles on SE(2). Controllability of the systems given by (2) is confirmed using the Lie brackets and the sufficiency condition provided by [29].

3.3 Linear functionals on $T_g SE(2)$ and $\mathfrak{se}(2)$

In order to deal with the optimal control problem on SE(2), we give the definitions of linear functionals on $T_g SE(2)$ and $\mathfrak{se}(2)$. The linear functionals are defined by inner products on linear space $T_g SE(2)$ and $\mathfrak{se}(2)$. More details of inner products and linear functionals are given in [6].

Definition 1 For the given $\hat{p} \in \mathfrak{se}(2)$, the linear functional $\hat{p}^* : \mathfrak{se}(2) \longrightarrow \mathbb{R}$ is defined as

$$\hat{p}^*(\hat{X}) \triangleq \langle \hat{p}, \hat{X} \rangle_I, \quad \hat{X} \in \mathfrak{se}(2),$$

where $\langle \cdot, \cdot \rangle_I$ is the inner product on $\mathfrak{se}(2)$ and given by

$$\langle \hat{X}, \hat{Y} \rangle_I = \operatorname{tr}\left(\operatorname{diag}\left(\frac{1}{2}, \frac{1}{2}, 1\right)\hat{X}^T\hat{Y}\right), \quad \hat{X}, \hat{Y} \in \mathfrak{se}(2).$$

Let

$$\hat{X} = \sum_{k=1}^{3} \hat{i}_k X_k, \qquad \hat{Y} = \sum_{k=1}^{3} \hat{i}_k Y_k.$$

One can obtain

$$\langle \hat{X}, \hat{Y} \rangle_I = \langle X, Y \rangle,$$

where $\langle \cdot, \cdot \rangle$ represents the inner product on $\mathbb{R}^3$.

Similarly, the following definition of linear functionals on $T_g SE(2)$ is given.

Definition 2 For the given $P_g \in \mathrm{T}_g\mathrm{SE}(2)$, the linear functional $P_g^*: \mathrm{T}_g\mathrm{SE}(2) \longrightarrow \mathbb{R}$ is defined as

$$P_g^*(X_g) \triangleq \langle P_g, X_g\rangle_g, \quad X_g \in \mathrm{T}_g\mathrm{SE}(2),$$

where $\langle\cdot,\cdot\rangle_g$ is the inner product on $\mathrm{T}_g\mathrm{SE}(2)$ and is given by

$$\langle X_g, Y_g\rangle_g = \mathrm{tr}\left(\mathrm{diag}\left(\frac{1}{2}, \frac{1}{2}, 1\right) X_g^T Y_g\right), \quad X_g, Y_g \in \mathrm{T}_g\mathrm{SE}(2).$$

All the linear functionals constitute the cotangent space $\mathrm{T}_g^*\mathrm{SE}(2)$, which is dual to $\mathrm{T}_g\mathrm{SE}(2)$. Similarly, we have the dual space $\mathfrak{se}^*(2)$ to $\mathfrak{se}(2)$. The dual basis of $\mathfrak{se}^*(2)$ to $\mathfrak{se}(2)$ are denoted by $\hat{i}_j^*$ $(j = 1, 2, 3)$ such that

$$\hat{i}_j^*(\hat{i}_k) = \delta_{jk} \quad (j, k = 1, 2, 3), \tag{3}$$

where δ_{jk} is the Kronecker delta.

For any $g \in \mathrm{SE}(2)$, $\mathrm{T}_I\mathrm{L}_g\hat{i}_k = g\hat{i}_k$ $(k = 1, 2, 3)$ constitute the basis of $\mathrm{T}_g\mathrm{SE}(2)$, where $\mathrm{L}_g(\cdot)$ denotes the left group action, and $\mathrm{T}_I\mathrm{L}_g : \mathfrak{se}(2) \to \mathrm{T}_g\mathrm{SE}(2)$ is the tangent mapping between $\mathfrak{se}(2)$ and $\mathrm{T}_g\mathrm{SE}(2)$. It follows from (3) that

$$\begin{aligned}\hat{i}_j^*(\hat{i}_k) &= \hat{i}_j^*\left(g^{-1}g\hat{i}_k\right) = \left(g^{-1}\right)^*\hat{i}_j^*(g\hat{i}_k)\\ &= \left((\mathrm{T}_I\mathrm{L}_{g^{-1}})^*\hat{i}_j^*\right)\left((\mathrm{T}_I\mathrm{L}_g)\hat{i}_k\right)\\ &= \delta_{jk}, \quad (j, k = 1, 2, 3).\end{aligned}$$

Therefore, $(g^{-1})^*\hat{i}_j^* \in \mathrm{T}_g^*\mathrm{SE}(2)$ $(j = 1, 2, 3)$ are the dual basis of $\mathrm{T}_g^*\mathrm{SE}(2)$ to $\mathrm{T}_g\mathrm{SE}(2)$. Further, any $P_g^* \in \mathrm{T}_g^*\mathrm{SE}(2)$ can be written as the linear combination of the basis $(g^{-1})^*\hat{i}_j^*$ $(j = 1, 2, 3)$:

$$P_g^* = \sum_{j=1}^{3}\left(g^{-1}\right)^*\hat{i}_j^* p_j = \left(g^{-1}\right)^*\sum_{j=1}^{3}\hat{i}_j^* p_j = \left(g^{-1}\right)^*\hat{p}^*. \tag{4}$$

For the adjoint operator $\mathrm{Ad}_g : \mathfrak{se}(2) \to \mathfrak{se}(2)$, it follows from (3) that

$$\hat{i}_j^*(\hat{i}_k) = \hat{i}_j^*(\mathrm{Ad}_{g^{-1}}\mathrm{Ad}_g\hat{i}_k) = \mathrm{Ad}_{g^{-1}}^*\hat{i}_j^*(\mathrm{Ad}_g\hat{i}_k) = \delta_{jk}.$$

Thus,

$$\mathrm{Ad}_{g^{-1}}^* : \mathfrak{se}^*(2) \to \mathfrak{se}^*(2).$$

For $\hat{p} \in \mathfrak{se}(2)$,

$$\hat{p} = \sum_{k=1}^{3} p^k\hat{i}_k, \quad \mathrm{Ad}_g\hat{p} = \sum_{k=1}^{3} p^k\mathrm{Ad}_g\hat{i}_k.$$

Let $\hat{p}^* = \sum_{k=1}^{3} p^k\hat{i}_k^*$. We call $\hat{p}$ and $\hat{p}^*$ mutually dual. Therefore, for

$$\mathrm{Ad}_{g^{-1}}^*\hat{p}^* = \sum_{k=1}^{3} p^k\mathrm{Ad}_{g^{-1}}^*\hat{i}_k^*,$$

$\mathrm{Ad}_g\hat{p}$ and $\mathrm{Ad}_{g^{-1}}^*\hat{p}^*$ are dual. Similarly, $\mathrm{Ad}_{g^{-1}}\hat{p}$ and $\mathrm{Ad}_g^*\hat{p}^*$ are dual. Then, it is easy to obtain

$$\mathrm{Ad}_{g^{-1}}^*\hat{p}^* = \left(g^{-1}\right)^*\hat{p}^*g^*, \qquad \mathrm{Ad}_g^*\hat{p}^* = g^*\hat{p}^*\left(g^{-1}\right)^*.$$

Definition 3 [22] The relative configuration on SE(2) of vehicle j with respect to vehicle k is $g_{jk} = g_k^{-1}g_j$.

