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A novel fractional-order hyperchaotic system stabilization via fractional sliding-mode control

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Abstract In this paper we numerically investigate the fractional-order sliding-mode control for a novel fractional-order hyperchaotic system. Firstly, the dynamic analysis approaches of the hyperchaotic system involving phase portraits, Lyapunov exponents, bifurcation diagram, Lyapunov dimension, and Poincaré maps are investigated. Then the fractional-order generalizations of the chaotic and hyperchaotic systems are studied briefly. The minimum orders we found for chaos and hyperchaos to exist in such systems are 2.89 and 3.66, respectively. Finally, the fractional-order sliding-mode controller is designed to control the fractional-order hyperchaotic system. Numerical experimental examples are shown to verify the theoretical results.

Keywords Hyperchaos · The minimum order · Fractional-order sliding-mode control · Numerical simulation

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1 Introduction

The hyperchaotic system is a higher-dimensional chaotic system. Having more than one positive Lyapunov exponents causes the system to show behaviors with a high degree of disorder and randomness. It has the virtue of wide bandwidths and exhibits more complex and richer dynamical behaviors. So it has great potential in technological applications, such as secure communication, lasers, neural networks, biological system, and so on [1–7].

In recent years, there is increasing interest in fractional calculus, which allows one describe a real object more accurately and more adequately than the integer methods [8–12] because of the unlimited memory of a fractional-order operator [13, 14]. It has been found that many physical systems can be properly described by using the fractional-order system theory [15–18]. During the past several years, a large number of various fractional-order systems have been proposed, such as the fractional-order Chua system, the fractional-order Rossler equation, the fractional-order Chen system, and the fractional-order Liu system [19–23].

The effective dimension is defined as the sum of orders of all involved derivatives. The minimum order has been numerically calculated for various systems, such as the fractional-order Chua system of order as low as 2.7 can produce a chaotic attractor, chaos can exist in the fractional-order Rossler equation with order as low as 2.4, and hyperchaos can also exist in the fractional-order hyperchaotic Rossler system with order as low as 3.8 [20], the lowest order to have chaos in fractional-order Chen system is 0.3, and so on. In this article we will numerically calculate the minimum orders of the fractional-order chaotic and hyperchaotic systems.

In 1988, Oustaloup proposed the idea of designing a fractional-order controller, called CRONE, which is a robust fractional-order control scheme [24–26]. Then some fractional-order control strategies are proposed one after another, such as the fractional-order PID controller[27], sliding-mode controllers [28–31], optimal controllers [32, 33], adaptive controllers [34–37], and so on.

The sliding-mode control (SMC) technique as one of the most attractive robust nonlinear control methods has been widely applied for both linear and nonlinear systems [38–44]. SMC is an effective robust control strategy with the feature of switching the control law to force the state trajectories of the system from the initial states onto some predefined sliding manifold. In Ref. [40], the authors have proposed a sliding-mode nonlinear PI control scheme. Wang et al. [41] design the sliding-mode controller of the uncertain chaotic system that contains sector nonlinearity and dead zone inputs, to achieve stabilization for the equilibrium points. Particularly, in order to determine the convergence rate, a class of proportional integral switching surface is introduced in Ref. [42]. The authors proposed an observer-based fuzzy neural sliding mode control scheme for interconnected unknown chaotic systems [43]. In Ref. [44], to solve the constraint of the maximum admissible values of the control inputs, the authors introduced the time-varying sliding-mode control that does not violate environmental and technical constraints by selecting the switching line parameters, to obtain the best possible control quality. In recent years, some works have been done dealing with the fractional-order sliding-mode control that combine the merits of fractional-order controllers and sliding-mode controllers [31, 45–47]. Tavazoei and Haeri [46] proposed an active SMC to synchronize fractional-order chaotic systems. In Ref. [47], the authors designed a SMC to control a class of fractional-order chaotic systems. In Ref. [31], based on the stability theorems for fractional-order linear systems, an active sliding-mode controller that has the integer-order sliding-mode surface is proposed to consider the modified projective synchronization for two different fractional-order systems. In this article a fractional-order sliding manifold, which is the combination of fractional calculus theory and the SMC technique, is designed. The system on the fractional-order sliding manifold has desired properties such as good stability, disturbance rejection ability, and tracking capability.

In the present paper, we will numerically investigate the hyperchaotic system, the minimum orders of the fractional-order systems, and the fractional-order SMC for the fractional-order hyperchaotic system. The rest of the paper is organized as follows. In Sect. 2, we introduce the basic dynamical properties of a novel hyperchaotic system in detail. Section 3 is on the minimum orders of the fractional-order chaotic and hyperchaotic systems. In Sect. 4, the design procedure of the fractional-order sliding mode approach is presented. Section 5 concludes this paper with some additional remarks.

2 A novel hyperchaotic system

Consider the following simple three-dimensional (3D) quadratic smooth autonomous system:

$$\begin{cases} \frac{dx}{dt} = a(y-x), \\ \frac{dy}{dt} = bx + lxz, \\ \frac{dz}{dt} = -hx^2 - ky^2 - cz, \end{cases} \tag{1}$$

where $[x, y, z]^T \in R^3$ is the state vector, and a, b, c, h, k, and l are positive constant parameters of the system.

Adding an additional state w to the 3D chaotic system(1),a novel hyperchaotic system can be generated. The differential equations are shown as follows:

$$\begin{cases} \frac{dx}{dt} = a(y-x), \\ \frac{dy}{dt} = bx + lxz + ew, \\ \frac{dz}{dt} = -hx^2 - ky^2 - cz - nw, \\ \frac{dw}{dt} = -dx. \end{cases} \tag{2}$$

When $a = 10$, $b = 40$, $c = 2.5$, $d = 10$, $e = 1$, $h = 2$, $k = 2$, $l = 1$, and $n = 1$, system (2) has two hyperchaotic strange attractors as shown in Figs. 1 and 2, where the initial values are appointed as (0.3, 0.6, 0.9, 1).

