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Exact solutions for a perturbed nonlinear Schrödinger equation by using Bäcklund transformations

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Abstract The Bäcklund transformation from the Riccati form of inverse method is presented for the Perturbed Nonlinear Schrödinger Equation. Consequently, the exact solutions for Perturbed Nonlinear Schrödinger equation can be obtained by the AKNS class. The technique developed relies on the construction of the wave functions which are solutions of the associated AKNS; that is, a linear eigenvalues problem in the form of a system of PDE. Moreover, we construct a new soliton solution from the old one and its wave function.

Keywords Bäcklund transformation · Perturbed nonlinear Schrödinger equation · Soliton solution · AKNS class

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1 Introduction

In the recent years, there has made noticeable progress in the construction of the exact solutions for nonlinear partial differential equations, which has long been a major concern for both mathematicians and physicists.

The effort in finding exact solutions to nonlinear differential equation (NPDE), when they exist, is very important for the understanding of most nonlinear physical phenomena. For instance, the nonlinear wave phenomena observed in fluid dynamics, plasma and optical fibres are often modelled by the bell shaped sech solutions and the kink shaped tanh solution. Many powerful methods for finding soliton solutions such as the Darboux transformation [13], Hirota bilinear method [14], Lie group method [15], the homogeneous balance method [16].

Nonlinear partial differential integrable by the inverse scattering transform (IST) method form a wide class of soliton solutions. The Bäcklund transformation (BT) technique is one of the direct methods to generate a new solution of a nonlinear evolution equation from a known solution of that equation [11, 15–17]. These BTs explicitly express the new solutions in terms of the known solutions of the nonlinear partial differential equations and the corresponding wave functions which are problem in the form of a system of first-order partial differential equations (PDEs). The basic aim of this paper is to construct the exact solutions for a Perturbed Nonlinear Schrödinger Equation (PNLSE):

$$i\frac{\partial q}{\partial t}+\frac{1}{2}\frac{\partial^2 q}{\partial x^2}+|q|^2 q + i\varepsilon\left(\beta_1\frac{\partial^3 q}{\partial x^3}+\beta_2|q|^2\frac{\partial q}{\partial x}+\beta_3 q\frac{\partial |q|^2}{\partial x}\right)=0, \quad (1)$$

where q represents a normalised complex amplitude of the pulse envelope, t is a normalised distance along the fibre, x is the normalised retarded time, ε is a small parameter and β_1,β_2,β_3 are the real normalised parameters which depend on the fibre characteristics (β_1 is the coefficient of the linear higher-order dispersion effect and β_2,β_3 are overlap integrals [2]). A new model, to include saturation effects of the Kerr nonlinearity, has been recently derived [3], in which the governing equation is a combination of the exponential nonlinear Schrödinger equation and the derivative one. For $\varepsilon=0$ in Eq. (1) we obtain the standard nonlinear Schrödinger equation (NLSE), which is one of the complete integrable nonlinear partial differential equations (NLPDEs). Its solutions can be obtained by different methods, e.g., by the inverse scattering transform [4], the Lie group theory [5]. To the best of our knowledge for arbitrary parameters β_1,β_2,β_3 Eq. (1) is not completely integrable, but for an appropriate choice of the these parameters it can be integrated by the IST. Thus the cases when $\beta_1{:}\beta_2{:}\beta_3=0{:}1{:}1$ (the derivative nonlinear Schrödinger equation (NLSE) type I) was solved in [6], $\beta_1{:}\beta_2{:}\beta_3=0{:}1{:}0$ (the derivative nonlinear Schrödinger equation (NLSE) type II) was solved in [7], $\beta_1{:}\beta_2{:}\beta_3=1{:}6{:}0$ (the Hirota equation) was solved in [8] and $\beta_1{:}\beta_2{:}\beta_3=1{:}6{:}3$ was solved in [9], [10]. With choice $\beta_2=6\beta_1$ and $\beta_3=0$, we have

$$i\frac{\partial q}{\partial t}+\frac{1}{2}\frac{\partial^2 q}{\partial x^2}+|q|^2 q+i\varepsilon\beta_1\left(\frac{\partial^3 q}{\partial x^3}+6|q|^2\frac{\partial q}{\partial x}\right)=0. \quad (2)$$

The article is organised as follows: this introduction in Sect. 1. In Sect. 2, the Ablowitz–Kaup–Newell–Segur (AKNS) system and the general form of the Bäcklund transformations (BTs) for the nonlinear evolution equations (NLEEs) are illustrated. In Sect. 3, a new exact solution class from a known constant solution is obtained for (2). In Sect. 4, a new exact soliton solution class from a known solution (simple function) for (2). In Sect. 5, a new exact soliton solution class from a known solution of a travelling wave for (2).