Remark 2 In the present paper, the left-invariant relative configuration $g_{jk} = g_k^{-1}g_j$, i.e., the relative configuration which is invariant under the same left action on the individual configurations, respectively, is considered. However, the proposed design methods can also be applied in the case of the right-invariant relative configuration $\rho_{jk} = g_jg_k^{-1}$, which is invariant under the same right action on the individual configurations, respectively.

4 Main results

In this section, the formation tracking problem for vehicles evolving on SE(2) is addressed. We begin with the problem formulation.

4.1 Problem formulation

The problem of optimal formation tracking control for vehicles in horizontal plane is studied. Consider a group of n identical vehicles evolving on Lie group SE(2). The holonomic model of vehicle k is described by

$$\dot{g}_k = g_k\hat{\xi}_k = g\hat{i}_1v_{x_k} + g\hat{i}_2v_{y_k} + g\hat{i}_3\omega_k, \quad k = 1, \ldots, n. \tag{5}$$

And the non-holonomic models are given by

$$\dot{g}_k = g_k\hat{\xi}_k = g_k\hat{i}_1v_{x_k} + g_k\hat{i}_3\omega_k, \quad k = 1, \ldots, n. \tag{6}$$

The reference trajectory $g_d \in \mathrm{SE}(2)$ is given by the configuration of system:

$$\dot{g}_d = g_d\hat{\xi}_d. \tag{7}$$

Throughout this article, it is assumed that the control law of reference trajectory is known.

Let $e_k = {g_d}^{-1} g_k$ denote the relative configuration of vehicle k with respect to the reference trajectory. Take the derivative of relative configuration e_k with respect to time. One has

$$\begin{aligned}\dot{e}_k &= \frac{d}{dt}({g_d}^{-1})g_k + {g_d}^{-1}\dot{g}_k \\ &= -{g_d}^{-1} g_d \hat{\xi}_d {g_d}^{-1} g_k + {g_d}^{-1} g_k \hat{\xi}_k \\ &= e_k(\hat{\xi}_k - \mathrm{Ad}_{{e_k}^{-1}} \hat{\xi}_d).\end{aligned}$$

Let $\hat{\xi}_{e_k} = \hat{\xi}_k - \mathrm{Ad}_{{e_k}^{-1}} \hat{\xi}_d$ be the relative control input, and it is derived that

$$\dot{e}_k = e_k \hat{\xi}_{e_k}, \quad k = 1, \ldots, n. \tag{8}$$

Thus, the tracking problem concerned is converted into the planning problem for the above systems (8).

Suppose that the cost function is given by

$$J = \frac{1}{2}\int_{t_0}^{t_f} \sum_{k=1}^{n} \big(\xi_{e_k}^T(t)\xi_{e_k}(t)\big)\,dt, \tag{9}$$

where t_0 and t_f are the initial time and terminal time of the maneuvering, respectively, and they are given in advance according to task requirements. In practice, the minimization of the above cost function is to minimize the length of geodesics or the control energy.

Let e_k^* denote the desired relative configuration of vehicle k with respect to the reference trajectory at the terminal time t_f, and it is specified by task requirements. For the case that e_k^* equals the identity matrix, i.e. $e_k^* = I$, the corresponding problem is called the configuration consensus tracking problem. Otherwise, it is the formation tracking problem.

In this paper, our objective is to design control laws for systems (5) to formation track the reference trajectory at the given terminal time t_f, and minimize the cost function (9) simultaneously. Then, the nonholonomic conditions are proposed to guarantee that the adjoint orbits of vehicles after terminal time t_f is non-holonomic.

Remark 3 For systems (5), the familiar performance index is given by

$$\tilde{J} = \frac{1}{2}\int_{t_0}^{t_f} \sum_{k=1}^{n} \big(\xi_k^T(t)\xi_k(t)\big)\,dt.$$

However, when the tracking problem is considered, the designed tracking control includes the control information of leader, which cannot be optimized. Therefore, it is meaningful and practical to optimize the performance index (9).

4.2 The finite-time optimal formation tracking control for holonomic models on SE(2)

As mentioned above, the problem concerned has been converted into the planning problem of systems (8). Thus, we begin by studying the planning problem concerned for (8) and the corresponding cost function (9) is considered. For simplicity, the model and the cost function to be optimized are given by

$$\dot{g}_k = g_k \hat{\xi}_k, \quad k = 1, \ldots, n, \tag{10}$$

$$J = \frac{1}{2}\int_{t_0}^{t_f} \sum_{k=1}^{n} \big(\xi_k^T(t)\xi_k(t)\big)\,dt, \tag{11}$$

where t_0 and t_f are the initial time and terminal time of the planning problem, respectively, and $\xi_k \in \mathbb{R}^3$ is the control input. The following task is to design the control law $\xi_k \in \mathbb{R}^3$ that steers the system (10) to the terminal configuration $g_k(t_f)$ at the given terminal time t_f, and the cost function (11) is minimized.

The Hamiltonian function is written as

$$H = -\frac{1}{2}\sum_{k=1}^{n}\big(\xi_k^T(t)\xi_k(t)\big) + \sum_{k=1}^{n} P_{g_k}^*(g_k\hat{\xi}_k), \tag{12}$$

where $P_{g_k}^* \in \mathrm{T}_{g_k}^*\mathrm{SE}(2)$. Using Definition 2 gives

$$\begin{aligned}H = &-\frac{1}{2}\sum_{k=1}^{n}\big(\xi_k^T(t)\xi_k(t)\big) \\ &+ \sum_{k=1}^{n}\mathrm{tr}\bigg(\mathrm{diag}\Big(\frac{1}{2},\frac{1}{2},1\Big)P_{g_k}^T g_k\hat{\xi}_k\bigg).\end{aligned}$$

Let $\widetilde{P}_{g_k} = P_{g_k}\mathrm{diag}(\frac{1}{2},\frac{1}{2},1)$, and $\widetilde{P}_{g_k}$ represents the costate. Then, one has

$$H = -\frac{1}{2}\sum_{k=1}^{n}\big(\xi_k^T(t)\xi_k(t)\big) + \sum_{k=1}^{n}\mathrm{tr}\big(\widetilde{P}_{g_k}^T g_k\hat{\xi}_k\big).$$

It follows from Pontryagin's maximum principle (PMP) on a Lie group [30] that the necessary conditions of optimality are written as

$$\dot{g}_k = \frac{\partial H}{\partial \widetilde{P}_{g_k}} = g_k\hat{\xi}_k, \tag{13}$$

$$\dot{\widetilde{P}}_{g_k} = -\frac{\partial H}{\partial g_k} = -\widetilde{P}_{g_k}\hat{\xi}_k^T. \tag{14}$$

To solve the Hamiltonian equations (13) and (14), we need to integrate Eq. (14). For this purpose, the following lemma is given.