By calculation, the unique equilibrium point of the system is $O(0, 0, 0, 0)$. The Jacobian matrix is

$$J = \begin{bmatrix} -a & a & 0 & 0 \\ b + lz & 0 & lx & e \\ -2hx & -2ky & -c & -n \\ 27 - d & 0 & 0 & 0 \end{bmatrix}$$

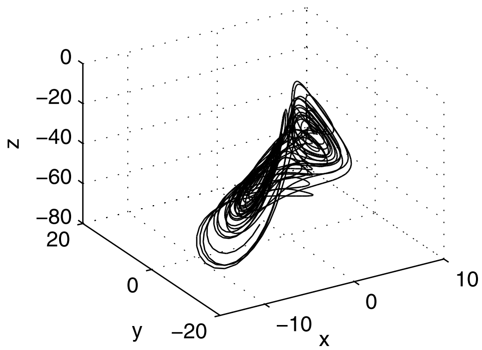
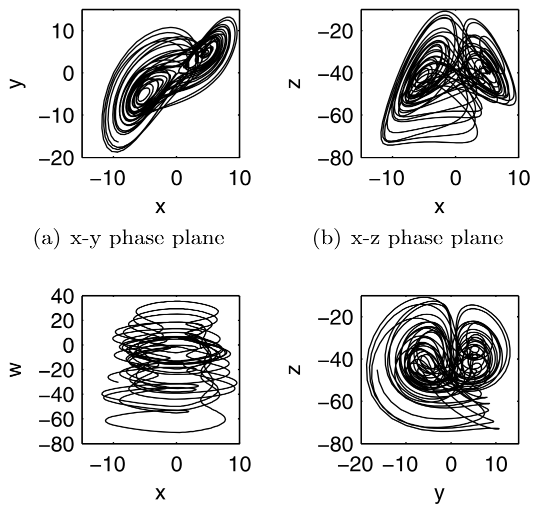


Fig. 1 Three-dimensional (x, y, z) view



(a) x-y phase plane (b) x-z phase plane

(c) x-w phase plane (d) y-z phase plane

Fig. 2 Phase plane strange attractors

$$= \begin{bmatrix} -10 & 10 & 0 & 0 \\ 40 & 0 & 0 & 1 \\ 0 & 0 & -2.5 & -1 \\ -10 & 0 & 0 & 0 \end{bmatrix}. \tag{3}$$

Then letting $|J - \lambda I| = 0$, the eigenvalues corresponding to $O(0, 0, 0, 0)$ are

$$\lambda_1 = -2.5, \quad \lambda_2 = -25.71, \quad \lambda_3 = 15.46, \quad \lambda_4 = 0.25. \tag{4}$$

Here $\lambda_{1,2} < 0$ are negative roots, and $\lambda_{3,4} > 0$ are positive roots. Therefore, the equilibrium $O(0, 0, 0, 0)$ is a saddle.

Then we analyze the Poincaré mapping of this nonlinear system. As we can see in Fig. 3, the Poincaré mappings are in confusion at these points.

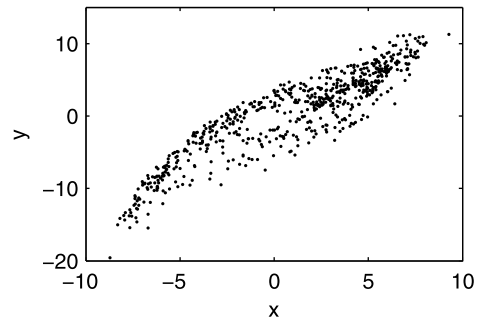


Fig. 3 Poincaré map of the x–y plane of the system

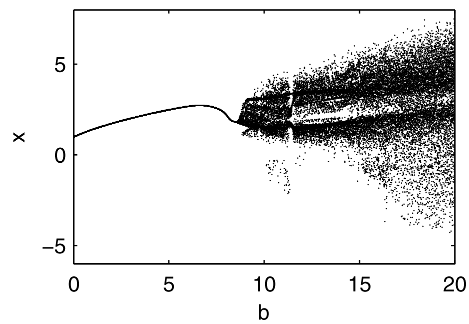


Fig. 4 Bifurcation diagram versus b

The bifurcation diagram of x with increasing b is given in Fig. 4, and it shows abundant and complex dynamical behaviors.

In general, the hyperchaotic system has two positive Lyapunov exponents. According to the chaos theory, the Lyapunov exponents measure the exponential rates of convergence and divergence of the nearby trajectories in the phase space of the system. This implies that the dynamics of the hyperchaotic system are expanded in several different directions simultaneously.

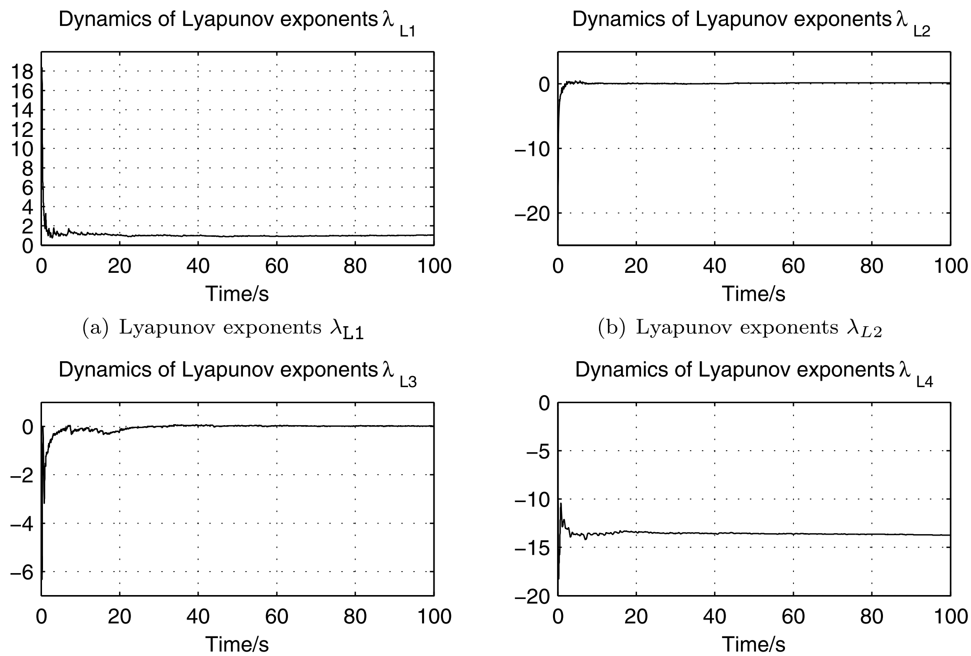
The Lyapunov exponents of this system are respectively obtained from Fig. 5 as