2 The AKNS system and the BTs for the NLEEs

Consider the AKNS eigenvalues problem defined in the form

$$\Phi_x=P\Phi,\qquad \Phi_t=Q\Phi, \quad (3)$$

where $\Phi=\left[\begin{smallmatrix}\varphi_1\\ \varphi_2\end{smallmatrix}\right]$, P and Q are two 2×2 null-trace matrices

$$P=\begin{bmatrix}\eta & q\\ r & -\eta\end{bmatrix},\qquad Q=\begin{bmatrix}A & B\\ C & -A\end{bmatrix}, \quad (4)$$

where η is a parameter, independent of x and t while q and r are functions of x and t.

The integrability condition reads

$$P_t-Q_x+PQ-QP=0 \quad (5)$$

or in component form

$$-A_x+qC-rB=0, \quad (6)$$

$$q_t-B_x-2Aq+2\eta B=0, \quad (7)$$

$$r_t-C_x-2\eta C+2Ar=0, \quad (8)$$

where A, B and C are functions of η, q and r.

Konno and Wadati [1], introduced the function

$$\Gamma=\frac{\varphi_1}{\varphi_2}. \quad (9)$$

Equations (3) are reduced to the Riccati equations:

$$\frac{\partial\Gamma}{\partial x}=2\eta\Gamma+q-r\Gamma^2, \quad (10)$$

$$\frac{\partial\Gamma}{\partial t}=B+2A\Gamma-C\Gamma^2 \quad (11)$$

we construct a transformation Γ' satisfying the same equation as with a potential $q'(x)$ and, for any of the NLEE, derived a BTs with the following form:

$$q'(x)=q(x)+F(\Gamma,\eta), \quad (12)$$

where q is the old solution and q' is the a new solution of the corresponding NLEEs. In the following section, we expound the new form A, B, C and r.

Consider

$$P=\begin{bmatrix}\eta & q\\ -q^* & -\eta\end{bmatrix},\qquad Q=\begin{bmatrix}A & B\\ C & -A\end{bmatrix},\tag{13}$$

where

$$A=\frac{i}{2}|q|^2+\varepsilon\beta_1\left(qq_x^*-q_xq^*\right)-2\varepsilon\beta_1\eta|q|^2 + i\eta^2-4\varepsilon\beta_1\eta^3,\tag{14}$$

$$B=\frac{i}{2}q_x-\varepsilon\beta_1\left(2|q|^2q+q_{xx}\right)-2\varepsilon\beta_1\eta q_x + i\eta q-4\varepsilon\beta_1\eta^2q,\tag{15}$$

$$C=\frac{i}{2}q_x^*+\varepsilon\beta_1\left(2|q|^2q^*+q_{xx}^*\right)-2\varepsilon\beta_1\eta q_x^* - i\eta q^*+4\varepsilon\beta_1\eta^2q^*\tag{16}$$

substitute from Eqs. (13)–(16) into Eqs. (6)–(8), then Eq. (8) gives a perturbed nonlinear Schrödinger equation (PNLSE) (2). To derive the new solution q' from the known solution q, Eq. (10) becomes

$$\frac{\partial\Gamma}{\partial x}=2\eta\Gamma+q+q^*\Gamma^2.\tag{17}$$

If we choose Γ' and q' as

$$\Gamma'=\frac{1}{\Gamma^*}\tag{18}$$

$$q'(x)=q(x)+2\frac{\Gamma^2(\frac{\partial\Gamma^*}{\partial x})-(\frac{\partial\Gamma}{\partial x})}{1-|\Gamma|^4}\tag{19}$$

then Γ' with $q'(x)$ satisfies Eq. (17) for real η. Equation (19) reduces to the following BTs form:

$$q'(x)=-q(x)-4\eta\frac{\Gamma}{1+|\Gamma|^2}.\tag{20}$$

3 The know solution is a constant

Let $q=0$ be a solution of Eq. (2), then, the matrices P and Q take the following form:

$$P=\begin{bmatrix}\eta & 0\\ 0 & -\eta\end{bmatrix},\quad Q=\begin{bmatrix}i\eta^2-4\varepsilon\beta_1\eta^3 & 0\\ 0 & -(i\eta^2-4\varepsilon\beta_1\eta^3)\end{bmatrix}.\tag{21}$$