Lemma 1 *$\widetilde{P}_{g_k}(t)g_k^T(t)$ is a constant.*

Proof From (13), the twist is written as

$$\hat{\xi}_k = g_k^{-1}\dot{g}_k.$$

Substituting the above equality into (14), one obtains

$$\dot{\widetilde{P}}_{g_k} = -\widetilde{P}_{g_k}\big(g_k^{-1}\dot{g}_k\big)^T = -\widetilde{P}_{g_k}\dot{g}_k^T\big(g_k^T\big)^{-1}.$$

Then,

$$\dot{\widetilde{P}}_{g_k}g_k^T + \widetilde{P}_{g_k}\dot{g}_k^T = 0, \qquad \frac{d}{dt}\big(\widetilde{P}_{g_k}g_k^T\big) = 0.$$

Consequently, $\widetilde{P}_{g_k}g_k^T$ is a constant.

Using the definition of inner product on $\mathfrak{se}(2)$, we find that for any $\hat{\zeta} \in \mathfrak{se}(2)$,

$$\begin{aligned}\operatorname{tr}\big(\big(\widetilde{P}_{g_k}g_k^T\big)^T\hat{\zeta}\big) &= \operatorname{tr}\left(\operatorname{diag}\left(\frac{1}{2},\frac{1}{2},1\right)P_{g_k}^T\hat{\zeta}g_k\right)\\ &= \langle P_{g_k},\hat{\zeta}g_k\rangle_{g_k} = P_{g_k}^*(\mathrm{T}_I\mathrm{R}_{g_k}\hat{\zeta})\\ &= \big((\mathrm{T}_I\mathrm{R}_{g_k})^*P_{g_k}^*\big)(\hat{\zeta}) = \big(P_{g_k}^*{g_k}^*\big)(\hat{\zeta}),\end{aligned} \tag{15}$$

where $\mathrm{R}_{g_k}(\cdot)$ denotes the right group action, and $\mathrm{T}_I\mathrm{R}_{g_k} : \mathfrak{se}(2) \longrightarrow \mathrm{T}_{g_k}\mathrm{SE}(2)$ is the tangent map of the right group action at identity element I. It follows from (15) that $P_{g_k}^*{g_k}^* \in \mathfrak{se}^*(2)$ is invariant with respect to time t. Thus, it can be assumed that

$$P_{g_k}^*{g_k}^* = \hat{c}_k^*, \quad k = 1,\dots,n, \tag{16}$$

where $\hat{c}_k^*$ is an unknown constant. Considering that

$$P_{g_k}^* = \big(g_k^{-1}\big)^*\hat{p}_k^*,$$

one gets

$$\big(g_k^{-1}(t)\big)^*\hat{p}_k^*(t)g_k^*(t) = \hat{c}_k^*,$$
$$\hat{p}_k^*(t) = g_k^*(t)\hat{c}_k^*\big(g_k^*(t)\big)^{-1}.$$

According to the duality relation, one has

$$\hat{p}_k(t) = \mathrm{Ad}_{{g_k}^{-1}(t)}\hat{c}_k, \quad k = 1,\dots,n. \tag{17}$$

In a similar manner to [31], the Hamiltonian equation (12) is a function on the cotangent bundle $\mathrm{T}^*\mathrm{SE}(2)$, which can be identified with $\mathrm{SE}(2)\times\mathfrak{se}^*(2)$. Therefore, the appropriate Hamiltonian is a function on $\mathfrak{se}^*(2)$. Equation (12) can be pulled back by the left transformation and written as

$$H = -\frac{1}{2}\sum_{k=1}^{n}\langle\xi_k,\xi_k\rangle + \sum_{k=1}^{n}\langle p_k,\xi_k\rangle.$$

Considering the new Hamiltonian, we get the following lemma for the optimal planning problem considered.

Lemma 2 *For systems* (10), *the control laws*

$$\hat{\xi}_k^{\mathrm{op}}(t) = \frac{1}{t_f - t_0}\mathrm{Ad}_{g_k(t)^{-1}}\log\big(g_k(t_f)g_k^{-1}(t_0)\big),$$
$$k = 1,\dots,n \tag{18}$$

steer the systems from the initial configurations $g_k(t_0)$ to the terminal configurations $g_k(t_f)$ at the given terminal time t_f, and the cost function (11) *is minimized.*

Proof It follows from the PMP [32] that the optimal control laws are determined from the following condition:

$$\frac{\partial H}{\partial \xi_k} = -\xi_k + p_k = 0.$$

Thus

$$\xi_k^{\mathrm{op}} = p_k.$$

Using (17), we have

$$\hat{\xi}_k^{\mathrm{op}}(t) = \hat{p}_k(t) = \mathrm{Ad}_{g_k^{-1}(t)}\hat{c}_k,$$

where $\hat{c}_k \in \mathfrak{se}(2)$ is a constant, which is determined by the boundary conditions. Substituting the above equation into the system (10) and integrating the system from t_0 to t_f, one obtains

$$g_k(t_f) = e^{\hat{c}_k(t_f - t_0)}g_k(t_0).$$

Then,

$$\hat{c}_k = \frac{1}{t_f - t_0}\log\big(g_k(t_f)g_k^{-1}(t_0)\big).$$

Therefore, the control is given by

$$\hat{\xi}_k^{\mathrm{op}}(t) = \frac{1}{t_f - t_0}\mathrm{Ad}_{g_k(t)^{-1}}\log\big(g_k(t_f)g^{-1}(t_0)\big).$$

Remark 4 During the derivation of control laws (18), integrating the system from t to t_f gives the following real-time feedback control law:

$$\hat{\xi}_k^{\mathrm{op}}(t) = \frac{1}{t_f - t}\mathrm{Ad}_{g_k(t)^{-1}}\log\big(g_k(t_f){g_k}^{-1}(t)\big),$$

which depends on the current states instead of the initial states. Thus, it can achieve the desired terminal configuration even if disturbance exists in the initial conditions and/or control inputs. When there is no disturbance, the above new control law is equivalent to (18).

Based on the above derivations, the optimal planning control laws for systems (8) are given as follows:

$$\hat{\xi}_{e_k}^{\mathrm{op}}(t) = \frac{1}{t_f - t_0}\mathrm{Ad}_{e_k(t)^{-1}}\log\bigl(e_k^* e_k^{-1}(t_0)\bigr),\quad k = 1, \dots, n, \tag{19}$$

which steer systems (8) to the desired terminal configurations e_k^* $(k = 1, \dots, n)$ at the given terminal time t_f, and the cost function (9) can be minimized. Then, go back to the original tracking problem for system (5). Considering that

$$\hat{\xi}_{e_k} = \hat{\xi}_k - \mathrm{Ad}_{e_k^{-1}}\hat{\xi}_d,$$

the following theorem is proposed.