$$\lambda_{L1} = 1.0406, \quad \lambda_{L2} = 0.1847, \quad \lambda_{L3} = 0.0164, \quad \lambda_{L4} = -13.7416. \tag{5}$$

The Lyapunov dimension of system (2) is

$$D_L = j + \frac{1}{\lambda_{j+1}} \sum_{i=1}^{j} \lambda_i = 3 + \frac{1}{\lambda_{L4}} (\lambda_{L1} + \lambda_{L2} + \lambda_{L3}) = 2.91. \tag{6}$$

We fix the parameters at $a = 10$, $c = 2.5$, $d = 10$, $e = 1$, $h = 2$, $k = 2$, $l = 1$, $n = 1$, and let b vary. Then



(a) Lyapunov exponents λ_{L1} (b) Lyapunov exponents λ_{L2}

(c) Lyapunov exponents λ_{L3} (d) Lyapunov exponents λ_{L4}

Fig. 5 Time evolution of Lyapunov exponents for $b = 40$

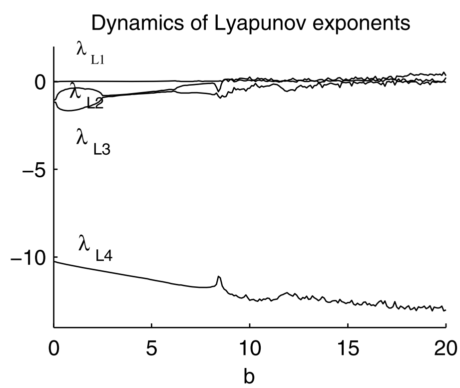


Fig. 6 Lyapunov exponents of the system for varying b

we get the evolution of the Lyapunov exponents as shown in Fig. 6.

The designed circuit, which is shown in Fig. 7, consists of operational amplifiers, multipliers, resistors, and capacitors. Op-Amps is AD741, and we select AD633 for multipliers $C_1 = C_2 = C_3 = C_4 = 1\mu$, $R_{13} = R_{18} = R_{19} = 1k$, $R_5 = R_6 = R_{12} = 1k$, $R_7 = R_{16} = 40k$, $R_{21} = 2.5k$, $R_1 = R_3 = R_9 = R_{14} = 100k$, $R_2 = R_4 = R_8 = 10k$, $R_{10} = R_{11} = R_{15} = 10k$, and $R_{17} = R_{20} = R_{22} = 10k$. The four state variables x, y, z, and w are respectively obtained from the voltage outputs of V_{c1}, V_{c2}, V_{c3}, and V_{c4}.

The circuit is simulated using Pspice. The phase diagrams are shown in Fig. 8. It is in agreement with the numerical simulations in Matlab environment, which are mentioned above.

3 The fractional-order chaotic and hyperchaotic systems

In fractional calculus, ${}_aD_t^\alpha$ denotes a noninteger-order differ-integral operator. It is a notation for taking both the fractional derivative and integral in a single expression and is defined by

$$ {}_aD_t^\alpha = \begin{cases} \frac{d^\alpha}{dt^\alpha}, & \alpha > 0, \\ 1, & \alpha = 0, \\ \int_a^t (d\tau)^{-\alpha}, & \alpha < 0. \end{cases} \tag{7} $$

There exist some different definitions for fractional derivatives [16]. Riemann–Liouville, Caputo, and Grunwald–Letnikov definitions are commonly