From Eqs. (3)–(4)

$$d\Phi=\Phi_x\,dx+\Phi_t\,dt=(P\,dx+Q\,dt)\Phi,\tag{22}$$

from Eq. (21), we get

$$Q=\left(i\eta-4\varepsilon\beta_1\eta^2\right)P,\tag{23}$$

substitute from Eq. (23) into Eq. (22), we get

$$d\Phi=PQ\,d\rho,\tag{24}$$

where

$$\rho=x+kt;\quad k=i\eta-4\varepsilon\beta_1\eta^2.\tag{25}$$

By solving Eq. (24), we obtain the following solution:

$$\Phi=\Phi_0e^{P\rho} = \left[I+\rho P+\frac{1}{2!}\rho^2P^2+\frac{1}{3!}\rho^3P^3+\cdots\right]\Phi_0,\tag{26}$$

$$\Phi=\begin{bmatrix}\cosh\eta\rho+\sinh\eta\rho & 0\\ 0 & \cosh\eta\rho-\sinh\eta\rho\end{bmatrix}\Phi_0,\tag{27}$$

$$\Phi=\begin{bmatrix}e^{\eta\rho} & 0\\ 0 & e^{-\eta\rho}\end{bmatrix}\Phi_0,\tag{28}$$

where Φ_0 is a constant column vector, now we choose $\Phi_0=(1,1)^T$ in Eq. (28), we obtain

$$\Phi=\begin{bmatrix}\varphi_1\\ \varphi_2\end{bmatrix}=\begin{bmatrix}e^{\eta\rho}\\ e^{-\eta\rho}\end{bmatrix}.\tag{29}$$

By using Eqs. (9) and (29), then the Bäcklund transformations (20) gives the new solution of the PNLSE (2) corresponding to the know constant PNLSE solution $q=0$ as follows:

$$q'(x)=-2\eta e^{2i\eta^2t}\sec h\left[2\eta(\rho-i\eta t)\right];\quad \rho=x+\left(i\eta-4\varepsilon\beta_1\eta^2\right)t.\tag{30}$$

4 The know solution $q=q(x,t)$ is a simple function

In this case, Eqs. (3)–(4) cannot be solved for the vector Φ as a whole, but it can be solved in the components φ_1,φ_2 separately. After substituting for the known solution $q(x,t)$ of the PNLSE into the corresponding matrices P and Q, we will have the following system for the unknowns φ_1, φ_2:

$$\varphi_{1x} = \eta\varphi_1 + q\varphi_2, \tag{31}$$

$$\varphi_{2x} = r\varphi_1 - \eta\varphi_2, \tag{32}$$

$$\varphi_{1t} = A\varphi_1 + B\varphi_2, \tag{33}$$

$$\varphi_{2t} = C\varphi_1 - A\varphi_2. \tag{34}$$

These equations are compatible under the conditions of the assumed values of matrices P and Q connected with the PNLSE under consideration. Solve for φ_1 from Eq. (32), we get

$$\varphi_1 = \frac{1}{r}(\varphi_{2x} + \eta\varphi_2). \tag{35}$$

Substituting from Eq. (35) of φ_1 and Eq. (8) into Eq. (34), we get

$$C\varphi_{2x} - r\varphi_{2t} = \frac{1}{2}(C_x - r_t)\varphi_2. \tag{36}$$

Equation (36) is a linear first-order partial differential equation with φ_2 and we can be solved by the method of characteristics. After, occurrence φ_2 from Eq. (36) and substituting it into Eq. (35), we will obtain φ_1. Thus we have obtained two general solutions φ_1 and φ_2, which contain an arbitrary function F. This arbitrary function can be determined by substitution for φ_1, φ_2 into Eqs. (31) or (33) which will yield a second-order linear ordinary differential equation with function F as its unknown. If we can solve for the function F, we will obtain the two particular solutions φ_1 and φ_2. Finally, by applying (9) and the Bäcklund transformations for the PNLSE (20), we shall obtain a new solution of the PNLSE, we will apply this technique for the following example:

Example Let

$$q(x,t) = a_0 \exp\left[i\sqrt{2}a_0\left(x - 4\varepsilon\beta_1 a_0^2 t\right) + a_3\right] \tag{37}$$

be a solution of the perturbed nonlinear Schrödinger equation (2), substituting from Eq. (37) into (36) with Eqs. (13)–(16), we obtain

$$h\varphi_{2x} + \varphi_{2t} = -\frac{i\sqrt{2}}{2}a_0\left(h - 4\varepsilon\beta_1 a_0^2\right)\varphi_2, \tag{38}$$

where

$$h = \frac{\sqrt{2}}{2}a_0 + 2\sqrt{2}i\varepsilon\beta_1 a_0\eta - i\eta + 4\varepsilon\beta_1\eta^2, \tag{39}$$

Equation (39) has the following system of ordinary differential equations (ODEs) as its characteristic equations:

$$\frac{dx}{dt} = h, \tag{40}$$

$$\frac{d\varphi_2}{dt} = -\frac{i\sqrt{2}}{2}a_0\left(h - 4\varepsilon\beta_1 a_0^2\right)\varphi_2. \tag{41}$$

Solving the two Eqs. (40), (41) gives the general solution of φ_2, which reads

$$\varphi_2 = F(\xi)\exp\left[-\frac{i\sqrt{2}}{2}a_0\left(h - 4\varepsilon\beta_1 a_0^2\right)t\right], \tag{42}$$

$$\xi = x - ht, \tag{43}$$

where $F(\xi)$ is an arbitrary function. Substituting Eqs. (37), (42) and (43) into (35) gives the general solution of φ_1, which reads

$$\varphi_1 = -\frac{1}{a_0}\left(F' + \eta F\right)e^{\gamma} \tag{44}$$

where

$$\gamma = i\sqrt{2}a_0\left(x - 2\varepsilon\beta_1 a_0^2 t\right) - \frac{i\sqrt{2}}{2}ha_0 t + a_3. \tag{45}$$

To determine the arbitrary function $F(\xi)$, we substitute from Eqs. (37), (42), (43) and (44) into (31); then $F(\xi)$ must satisfy the following second-order linear ODE:

$$F'' + 2\alpha F' + \beta^2 F = 0, \tag{46}$$

where "$'$" denotes to $\frac{d}{d\xi}$ and

$$\alpha = \frac{i\sqrt{2}}{2}a_0, \qquad \beta^2 = a_0^2 + i\sqrt{2}a_0\eta - \eta^2. \tag{47}$$

The general solution of Eq. (46) for the arbitrary function $F(\xi)$ is

$$F(\xi) = \left[k_3\cosh\left(\sqrt{\alpha^2 - \beta^2}\xi\right) + k_4\sinh\left(\sqrt{\alpha^2 - \beta^2}\xi\right)\right]e^{-\alpha\xi}, \tag{48}$$

where k_3, k_4 are arbitrary constants.

Substituting Eq. (48) into Eqs. (42), (44) and by applying (9), we obtain

$$\Gamma = -\frac{1}{a_0}\left[\frac{d(\log F)}{d\xi} + \eta\right] \times \exp\left[i\sqrt{2}a_0\left(x - 4\varepsilon\beta_1 a_0^2 t\right) + a_3\right]. \tag{49}$$

Then substituting this Γ and (37) into BTs (20) gives the new solution of the PNLSE corresponding to a simple function (37):

$$q' = -a_0\left[1 - 4\eta\frac{\frac{d(\log F)}{d\xi} + \eta}{a_0^2 + [\frac{d(\log F)}{d\xi} + \eta][\frac{d(\log F^*)}{d\xi^*} + \eta]}\right] \times \exp\left[i\sqrt{2}a_0\left(x - 4\varepsilon\beta_1 a_0^2 t\right) + a_3\right], \tag{50}$$

where ξ^* is the conjugate of ξ which is defined in Eq. (43).

5 The know solution is a travelling wave

In this case we suppose that the components of q and r of the matrix P are functions of ρ:

$$q = q(\rho), \qquad r = r(\rho), \quad \text{where } \rho = x - kt. \tag{51}$$

Then the components of A, B and C of the matrix Q determined by Eqs. (6)–(8) are functions of ρ:

$$A = A(\rho), \qquad B = B(\rho) \quad \text{and} \quad C = C(\rho). \tag{52}$$

We require the quantity

$$\delta = (A + k\eta)^2 + (B + kq)(C + kr) \tag{53}$$

to be constant with respect to x and t. Solving Eqs. (31)–(34) by applying the method of characteristics. The partial differential equations (31)–(34) possesses the following characteristics equations:

$$\frac{dx}{C} = \frac{dt}{-r} = \frac{2d\varphi_2}{(C_x - r_t)\varphi_2}. \tag{54}$$