Theorem 1 *Consider the multi-vehicle systems* (5) *and the desired trajectory* $g_d(t)$. *Then, for the given formation tracking time* t_f *and desired formation configurations* e_k^* $(k = 1, \dots, n)$, *the control laws*

$$\hat{\xi}_k^{\mathrm{op}} = \mathrm{Ad}_{e_k^{-1}(t)}\Bigl(\hat{\xi}_d(t) + \frac{1}{t_f - t_0} \times \log\bigl(e_k^* g_k^{-1}(t_0) g_d(t_0)\bigr)\Bigr),\tag{20}$$

$$e_k(t) = g_d^{-1}(t) g_k(t),\quad k = 1, \dots, n,$$

make sure the vehicles to formation track the desired trajectory g_d *at the given terminal time* t_f, *and the corresponding cost function* (9) *is minimized.*

Similarly, one can get the following real-time tracking feedback control laws:

$$\hat{\xi}_k^{\mathrm{op}} = \mathrm{Ad}_{e_k^{-1}(t)}\Bigl(\hat{\xi}_d(t) + \frac{1}{t_f - t}\log\bigl(e_k^* g_k^{-1}(t) g_d(t)\bigr)\Bigr),\quad k = 1, \dots, n. \tag{21}$$

It can achieve the desired formation tracking even if disturbance exists in the initial conditions and/or control inputs.

For the formation tracking problem, it is desirable to keep the formation tracking after the terminal time t_f. Thus, the following stabilization problem is studied. Likewise, we begin by considering the problem of asymptotically stabilizing for system (8). For the desired relative configuration e_k^*, we have

$$\bigl(\dot{e}_k^{*-1} e_k\bigr) = \bigl(e_k^{*-1} e_k\bigr)\hat{\xi}_{e_k}.$$

The asymptotic tracking problem concerned can be converted into designing $\hat{\xi}_{e_k}$ for the above system such that $e_k^{*-1} e_k \to I$. For a fixed value $\alpha > 0$, it follows from [33] that

$$\hat{\xi}_{e_k} = -\alpha\log\bigl(e_k^{*-1} e_k\bigr) = \alpha\mathrm{Ad}_{e_k^{-1}(t)}\log\bigl(e_k^* e_k{}^{-1}\bigr),$$

which converges asymptotically to the identity matrix I. Therefore, we get the following asymptotic formation tracking control laws for systems (5):

$$\hat{\xi}_k{}^{\mathrm{op}} = \mathrm{Ad}_{e_k^{-1}(t)}\bigl(\hat{\xi}_d(t) + \alpha\log\bigl(e_k^* g_k^{-1}(t) g_d(t)\bigr)\bigr),\quad k = 1, \dots, n. \tag{22}$$

In practical applications, the switching control laws are proposed as follows:

$$\hat{\xi}_k(t) = \begin{cases} \mathrm{Ad}_{e_k^{-1}(t)}(\hat{\xi}_d(t) + \frac{1}{t_f - t}\log(e_k^* g_k^{-1}(t) g_d(t))), & 0 \le t < t_f \\ \mathrm{Ad}_{e_k^{-1}(t)}(\hat{\xi}_d(t) + \alpha\log(e_k^* g_k^{-1}(t) g_d(t))), & t \ge t_f \end{cases}, \tag{23}$$

which achieve formation tracking at the given terminal time t_f, and then switch to the general asymptotic control laws (22) to keep formation tracking.

Note that the switching control laws are discontinuous at time t_f. Theoretically, the switching represents that the control input is infinite. For the low speed vehicles, the implementation of this control law can be approximated by a jump of the twist with a finite slope. However, for the high speed vehicles, this approximation will be badly limited by the ability of the magnitude of the control input. Therefore, we consider the following suboptimal strategy:

$$\hat{\xi}_k(t) = \begin{cases} \mathrm{Ad}_{e_k^{-1}(t)}(\hat{\xi}_d(t) + \frac{1}{t_f + \tau - t} \times \log(e_k^* g_k^{-1}(t) g_d(t))), & 0 \le t < t_f \\ \mathrm{Ad}_{e_k^{-1}(t)}(\hat{\xi}_d(t) + \frac{1}{\tau}\log(e_k^* g_k^{-1}(t) g_d(t))), & t \ge t_f \end{cases}, \tag{24}$$

where $0 < \tau < t_f$. Actually, the suboptimal strategy is to design the optimal control law for the terminal time $t_f + \tau$ during $0 \le t < t_f$, and then switch to the asymptotic control law for $t \ge t_f$. This control law guarantees the continuity of switching control strategy aforementioned. Besides, τ represents the ability of dynamic implementation. When τ goes to zero, (24) will be a discontinuous optimal control.

Note that the argument of Theorem 1 dose not rely on the information-exchange topology among the vehicles. The control laws (20) solve the formation tracking problem under condition that each vehicle has access to g_d. This argument is rather restricted in sense that each vehicle must have access to the desired reference trajectory. Therefore, we assume that only a portion of the vehicles have access to g_d and the topology corresponds to an information-exchange graph is a directed tree. Let e_{kj}^* denote the desired relative configuration of vehicle k with respect to vehicle j at the terminal time t_f. Then, the following algorithms are proposed:

$$\begin{aligned}\hat{\xi}_k^{\mathrm{op}} &= \sum_{j=1}^{n} a_{kj} \mathrm{Ad}_{e_{kj}^{-1}(t)} \Big(\hat{\xi}_j(t) \\ &\quad + \frac{1}{t_f - t_0} \log\big(e_{kj}^* g_k^{-1}(t_0) g_j(t_0)\big) \Big) \\ &\quad + a_{k(n+1)} \mathrm{Ad}_{e_k^{-1}(t)} \Big(\hat{\xi}_d(t) \\ &\quad + \frac{1}{t_f - t_0} \log\big(e_k^* g_k^{-1}(t_0) g_d(t_0)\big) \Big), \\ k &= 1, \ldots, n, \end{aligned} \tag{25}$$

where $e_{kj}(t) = g_j^{-1}(t) g_k(t)$, $e_k(t) = g_d^{-1}(t) g_k(t)$, $a_{kk} \triangleq 0$ and a_{kj} is 1 if information flows from vehicle j to vehicle k and 0 otherwise, $\forall k, j \in 1, \ldots, n$, and $a_{k(n+1)}$ is 1 if vehicle k has access to g_d and 0 otherwise.

We have the following theorem for finite-time formation tracking of multi-vehicle system using algorithm (25).

Theorem 2 *Consider the multi-vehicle system* (5). *Let the communication graph be a directed tree with the root node* (7). *The algorithms* (25) *solve the formation tracking problem in finite time.*

Proof Let S_1 denote the set of the nodes that receive information directly from the root node (7). Assume $k \in S_1$. Thus, $a_{kj} = 0$, $\forall j \in \{1, \ldots, n\}$ and $a_{k(n+1)} = 1$. Rewrite Eq. (25) as

$$\begin{aligned}\hat{\xi}_k^{\mathrm{op}} &= \mathrm{Ad}_{e_k^{-1}(t)} \Big(\hat{\xi}_d(t) \\ &\quad + \frac{1}{t_f - t_0} \log\big(e_k^* g_k^{-1}(t_0) g_d(t_0)\big) \Big). \end{aligned} \tag{26}$$

It follows from Theorem 1 that the desired relative configuration e_k^* is achieved at the terminal time t_f.