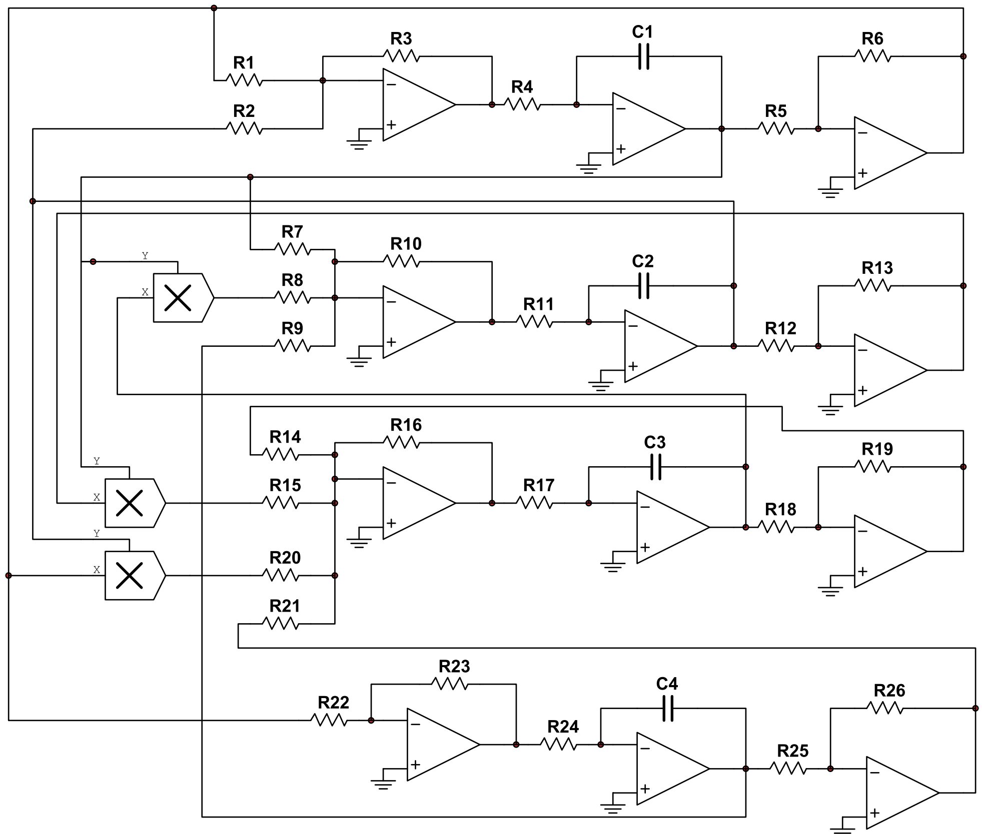


Fig. 7 Analog electronic implementation of the hyperchaotic system Op-Amps: AD741, Multiplier: AD633

used. The well-known Riemann–Liouville definition of the fractional differential operator is given as

$$\frac{d^\alpha f(t)}{dt^\alpha} = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t \frac{f(\tau)}{(t-\tau)^{\alpha-n+1}} d\tau, \qquad (8)$$

where $n-1 < \alpha < n$, i.e., n is the first integer which is not less than α, and Γ is the gamma function. Another alternative definition of the Riemann–Liouville function was reported by Caputo as follows:

$$\frac{d^\alpha f(t)}{dt^\alpha} = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha-n+1}} d\tau. \qquad (9)$$

In this paper, the operator D^α is used to denote the Caputo fractional derivative of order α.

To obtain the minimum order for the fractional-order chaotic system, the following lemma is presented.

Lemma 1 *For a given fractional-order linear system* $D^\alpha X(t) = AX(t)$, $X(0) = X_0$, *where* $0 < a < 1$, $X(t) \in R^n$, *and* $A \in R^{n\times n}$, *the equilibrium points are asymptotically stable for* $\alpha_1 = \alpha_2 = \cdots = \alpha_n \equiv \alpha$ *if all the eigenvalues* λ_i $(i = 1, 2, \ldots, n)$ *of the Jacobian matrix* $\mathrm{J} = \partial\mathrm{f}/\partial\mathrm{x}$, *where* $\mathrm{f} = [f_1, f_2, \ldots, f_n]^T$ *is evaluated at the equilibrium, satisfy the following condition* [48]: $|\arg(\lambda_i(A))| > \alpha\pi/2$ $(i = 1, 2, \ldots, n)$. *The stable and unstable regions for* $0 < a < 1$ *are shown in Fig.* 9.

Suppose that a 3D chaotic system has the unstable eigenvalues $\lambda_{1,2} = a_{1,2} + ib_{1,2}$ of equilibrium points. According to Lemma 1, if the condition for commensurate derivative order is

$$\alpha > \frac{2}{\pi} \operatorname{atan}(b_j/a_j), \quad j = 1, 2, \qquad (10)$$

the system will exhibit double-scroll attractors. In other words, a necessary condition for fractional-order systems to remain chaotic is keeping at least one eigenvalue λ in the unstable region. We can use this condition to determine the minimum order for a