Substituting from (51), (52) into (54), we get

$$\frac{dt}{-r} = \frac{d\rho}{C + kr} = \frac{2d\varphi_2}{(C + kr)'_\rho \varphi_2}. \tag{55}$$

These equations gives the following system of ODEs:

$$\frac{d(\ln\varphi_2)}{d\rho} = \frac{(C + kr)'_\rho}{2(C + kr)}, \tag{56}$$

$$\frac{d\rho}{dt} = \frac{-(C + kr)}{r}. \tag{57}$$

These two equations gives the general solutions

$$\varphi_2 = k_2(C + kr)^{1/2}, \tag{58}$$

$$-t + k_1 = \int \frac{r\, d\rho}{(C + kr)}, \tag{59}$$

where k_1, k_2 are integration constants.

Denoting

$$\sigma(\rho) = \int \frac{r\, d\rho}{(C + kr)}. \tag{60}$$

Substituting from (60) into (59), we have

$$\sigma(\rho) + t = k_1 \tag{61}$$

and from (58) and (61), we obtain the general solution of Eq. (36)

$$\varphi_2 = F(\xi)(C + kr)^{1/2}, \tag{62}$$

$$\xi = \sigma(\rho) + t. \tag{63}$$

Substituting (62) into (35) gives the solution for φ_1:

$$\varphi_1 = (C + kr)^{-1/2}\left(F'_\xi + (A + k\eta)F\right). \tag{64}$$

To determine the function F, we substitute Eqs. (62) and (64) into (31), then $F(\xi)$ must satisfy the following second-order ODE:

$$F''_{\xi\xi} - \delta F = 0, \tag{65}$$

where δ is a constant defined in (53). According to the sign of δ, Eq. (65) have the following three different solutions:

$$F(\xi) = c_1\xi + c_2, \quad \text{when } \delta = 0, \tag{66}$$

$$F(\xi) = c_1 \sinh\omega(\xi + c_2), \quad \text{when } \delta \succ 0,\ \omega^2 = \delta, \tag{67}$$

$$F(\xi) = c_1 \sin\omega(\xi + c_2), \quad \text{when } \delta \prec 0,\ \omega^2 = -\delta, \tag{68}$$

where c_1 and c_2 are integrations constants. Substituting these solutions into (62) and (64), we obtain the corresponding different solutions of system (3) and (4):

- when $\delta = 0$

$$\begin{bmatrix}\varphi_1\\ \varphi_2\end{bmatrix} = \begin{bmatrix}(C+kr)^{-1/2}[(A+k\eta)(c_1\xi+c_2)+c_1]\\ (C+kr)^{1/2}(c_1\xi+c_2)\end{bmatrix}, \tag{69}$$

- when $\delta \succ 0$

$$\begin{bmatrix}\varphi_1\\ \varphi_2\end{bmatrix} = \begin{bmatrix}c_1(C+kr)^{-1/2}[(A+k\eta)\sinh\omega(\xi+c_2)+\omega\cosh\omega(\xi+c_2)]\\ c_1(C+kr)^{1/2}\sinh\omega(\xi+c_2)\end{bmatrix}, \tag{70}$$

- when $\delta \prec 0$

$$\begin{bmatrix}\varphi_1\\ \varphi_2\end{bmatrix} = \begin{bmatrix}c_1(C+kr)^{-1/2}[(A+k\eta)\sin\omega(\xi+c_2)+\omega\cos\omega(\xi+c_2)]\\ c_1(C+kr)^{1/2}\sin\omega(\xi+c_2)\end{bmatrix}. \tag{71}$$

Equations (69)–(71) are satisfy for any NLEE contained in the AKNS system (3) and (4), provided they satisfy assumptions (51)–(53).

Now, we apply these results and the known travelling wave solution of the perturbed nonlinear Schrödinger equation (PNLSE) to find a new solution of the corresponding (PNLSE) by using the BTs for the following example.