If vehicle k does not have access to g_d, $a_{k(n+1)} = 0$, then there is only one vehicle j such that $a_{kj} = 1$. So, Eq. (25) is rewritten as

$$\begin{aligned}\hat{\xi}_k^{\mathrm{op}} &= \mathrm{Ad}_{e_{kj}^{-1}(t)} \Big(\hat{\xi}_j(t) \\ &\quad + \frac{1}{t_f - t_0} \log\big(e_{kj}^* g_k^{-1}(t_0) g_j(t_0)\big) \Big). \end{aligned} \tag{27}$$

Similarly, the desired relative configuration e_{kj}^* of vehicle k with respect to vehicle j is achieved at the terminal time t_f.

Remark 5 In this section, the derived results are also available to the finite-time optimal formation tracking control for multi-vehicle systems on SE(3). There is no essential difference.

4.3 The finite-time optimal formation tracking control for non-holonomic models on SE(2)

Compared with the holonomic case, more challenges arise when dealing with the non-holonomic models. Before considering the optimal formation tracking problem, we study the controllability of the non-holonomic models (6). Thus, the following lemma is given.

Lemma 3 *The non-holonomic system* (6) *is controllable on the Lie group SE*(2).

Proof Note that

$$[\hat{i}_3, \hat{i}_1] = \hat{i}_3 \hat{i}_1 - \hat{i}_1 \hat{i}_3 = \hat{i}_2.$$

Therefore, the controllability rank condition is satisfied. It follows from the Group Test Theorem (see [30]) that the system (6) is controllable.

Although the non-holonomic system (6) is controllable, it is difficult and complex to directly design the tracking control law for system (6). In this paper, we mainly focus on the non-holonomic tracking control for the adjoint orbit. The formation tracking control of vehicles from the initial time to the terminal time is under study.

Considering the control law (21), it reduces to $\mathrm{Ad}_{e_k^{-1}(t_f)} \hat{\xi}_d(t)$ when the desired relative configuration is achieved. $\mathrm{Ad}_{e^{-1}(t_f)} \hat{\xi}_d(t)$ is called the adjoint orbit control. It is assumed that the control law for the reference trajectory is given by

$$\hat{\xi}_d = \begin{bmatrix} 0 & -\omega & v_x \\ \omega & 0 & v_y \\ 0 & 0 & 0 \end{bmatrix}, \tag{28}$$

where v_x and v_y are the velocities in the X-direction and Y-direction, respectively. In this paper, for the case with $v_y = 0$, we say that the above control law satisfies the non-holonomic condition. For non-holonomic vehicles, this condition indicates that vehicles cannot move laterally, which is common in practice. Based on the optimal formation tracking control of vehicles with holonomic dynamics, we consider how to design the non-holonomic adjoint orbit control for vehicles or derive the requirements for vehicles to satisfy the non-holonomic conditions. The following theorem is derived.

Theorem 3 *Assume that the desired relative configurations are given by*

$$e_k^* = \begin{bmatrix} \cos\theta_k^* & -\sin\theta_k^* & x_k^* \\ \sin\theta_k^* & \cos\theta_k^* & y_k^* \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} R_k^* & p_k^* \\ 0_{1\times 2} & 1 \end{bmatrix}, \quad k = 1, \dots, n. \tag{29}$$

The adjoint orbit control satisfies the non-holonomic condition if and only if

$$\Phi_{\hat{\xi}_d}(x_k^*, y_k^*, \theta_k^*) = (v_y + \omega x_k^*)\cos\theta_k^* - (v_x - \omega y_k^*)\sin\theta_k^* = 0. \tag{30}$$

Proof

$$\begin{aligned} \mathrm{Ad}_{e_k^{-1}(t_f)}\hat{\xi}_d(t) &= e_k^{-1}(t_f)\hat{\xi}_d(t)e_k(t_f) \\ &= \begin{bmatrix} (R_k^*)^T & -(R_k^*)^T p_k^* \\ 0 & 1 \end{bmatrix} \\ &\quad \times \begin{bmatrix} 0 & -\omega & v_x \\ \omega & 0 & v_y \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} R_k^* & p_k^* \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} (R_k^*)^T \hat{\omega} R_k^* & \Lambda_k \\ 0 & 0 \end{bmatrix}, \end{aligned}$$

where $\Lambda_k = (R_k^*)^T(\hat{\omega} p_k^* + [\begin{smallmatrix} v_x \\ v_y \end{smallmatrix}])$.

On further computations, we obtain

$$\Lambda_k = \begin{bmatrix} (v_x - \omega y_k^*)\cos\theta_k^* + (v_y + \omega x_k^*)\sin\theta_k^* \\ (v_y + \omega x_k^*)\cos\theta_k^* - (v_x - \omega y_k^*)\sin\theta_k^* \end{bmatrix}.$$

Therefore, the adjoint orbit control satisfies the non-holonomic condition if and only if

$$\Phi_{\hat{\xi}_d}(x_k^*, y_k^*, \theta_k^*) = (v_y + \omega x_k^*)\cos\theta_k^* - (v_x - \omega y_k^*)\sin\theta_k^* = 0.$$

The proof is completed.

Obviously, for arbitrary independent $(x_k^*, y_k^*, \theta_k^*)$, the above non-holonomic condition is not satisfied. In general, in order to keep the specified relative position (x_k^*, y_k^*), the attitude angle θ_k^* is not independent and is decided by

$$\theta_k^* = \arctan\left(\frac{v_y + \omega x_k^*}{v_x - \omega y_k^*}\right), \quad k = 1, \dots, n.$$

Considering the formation tracking problem for the case with non-holonomic reference trajectory, i.e. $v_y = 0$, we get the corollary below.

Corollary 1 *For the desired relative configurations* (29) *and non-holonomic reference trajectory, the adjoint orbit control satisfies the non-holonomic condition if and only if*

$$\Phi_{\hat{\xi}_d}(x_k^*, y_k^*, \theta_k^*) = \omega x_k^* \cos\theta_k^* - (v_x - \omega y_k^*)\sin\theta_k^* = 0, \quad k = 1, \dots, n. \tag{31}$$

For the case with $(x_k^* = 0, y_k^* = 0, \theta_k^* = 0)$, the adjoint orbit and the desired reference trajectory coincide. Thus, the adjoint orbit satisfies the non-holonomic condition. For the case with $(x_k^* = 0, \theta_k^* = 0)$, the vehicle k has the same attitude with the reference trajectory. Similarly, in order to keep the specified relative position, attitude angles are given as follows:

$$\theta_k^* = \arctan\left(\frac{\omega x_k^*}{v_x - \omega y_k^*}\right), \quad k = 1, \dots, n.$$

We refer to (30) and (31) as the non-holonomic conditions of adjoint orbit.