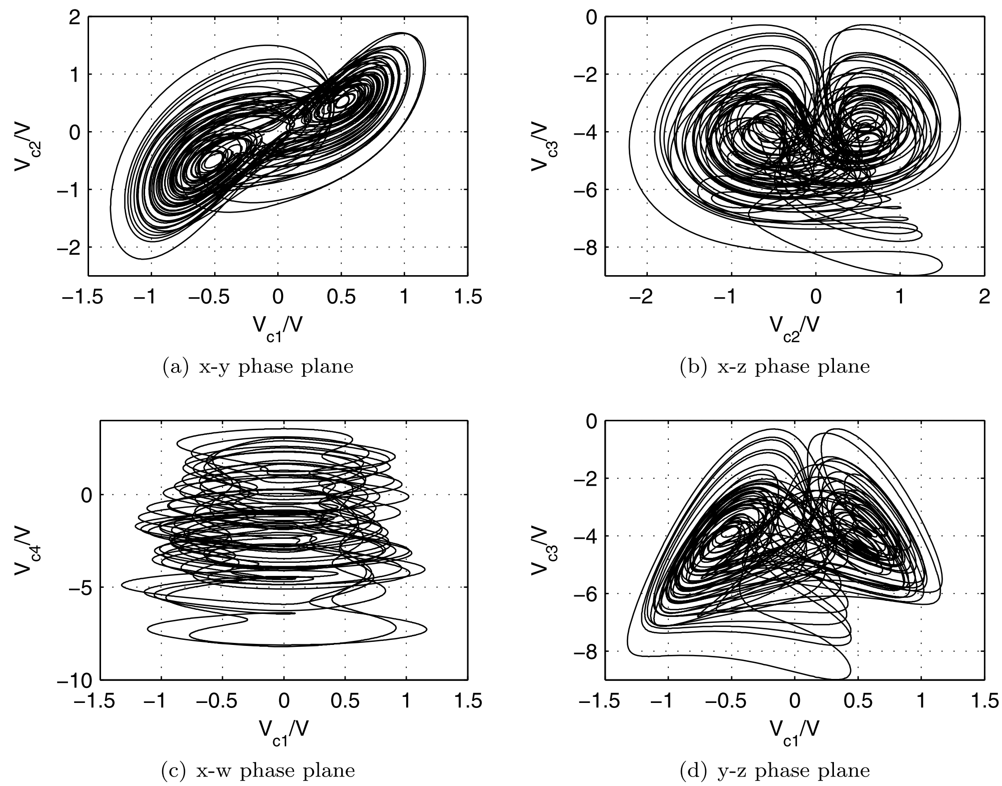


Fig. 8 Phase plane of hyperchaotic system using Pspice

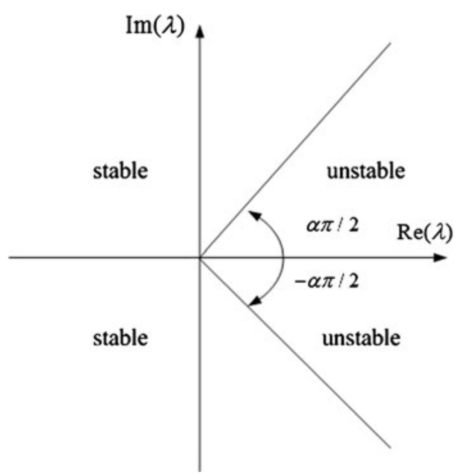


Fig. 9 Stability regions of the fractional order system

chaotic system. Consider a fractional-order generalization of the 3D chaotic system. Here, the conventional derivative is replaced by a fractional derivative as follows:

$$\begin{cases} \frac{d^\alpha x}{dt^\alpha} = a(y-x), \\ \frac{d^\alpha y}{dt^\alpha} = bx + lxz, \\ \frac{d^\alpha z}{dt^\alpha} = -hx^2 - ky^2 - cz. \end{cases} \tag{11}$$

By calculation, the unique equilibrium points of the system are $P_1(0,0,0)$, $P_2(\sqrt{\frac{bc}{lh+lk}}, \sqrt{\frac{bc}{lh+lk}}, -\frac{b}{l})$, and $P_3(-\sqrt{\frac{bc}{lh+lk}}, -\sqrt{\frac{bc}{lh+lk}}, -\frac{b}{l})$. Let $a=10$, $b=40$, $c=2.5$, $d=10$, $h=2$, $k=2$, and $l=1$. Then we get $P_1(0,0,0)$, $P_2(5,5,-40)$, and $P_3(-5,-5,-40)$.

Jacobian matrix is

$$\mathrm{J} = \begin{bmatrix} -10 & 10 & 0 \\ 40+z & 0 & x \\ -4x & -4y & -2.5 \end{bmatrix}. \tag{12}$$

The eigenvalues corresponding to $P_2(5,5,-40)$ and $P_3(-5,-5,-40)$ are $\lambda_1 = -13.8776$ and $\lambda_{2,3} = 0.6888 + 11.9851\mathrm{i}$. So, we have $\min(|\arg(\lambda_i)|) =$

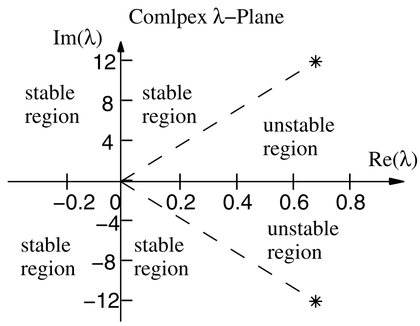


Fig. 10 Stability regions of the fractional-order system

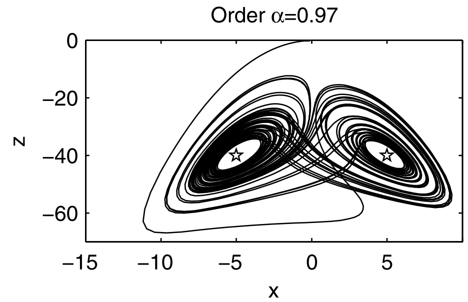


Fig. 11 x–y phase plane for $\alpha = 0.97$

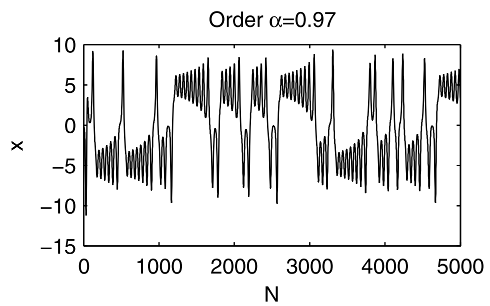


Fig. 12 Time evolutions of x for $\alpha = 0.97$

1.5134. As it has been mentioned, the criterion of instability for the system is

$$\alpha > 2\left|\arg(\lambda)\right|/\pi = 0.9635. \tag{13}$$

Then we get that the minimum order of the 3D chaotic system is 2.89 as shown in Fig. 10.

When $\alpha = 0.97$, the x–z phase plane and the time evolutions of x are shown in Figs. 11 and 12. System (11) still presents the chaotic state.

When $\alpha = 0.95$, system(11) is stable at the equilibrium point $P_2(5, 5, -40)$. Figures 13 and 14 show the x–z phase plane and the time evolutions of x.

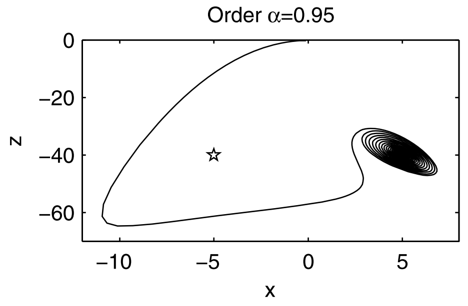


Fig. 13 x–y phase plane for $\alpha = 0.95$

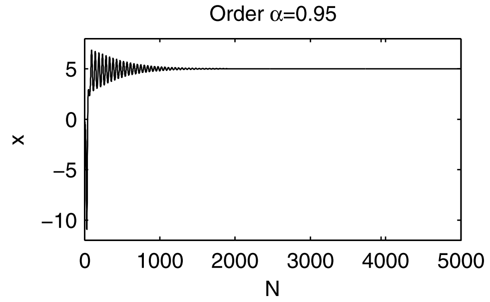


Fig. 14 Time evolutions of x for $\alpha = 0.95$

Next, we consider the fractional generalization of the hyperchaotic system

$$\begin{cases} \frac{d^\alpha x}{dt^\alpha} = a(y - x), \\ \frac{d^\alpha y}{dt^\alpha} = bx + lxz + ew, \\ \frac{d^\alpha z}{dt^\alpha} = -hx^2 - ky^2 - cz - nw, \\ \frac{d^\alpha w}{dt^\alpha} = -dx. \end{cases} \tag{14}$$