Example Let

$$q(x,t) = -2\eta\exp\big[2i\eta^2 t\big]\sec h[2\eta\rho] \tag{72}$$

where

$$\rho = x - kt; \quad k = 4\varepsilon\beta_1\eta^2 \tag{73}$$

be a solution of the perturbed nonlinear Schrödinger equation (2), substituting from (72) and (73) into Eqs. (13)–(16) to find A, B, C and r, we get

$$\begin{aligned}
A &= \eta^2(i-4\varepsilon\beta_1\eta)\big(2\sec h^2[2\eta\rho]+1\big),\\
B &= 2\eta^2\big[(i-4\varepsilon\beta_1\eta)\big(\tanh[2\eta\rho]-1\big)\\
&\quad + 4\varepsilon\beta_1\eta\big]\sec h[2\eta\rho]\exp\big[2i\eta^2 t\big],\\
C &= 2\eta^2\big[(i-4\varepsilon\beta_1\eta)\big(\tanh[2\eta\rho]+1\big)\\
&\quad - 4\varepsilon\beta_1\eta\big]\sec h[2\eta\rho]\exp\big[-2i\eta^2 t\big],\\
r &= -q^* = 2\eta\exp\big[-2i\eta^2 t\big]\sec h[2\eta\rho].
\end{aligned} \tag{74}$$

Substituting from Eq. (74) into (60)–(63), we obtain

$$\begin{aligned}
A+k\eta &= \eta^2\big[i+2(i-4\varepsilon\beta_1\eta)\sec h^2[2\eta\rho]\big],\\
C+kr &= 2\eta^2\big[(i-4\varepsilon\beta_1\eta)\big(1+\tanh[2\eta\rho]\big)\\
&\quad \times\sec h[2\eta\rho]\big]\exp\big[-2i\eta^2 t\big],\\
\xi &= \frac{1}{\eta}\int\frac{d\rho}{(i-4\varepsilon\beta_1\eta)(\tanh[2\eta\rho]+1)}+t\\
&= \frac{-1+4\eta\rho+(1+4\eta\rho)\tanh[2\eta\rho]}{8\eta^2(i-4\varepsilon\beta_1\eta)(1+\tanh[2\eta\rho])}+t.
\end{aligned} \tag{75}$$

Substituting from (74) and (75) into Eqs. (69)–(71), we have

- when $\delta = 0$

$$\begin{aligned}
\varGamma &= \frac{\exp[-2i\eta^2 t]}{2\eta^2[(i-4\varepsilon\beta_1\eta)(1+\tanh[2\eta\rho])\sec h[2\eta\rho]]}\\
&\quad\times\Big[\eta^2\big[i+2(i-4\varepsilon\beta_1\eta)\sec h^2[2\eta\rho]\big]\\
&\quad+\frac{1}{\xi+c}\Big];\\
c &= \frac{c_2}{c_1}
\end{aligned} \tag{76}$$

substituting Eq. (76) into the BTs (20) to find the new solution q' of the PNLSE (2) corresponding to the known solution (72):

$$q'(x) = 2\eta\exp\big[2i\eta^2 t\big]\sec h[2\eta\rho] - 4\eta\frac{\varGamma}{1+|\varGamma|^2}, \tag{77}$$

where Γ, ξ are defined into (76) and (75), respectively.

- when $\delta \succ 0$

$$\Gamma = \frac{\exp[-2i\eta^2 t]}{2\eta^2[(i-4\varepsilon\beta_1\eta)(1+\tanh[2\eta\rho])\sec h[2\eta\rho]]} \times \left[\eta^2\left[i+2(i-4\varepsilon\beta_1\eta)\sec h^2[2\eta\rho]\right] + \omega\coth\omega(\xi+c_2)\right]; \quad \delta=\omega^2 \tag{78}$$

substituting Eq. (78) into the BTs (20) to find the new solution q' of the PNLSE (2) corresponding to the known solution (72):

$$q'(x) = 2\eta\exp\left[2i\eta^2 t\right]\sec h[2\eta\rho] - 4\eta\frac{\Gamma}{1+|\Gamma|^2}, \tag{79}$$

where Γ, ξ are defined into (78) and (75), respectively.

- when $\delta \prec 0$

$$\Gamma = \frac{\exp[-2i\eta^2 t]}{2\eta^2[(i-4\varepsilon\beta_1\eta)(1+\tanh[2\eta\rho])\sec h[2\eta\rho]]} \times \left[\eta^2\left[i+2(i-4\varepsilon\beta_1\eta)\sec h^2[2\eta\rho]\right] + \omega\cot\omega(\xi+c_2)\right]; \quad \delta=-\omega^2 \tag{80}$$

substituting Eq. (80) into the BTs (20) to find the new solution q' of the PNLSE (2) corresponding to the known solution (72):

$$q'(x) = 2\eta\exp\left[2i\eta^2 t\right]\sec h[2\eta\rho] - 4\eta\frac{\Gamma}{1+|\Gamma|^2}, \tag{81}$$

where Γ, ξ are defined into (80) and (75), respectively.

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