5 Simulation examples

In this section, some numerical simulation examples of finite-time formation tracking for system (5) are given to illustrate the theoretical results. For simplicity, the initial time is given by $t_0 = 0$.

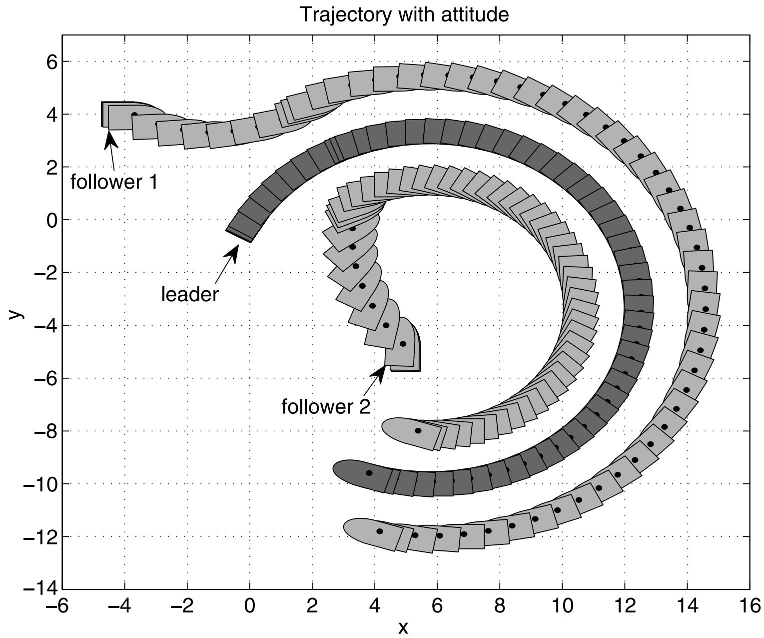

Fig. 1 Finite-time optimal formation tracking for vehicles in plane and $t_f = 5$

Example 1 Consider the system with two vehicles, and select the initial configurations and the desired relative configurations as follows:

$$g_1(0) = \begin{bmatrix} 1 & 0 & -4 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}, \qquad g_2(0) = \begin{bmatrix} 0 & -1 & 5 \\ 1 & 0 & -5 \\ 0 & 0 & 1 \end{bmatrix},$$

$$e_1^* = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}, \qquad e_2^* = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}.$$

The initial configuration and external input of the desired reference trajectory are given by

$$g_d(0) = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$\hat{\xi}_d = \begin{bmatrix} 0 & 0.15 & 1 \\ -0.15 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

i.e. the reference trajectory is a circular starting from origin of coordinate produced by a non-holonomic vehicle. Here we assume that each vehicle has access to the desired reference trajectory. The tracking time is 5. For the control laws (23), select $\alpha = 0.6$.

Figures 1 and 2 show the simulation results with the real-time feedback control laws (23). The two vehicles and the reference trajectory are denoted by follower 1, follower 2 and leader, respectively. Figure 1 describes the plane movement trajectories with attitude for vehicles. The curves in Fig. 2 present the control inputs, configurations, and relative configurations, respectively. It can be seen that the control laws (23) make sure vehicles to achieve the desired formation tracking at the given terminal time $t_f = 5$ and then follow the leader in a manner of given formation.

Example 2 Consider the system in Example 1. For that two planar vehicles and the reference trajectory, select the same initial conditions and external input, respectively. In order to keep the desired relative position $(-1, 2)$ and $(-1, -2)$, respectively, for the two vehicles, the attitude angles obtained by Corollary 1 are given by

$$\theta_1 = 0.1149, \qquad \theta_2 = 0.2111.$$

Thus, the relative configurations are

$$e_1^* = \begin{bmatrix} \cos(0.1149) & -\sin(0.1149) & -1 \\ \sin(0.1149) & \cos(0.1149) & 2 \\ 0 & 0 & 1 \end{bmatrix},$$

$$e_2^* = \begin{bmatrix} \cos(0.2111) & -\sin(0.2111) & -1 \\ \sin(0.2111) & \cos(0.2111) & -2 \\ 0 & 0 & 1 \end{bmatrix}.$$

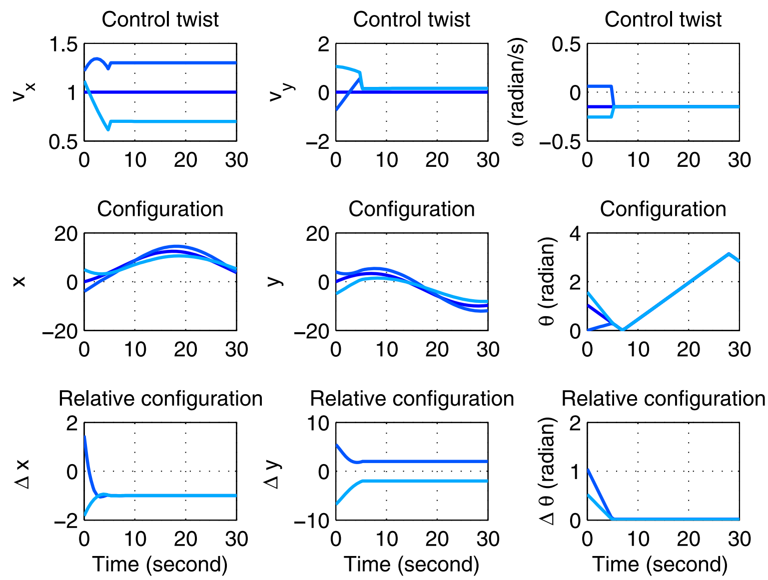

Fig. 2 From top to bottom: control input, configuration and relative configuration (with respect to the reference trajectory); from left to right: with respect to the x and y coordinate and the attitude θ

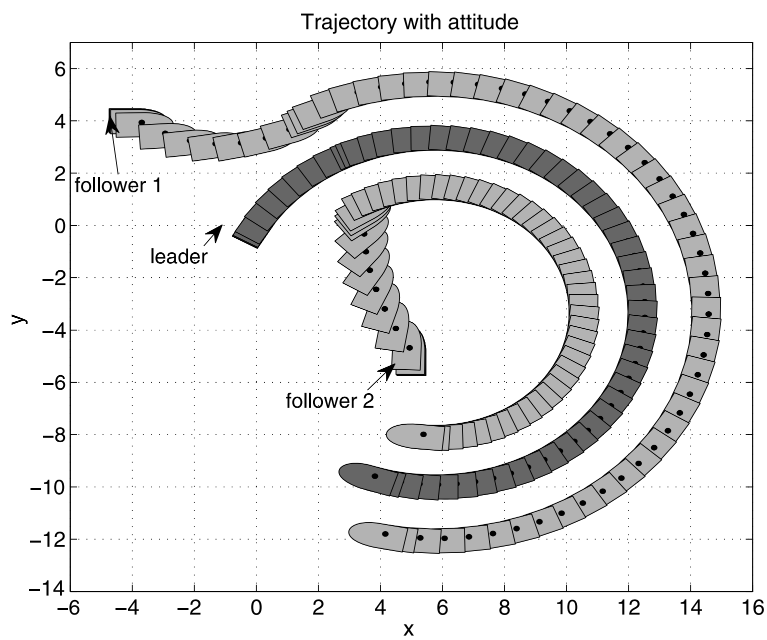

Fig. 3 Finite-time optimal formation tracking for vehicles with non-holonomic dynamics and $t_f = 5$

Figures 3 and 4 show the formation tracking results for vehicles, where the desired relative configurations satisfy the non-holonomic conditions (31). It can be seen from Fig. 4 that the relative configurations are obtained and the vehicles go into the non-holonomic adjoint orbits at the given terminal $t_f = 5$. Afterwards, the two followers have no motion in the Y-direction, which is shown in Fig. 3. This is in marked contrast to the situation in Fig. 1, which has no non-holonomic condition.