To our knowledge, there is no better theoretical method to calculate the minimum order of the fractional-order hyperchaotic systems. Since a hyperchaotic attractor is typically defined as chaotic behavior with at least two positive Lyapunov exponents, we use the predictor–corrector method to carry on the value simulation. The bifurcation diagram of α with $\alpha \in (0.8, 1)$ is given. As shown in Fig. 15, when α is around 0.9, a chaos occurs in the fractional-order hyperchaotic system.

As shown in Figs. 16, 17, and 18, the simulation results demonstrate that:

(1) When $\alpha = 0.9$, the system is in a periodic state.
(2) When $\alpha = 0.91$, the system is in the period-doubling bifurcation state.
(3) When $\alpha = 0.915$, the system is in a hyperchaotic state.

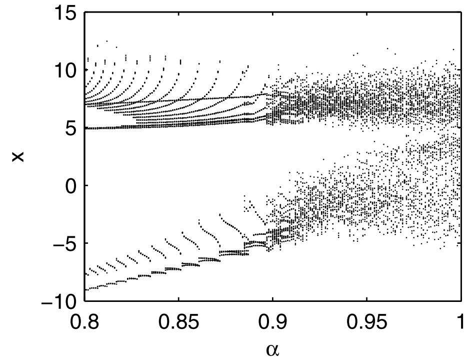
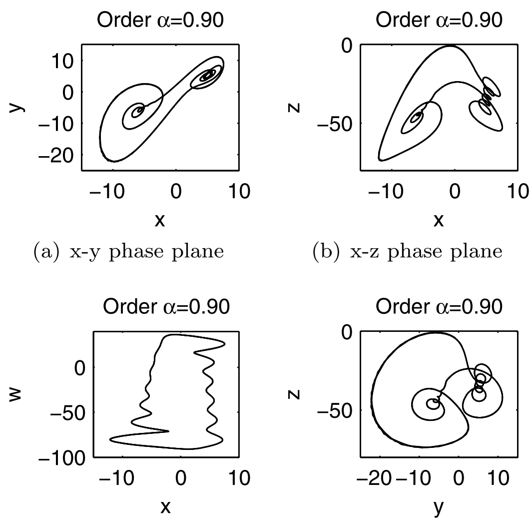


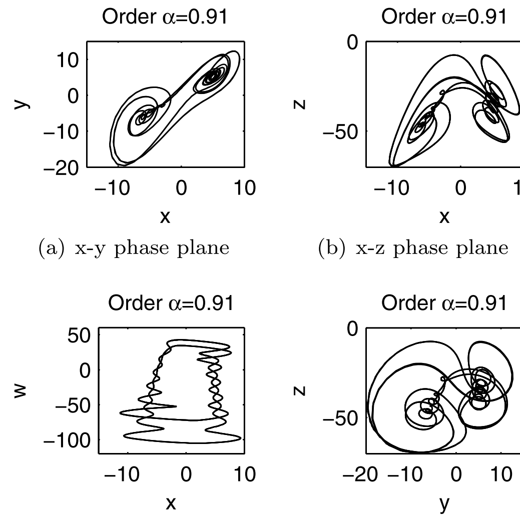
Fig. 15 Bifurcation diagram of the fractional-order hyperchaotic system versus α



(a) x-y phase plane (b) x-z phase plane

(c) x-w phase plane (d) y-z phase plane

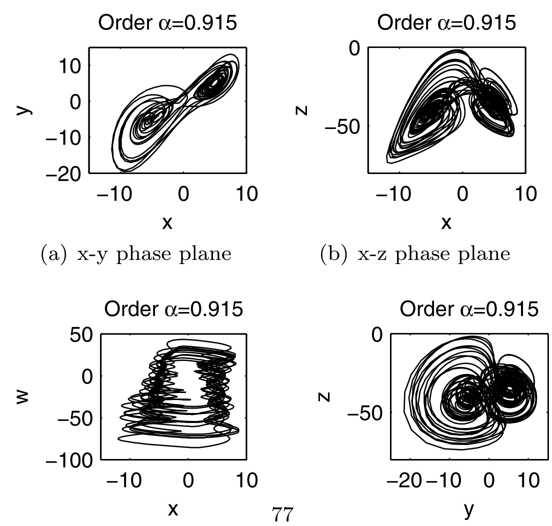
Fig. 16 Phase plane strange attractors



(a) x-y phase plane (b) x-z phase plane

(c) x-w phase plane (d) y-z phase plane

Fig. 17 Phase plane strange attractors



(a) x-y phase plane (b) x-z phase plane

(c) x-w phase plane (d) y-z phase plane

Fig. 18 Phase plane strange attractors

The hyperchaos exists in the fractional-order hyperchaotic system with order as low as 3.66.

4 Fractional-order sliding-mode control of the novel fractional-order hyperchaotic system

The fractional-order hyperchaotic system (14) may be expressed in the following matrix form:

$$D^{\alpha} X(t) = AX(t) + \mathrm{H}\big(X(t)\big), \tag{15}$$

where $X(t) \in R^n$ is the state vector of the four-dimensional system, $AX(t)$ represents the linear part, and $\mathrm{H}(X(t))$ is the nonlinear part of the system.

In order to stabilize the fractional-order hyperchaotic system to its unstable equilibrium point, we add the control input $u(t)$ to the state equation:

$$D^{\alpha} X(t) = AX(t) + \mathrm{H}\big(X(t)\big) + u(t). \tag{16}$$

Then, our aim is changed to design a fractional-order sliding-mode controller. The first step is constructing a fractional-order sliding manifold that represents a desired system dynamics, to be followed by developing a switching control law such that a sliding mode exists at every point of the sliding manifold. Any states outside the manifold are driven to reach the plane in a finite time.

The following control structure is considered:

$$u = Bu_{\text{SMC}} - \text{H}(X), \tag{17}$$

where $B = [b_1, b_2, b_3, b_4]^{\text{T}}$ is the control gain vector, and

$$u_{\text{SMC}} = u_{\text{eq}} + u_r, \tag{18}$$

where u_{eq} is the equivalent control for system (15), and u_r is the switching control.

We choose the fractional-order sliding manifold of the following form:

$$s(t) = CD^{\alpha-1}X(t), \tag{19}$$

where $C = [c_1, c_2, c_3, c_4]$ is the designed gain vector chosen so that the system dynamics have the desired closed-loop behavior on the sliding manifold. The equivalent control can make the system arrive at the sliding manifold.

Based on the theory of sliding-mode control, to ensure that, regardless of the initial condition, the controller would direct the trajectory to reach the sliding manifold, the controlled system must satisfy the hitting condition and existence condition, which can be expressed as $\lim_{s(t)\to\infty} s(t)\dot{s}(t) \leq 0$. Then, the equivalent control u_{eq} can be obtained by setting the derivative of Eq. (16) with respect to time to zero:

$$\dot{s}(t) = CD^{\alpha}X(t) = 0 \tag{20}$$

The equivalent control can make the system arrive at the sliding manifold:

$$u_{\text{eq}} = -(CB)^{-1}CAX(t). \tag{21}$$

The switching control can keep the system within the sliding manifold. To satisfy the sliding condition, the discontinuous reaching law is chosen as follows:

$$u_r = q\,\text{sign}(s), \tag{22}$$

where q is the gain of the controller, and

$$\text{sign}(s) = \begin{cases} +1, & s > 0, \\ 0, & s = 0, \\ -1, & s < 0. \end{cases} \tag{23}$$

Next, we will discuss whether all the required conditions, such as the reaching condition and stability condition, are met.

First, to verify the sliding-mode reaching condition, we find a Lyapunov function

$$V_{\text{SMC}} = \frac{1}{2}s^2. \tag{24}$$

Its time derivative is

$$\begin{aligned} \dot{V}_{\text{SMC}} &= s\dot{s} \\ &= sCD^{\alpha}X(t) \\ &= sC\big[AX + \text{H}(X) + B\big(u_{\text{eq}} + q\,\text{sign}(s)\big) \\ &\quad - \text{H}(X)\big] \\ &= s\big[CAX(t) - CB(CB)^{-1}CAX(t) \\ &\quad + CBq\,\text{sign}(s)\big] \\ &= s\,\text{sign}(s)qCB. \end{aligned} \tag{25}$$

We can find $\dot{V}_{\text{SMC}} < 0$ for $s(t) \neq 0$ because $q < 0$, $CB > 0$, and $s\,\text{sign}(s) > 0$. In other words, the controlled system satisfies the reaching condition.

Then, we will verify the sliding-mode stability condition. When the system arrives at the sliding mode manifold, we get

$$\begin{aligned} D^{\alpha}X(t) &= AX + \text{H}(X) + B\big(u_{\text{eq}} + q\,\text{sign}(s)\big) - \text{H}(X) \\ &= \big(I - B(CB)^{-1}C\big)AX + Bq\,\text{sign}(s). \end{aligned} \tag{26}$$

Let

$$A_{\text{SMC}} = \big(I - B(CB)^{-1}C\big)A. \tag{27}$$

The controlled fractional-order hyperchaotic system is changed to a linear system with bounded input (Bq for $s > 0$ and $-Bq$ for $s < 0$). According to the stability theory of the fractional-order system mentioned in the above chapter, the system is asymptotically stable when all the characteristic roots of A_{SMC} satisfy $|\arg(\text{eig}(A_{\text{SMC}}))| > \frac{\alpha\pi}{2}$. The controlled system will be asymptotically stable at an unstable equilibrium point when we select appropriate matrixes B and C.

In order to verify the effectiveness of the proposed control scheme, we use the sliding-mode controller mentioned above to make the controlled system asymptotically stable. Assuming the same orders of derivatives ($\alpha = 0.98$) in system (14), we get a commensurate-order system. The parameters of the designed controller are obtained as follows.

According to Eq. (16), we can define the controlled system by

$$\begin{bmatrix} \frac{d^\alpha x}{dt^\alpha} \\ \frac{d^\alpha y}{dt^\alpha} \\ \frac{d^\alpha z}{dt^\alpha} \\ \frac{d^\alpha w}{dt^\alpha} \end{bmatrix} = \begin{bmatrix} -a & a & 0 & 0 \\ b & 0 & 0 & e \\ 0 & 0 & -c & -n \\ -d & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} + \begin{bmatrix} 0 \\ lxz \\ -hx^2 - ky^2 \\ 0 \end{bmatrix} + u. \tag{28}$$

Let

$$A = \begin{bmatrix} -a & a & 0 & 0 \\ b & 0 & 0 & e \\ 0 & 0 & -c & -n \\ -d & 0 & 0 & 0 \end{bmatrix} \quad \text{and}$$

$$\mathrm{H}(x) = \begin{bmatrix} 0 \\ lxz \\ -hx^2 - ky^2 \\ 0 \end{bmatrix}.$$

In order to satisfy the condition in Lemma 1, $|\arg(\mathrm{eig}(A_{\mathrm{SMC}}))| > \frac{\alpha\pi}{2}$, we select the matrixes $B = [1\ 1\ 0\ 0]^{\mathrm{T}}$ and $C = [20\ 0\ 1\ 0]$. Then, the eigenvalues of A_{SMC} are $\lambda_1 = -10$, $\lambda_2 = -7.216 \times e^{-16}$, $\lambda_3 = -0.2192$, and $\lambda_4 = -2.2808$, which satisfy the condition $|\arg(\mathrm{eig}(A_{\mathrm{SMC}}))| > 0.49\pi$, so that the value of $|\arg(\mathrm{eig}(A_{\mathrm{SMC}}))|$ lies in the stable region.