Example 3 Now, let us consider the example of finite-time formation tracking control for four vehicles when the communication topology is modeled as a directed tree which is shown in Fig. 5. The initial configurations are given as follows:

$$g_1(0) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -6 \\ 0 & 0 & 1 \end{bmatrix},$$

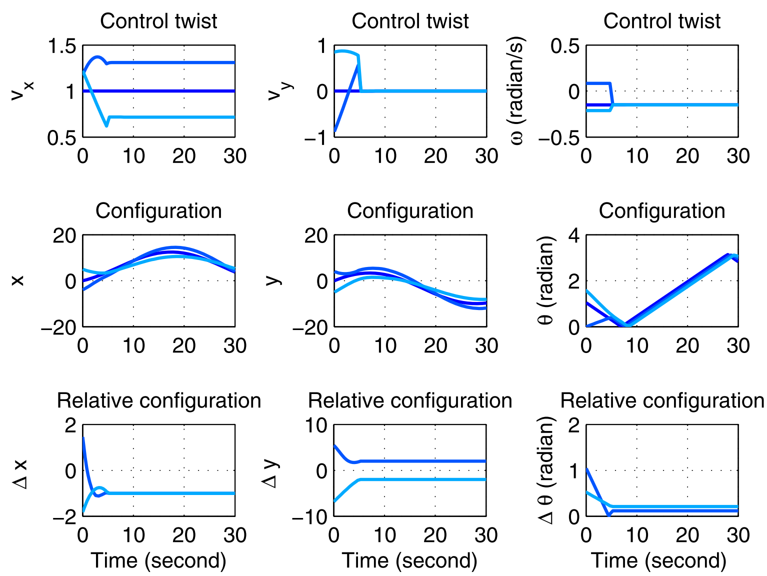

Fig. 4 From top to bottom: control input, configuration and relative configuration (with respect to the reference trajectory); from left to right: with respect to the x and y coordinate and the attitude θ

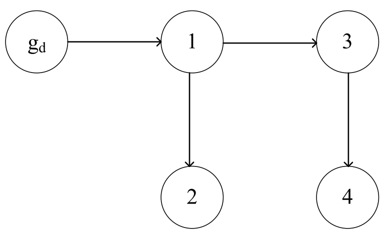

Fig. 5 The directed-tree communication topology

$$g_2(0) = \begin{bmatrix} 0 & 1 & -2 \\ -1 & 0 & 15 \\ 0 & 0 & 1 \end{bmatrix},$$

$$g_3(0) = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} & -10 \\ \frac{\sqrt{3}}{2} & \frac{1}{2} & 8 \\ 0 & 0 & 1 \end{bmatrix},$$

$$g_4(0) = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} & -15 \\ \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

For simplicity, let desired relative configurations be $e_1^* = e_{21}^* = e_{31}^* = e_{43}^* = I$, that is, all vehicles track the desired reference trajectory g_d at the terminal time t_f. The result of tracking control is shown in Figs. 6 and 7, and the vehicles consensus track the desired reference trajectory in finite time.

6 Conclusions

In this paper, we have studied the problem of finite-time optimal formation tracking for vehicles on Lie group SE(2). We first develop an optimal controllers for vehicles with holonomic dynamic to achieve the desired formation tracking at the given terminal time, which is given in advance according to the task requirements. And during the tracking, the given integral performance index is guaranteed to be optimal. Then, the finite-time formation tracking controllers are designed for multi-vehicle systems under a directed-tree communication topology. Furthermore, some sufficiency conditions are proposed for vehicles to guarantee the non-holonomic tracking after the formation time.

Nevertheless, there are still some problems remaining to be solved, such as finite-time optimal formation tracking control for vehicles with non-holonomic dynamic during formation tracking part, finite-time optimal tracking control for multiple vehicles with other network topologies. The solutions of these problems could be important both for theoretical research and for practical applications.

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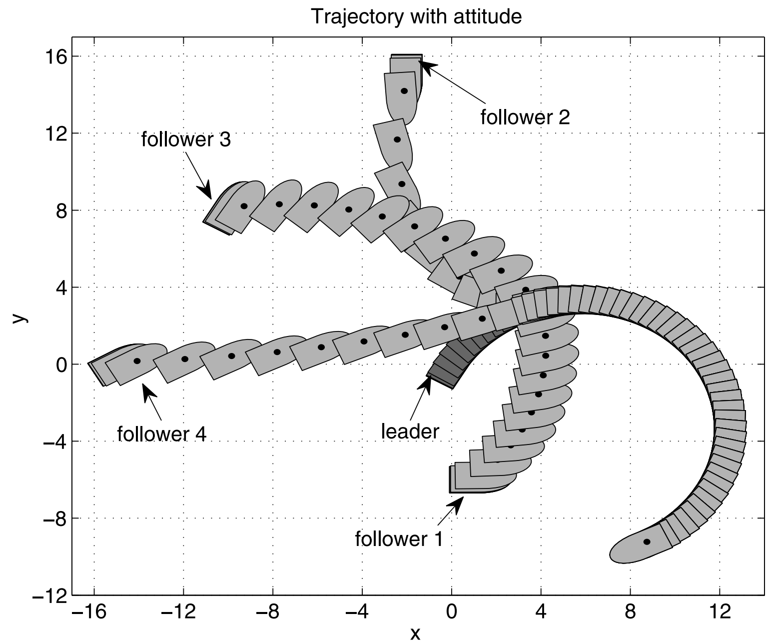

Fig. 6 Finite-time optimal formation tracking for multi-vehicle system with a directed-tree communication topology and $t_f = 5$