Now, substituting the matrixes A, B, and C into Eq. (21), we get the equivalent control

$$\begin{aligned} u_{\mathrm{eq}} &= -(CB)^{-1} CAX(t) \\ &= ax - ay + cz/20 + nw/20. \end{aligned} \tag{29}$$

Then the total control law can be defined as follows:

$$\begin{aligned} u &= B\big(u_{\mathrm{eq}} + q\,\mathrm{sign}(s)\big) - \mathrm{H}(x) \\ &= B\big(ax - ay + cz/20 + nw/20 + q\,\mathrm{sign}(s)\big) \\ &\quad - \mathrm{H}(x). \end{aligned} \tag{30}$$

As shown in Figs. 19 and 20, there are three stages of the controlled system. In the first 20 seconds, without controller, the system is chaotic as we can see in Fig. 18. In the second phase (known as reaching phase), after $t = 20$ s, the fractional-order hyperchaotic system is forced toward the sliding manifold by the sliding-mode controller. When the trajectory touches the sliding plane, the system enters the 3rd phase, which is called sliding-mode operation. In order to maintained the trajectory on the sliding plane, the system is controlled by the switching function, which is shown in Fig. 20, and moves toward the desired equilibrium, finally stabilized to its unstable equilibrium point $O(0, 0, 0, 0)$. It can be seen that state variables x, y, z, and w are closer to zero in Fig. 19.

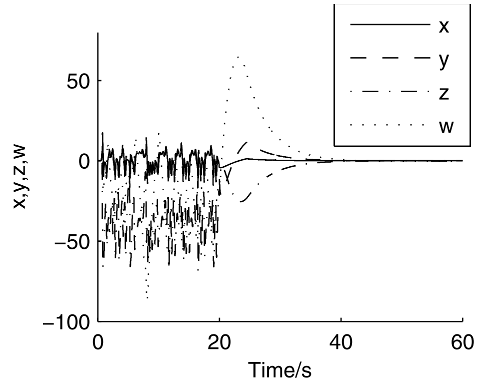


Fig. 19 Stabilization of the fractional-order hyperchaotic system with the controller started at $t = 20$ s

5 Conclusions

In this paper, a novel hyperchaotic system, its fractional-order generalizations, and the fractional-order SMC are investigated. First, some basic dynamical properties of the hyperchaotic system are studied. Then we analyzed the minimum orders of the chaotic and hyperchaotic systems. Simulation results show that orders as low as 2.89 and 3.66 can produce chaotic and hyperchaotic attractors, respectively. Finally, a fractional-order sliding-mode controller is designed for the novel fractional-order hyperchaotic system. A sliding manifold is determined using the SMC technique. The SMC law is derived to make the states of the fractional-order hyperchaotic system asymptotically stable. The designed control scheme is simple, theoretically rigorous and robust against the system uncertainty, and guarantees the property of asymptotical stability in the presence of an external disturbance. The illustrative simulation results are given to demonstrate the effectiveness of the proposed sliding-mode control design.

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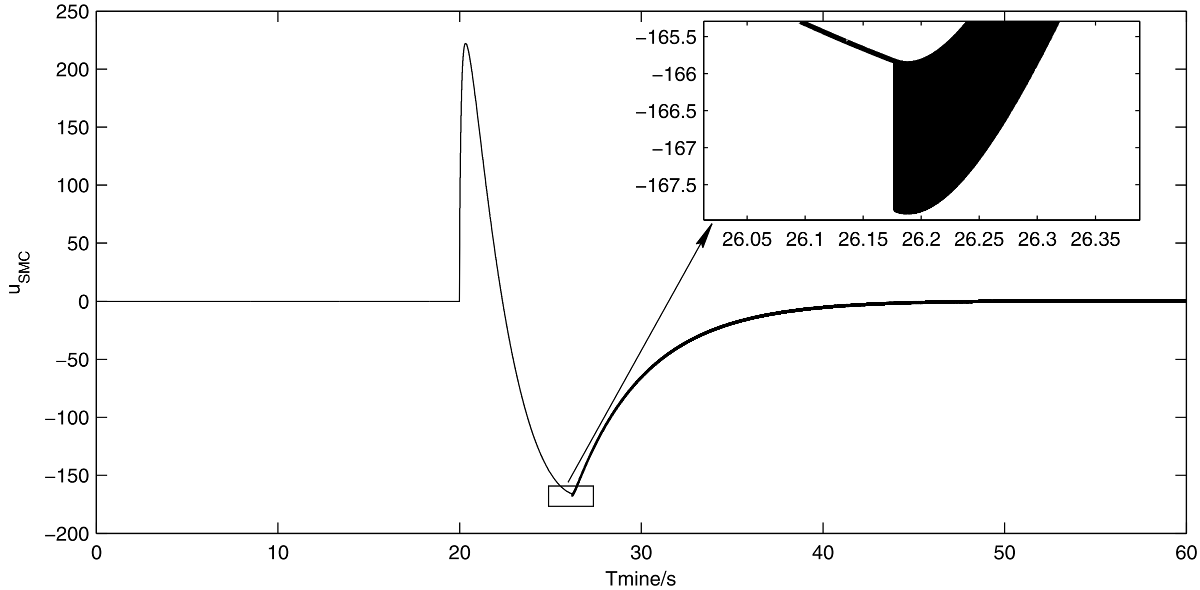

Fig. 20 The output of the sliding-mode controller u_{SMC}

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