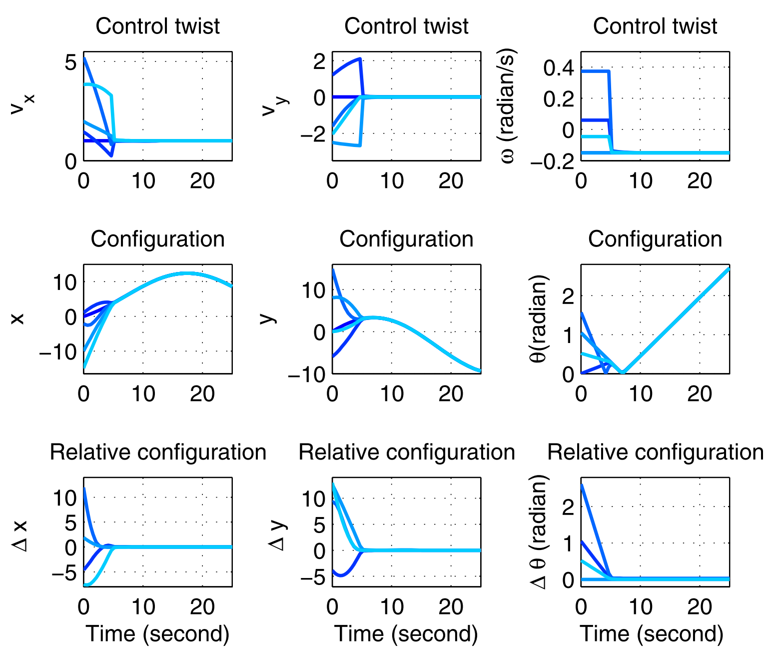

Fig. 7 From top to bottom: control input, configuration and relative configuration (with respect to the reference trajectory); from left to right: with respect to the x and y coordinate and the attitude θ

References

1. Kristiansen, R., Nicklasson, P.J.: Spacecraft formation flying: a review and new results on state feedback control. Acta Astronaut. **65**, 1537–1552 (2009)
2. Chen, J., Sun, D., Yang, J.: A receding-horizon formation tracking controller with leader-follower strategies. In: Proceedings of the 17th World Congress of the International Federation of Automatic Control, Seoul, Lorea, pp. 4400–4405 (2008)
3. Li, W.X., Chen, Z.Q., Liu, Z.X.: Leader-following formation control for second-order multiagent systems with time-varying delay and nonlinear. Nonlinear Dyn. **72**, 803–812 (2013)
4. Han, Y.: Simultaneous translational and rotational tracking in dynamic environments: theoretical and practical viewpoints. IEEE Trans. Robot. Autom. **20**, 309–318 (2004)

5. Dong, R.S., Geng, Z.Y.: Consensus based formation control laws for systems on Lie groups. Syst. Control Lett. **62**, 104–111 (2013)
6. Liu, Y.F., Geng, Z.Y.: Finite-time optimal formation control of multi-agent systems on SE(3). Int. J. Control. (???). doi: 10.1080/00207179.2013.792006
7. Olfati-Saber, R., Murray, R.: Consensus problems in networks of agents with switching topology and time-delays. IEEE Trans. Autom. Control **49**, 1520–1533 (2004)
8. Ren, W., Beard, R.: Consensus seeking in multiagent systems under dynamically changing interaction topologies. IEEE Trans. Autom. Control **50**, 655–661 (2005)
9. Tuna, S.: Conditions for synchronizability in arrays of coupled linear systems. IEEE Trans. Autom. Control **54**, 2416–2420 (2009)
10. Seo, J., Shim, H., Back, J.: Consensus of high-order linear systems using dynamic output feedback compensator: low gain approach. Automatica **45**, 2659–2664 (2009)
11. Qu, Z., Wang, J., Hull, R.: Cooperative control of dynamical systems with application to autonomous vehicles. IEEE Trans. Autom. Control **53**, 894–911 (2008)
12. Lin, Z., Francis, B., Maggiore, M.: State agreement for continuous-time coupled nonlinear systems. SIAM J. Control Optim. **46**, 288–307 (2008)
13. Lalish, E., Morgansen, K.A., Tsukamaki, T.: Formation tracking control using virtual structure and deconfliction. In: Proceedings of the 45th IEEE Conference on Decision and Control, San Diego, CA, USA, pp. 5699–5705 (2006)
14. Do, K.D.: Formation tracking control of unicycle-type mobile robots. In: 2007 IEEE International Conference on Robotics and Automation, Roma, Italy, pp. 2391–2396 (2007)
15. Abdessameud, A., Tayebi, A.: Attitude synchronization of a group of spacecraft without velocity measurements. IEEE Trans. Autom. Control **54**, 2642–2648 (2009)
16. Du, H.B., Li, S.H., Qian, C.J.: Finite-time attitude tracking control of spacecraft with application to attitude synchronization. IEEE Trans. Autom. Control **56**, 2711–2717 (2011)
17. Ou, M.Y., Du, H.B., Li, S.H.: Finite-time tracking control of multiple non-holonomic mobile robots. J. Franklin Inst. **349**, 2834–2860 (2012)
18. Zhao, D.Y., Zou, T.: Finite-time approach to formation control of multiple mobile robots with terminal sliding mode. Int. J. Inf. Syst. Sci. **43**, 1998–2014 (2012)
19. Justh, E.W., Krishnaprasad, P.S.: Equilibria and steering laws for planar formations. Syst. Control Lett. **52**, 25–38 (2004)
20. Nair, S., Leonard, N.E.: Stable synchronization of rigid body networks. Netw. Heterog. Media **2**, 595–624 (2007)
21. Sarlette, A.: Geometry and symmetries in coordination control. PhD dissertation, University of Liège, Belgium (2009)
22. Sarlette, A., Bonnabel, S., Sepulchre, R.: Coordination motion design on Lie groups. IEEE Trans. Autom. Control **55**, 1047–1058 (2011)
23. Bullo, F., Murray, R.M.: Tracking for fully actuated mechanical systems: a geometric gramework. Automatica **35**, 17–34 (1999)
24. Han, Y., Park, F.C.: Least squares tracking on the Euclidean group. IEEE Trans. Autom. Control **46**, 1127–1132 (2001)
25. Lee, T.: Geometric tracking control of the attitude dynamics of a rigid body on SO(3). In: Proceedings of the American Control Conference, San Francisco, CA, USA, pp. 1200–1205 (2011)
26. John, B.: Geometric methods for nonlinear optimal control problems. J. Optim. Theory Appl. **25**, 519–548 (1978)
27. Walsh, G.C., Montgomery, R., Sastry, S.S.: Optimal path planning on matrix Lie groups. In: Proceedings of the 33rd Conference on Decision and Control, Lake Buena Vista, pp. 1258–1263 (1994)
28. Spindlert, K.: Optimal control on Lie groups with applications to attitude control. Math. Control Signals Syst. **11**, 197–219 (1998)
29. Hermann, R., Krener, A.: Nonlinear controllability and observability. IEEE Trans. Autom. Control **22**, 728–740 (1977)
30. Sachkov, Y.L.: Control theory on Lie groups. Int. J. Math. Sci. **156**, 381–439 (2009)
31. Biggs, J., Holderbaum, W., Jurdjevic, V.: Singularities of optimal control problems on some 6-D Lie groups. IEEE Trans. Autom. Control **52**, 1027–1038 (2007)
32. Bryson, A.E., Ho, Y.C.: Applied Optimal Control. Hemisphere, London (1975)
33. Liu, Y.F., Geng, Z.Y.: Finite-time optimal formation control on Lie groups. In: Chinese Conference of Control, Hefei, China, pp. 715–720 (2012)