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Passivity analysis of uncertain neural networks with mixed time-varying delays

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Abstract This paper addresses the passivity problem for uncertain neural networks with both discrete and distributed time-varying delays. It is assumed that the parameter uncertainties are norm-bounded. By construction of an augmented Lyapunov–Krasovskii functional and utilization of zero equalities, improved passivity criteria for the networks are derived in terms of linear matrix inequalities (LMIs) via new approaches. Through three numerical examples, the effectiveness to enhance the feasible region of the proposed criteria is demonstrated.

Keywords Neural networks · Time-varying delays · Passivity · Lyapunov method

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1 Introduction

Since neural networks have been extensively applied in many areas such as reconstruction of moving image, signal processing, the tasks of pattern recognition, associative memories, fixed-point computations, and so on, the stability analysis of the concerned neural networks is a very important and prerequisite job because the application of neural networks heavily depends on the dynamic behavior of equilibrium points [1–5]. Also, due to the finite speed of information processing in the implementation of the network, time-delay occurs in many neural networks. It is well known that time-delay often causes undesirable dynamic behaviors such as oscillation and instability of the networks. Thus, delay-dependent stability and stabilization problem for neural networks with time-delay have been paid more attention than delay-independent ones because the information on the size of time-delays is utilized in delay-dependent criteria, which lead to reduce the conservatism of stability and stabilization criteria. To confirm this, see [6–23] and references therein.

In practice, it should be noted that the signal propagation is sometimes instantaneous and can be modeled with discrete delays. Also, it may be distributed during a certain time period so that the distributed delays should be incorporated in the model. That is, it is often the case that the neural network model possesses both discrete and distributed delays [24]. In this regard, the stability of cellular neural networks with discrete and distributed delays has been investigated in [25–28]. Furthermore, the norm-bounded parametric uncertainties, which sometimes affect the stability of systems, are considered in the works [26–28].

On the other hand, in various scientific and engineering problems, stability issues are often linked to the theory of dissipative systems. It postulates that the energy dissipated inside a dynamic system is less than the energy supplied from the external source [29]. Based on the concept of energy, the passivity is the property of dynamical systems and describes the energy flow through the system. It is also an input/output characterization and related to Lyapunov method. In the field of nonlinear control, the concept of dissipativeness was firstly introduced by Willems [30] in the form of inequality including supply rate and the storage function. The main idea of passivity theory is that passive properties of a system can keep the system internally.

In this regard, in [31–38], the passivity problem for the uncertain neural networks with both discrete and distributed time-varying delays was considered. Chen et al. [31] investigated the passivity problem of the neural networks by utilizing free-weighting matrices and the LMI framework. In [32], improved delay-dependent passivity criteria for the networks were proposed. Xu et al. [33] studied the problem of passivity analysis for neural networks with both time-varying delays and norm-bounded parameter uncertainties. In [34], improved passivity criteria for stochastic neural networks with interval time-varying delays and norm-bounded parameter uncertainties were proposed via an improved approximation method. In [35], for two types of time-varying delays, new delay-dependent passivity conditions of delayed neural networks were derived. Recently, by taking more information of states as augmented vectors, an augmented Lyapunov–Krasovskii functional was utilized in [36] to derive passivity criteria for uncertain neural networks with time-varying delays. Song and Cao [37] investigated the passivity problem for a class of uncertain neural networks with leakage delay and time-varying delays by employing Newton–Leibniz formulation and the free weighting matrix method. Very recently, in [38], by constructing a novel Lyapunov–Krasovskii functional including a tuning parameter in time-varying delays and introducing some proper free-weighting matrices, new passivity conditions for neural networks with both discrete and distributed time-varying delays were developed to guarantee the passivity performance of the networks. However, there are rooms for further improvement in enhancing the feasible regions of passivity criteria.

Motivated by above discussion, in this paper, the problem on delay-dependent passivity for uncertain neural networks with both discrete and distributed time-varying delays is addressed. The parameter uncertainties are assumed to be norm-bounded. The main contribution of this paper lies in two aspects:

1. Unlike the method of [38], no tuning parameters in a time-varying delay are utilized. Instead, by taking more information of states, a newly constructed Lyapunov–Krasovskii functional is proposed. Then, inspired by the work of [39–41], a passivity condition for neural networks with both discrete and distributed time-varying delays and parameter uncertainties is derived in terms of LMIs which will be introduced in Theorem 1.
2. A novel approach partitioning m-interval of the range of the activation function divided by state will be proposed. Through three numerical examples, it will be shown the maximum delay bounds for guaranteeing the passivity of the considered neural networks increase when the partitioning number of the bounding of activation function gets larger.

Based on the result of Theorem 1, a passivity criterion for uncertain neural networks with only discrete time-varying delays will be proposed in Theorem 2. Finally, three numerical examples are included to show the effectiveness of the proposed methods.

Notation $\mathbb{R}^n$ is the n-dimensional Euclidean space, and $\mathbb{R}^{m\times n}$ denotes the set of all $m \times n$ real matrices. For symmetric matrices X and Y, $X > Y$ (respectively, $X \geq Y$) means that the matrix $X - Y$ is positive definite (respectively, nonnegative). $X^{\perp}$ denotes a basis for the null-space of X. I_n, $0_{n,}$ and $0_{m\cdot n}$ denotes $n \times n$ identity matrix, $n \times n$ and $m \times n$ zero matrices, respectively. $\|\cdot\|$ refers to the Euclidean vector norm or the induced matrix norm. $\mathrm{diag}\{\cdots\}$ denotes the block diagonal matrix. For square matrix S, sym{S} means the sum of S and its symmetric matrix S^T; i.e., $\mathrm{sym}\{S\} = S + S^T$. $\star$ represents the elements below the main diagonal of a symmetric matrix.

2 Problem statements

Consider the following uncertain neural networks with both discrete and distributed time-varying delays:

$$\begin{aligned}\dot{x}(t) = {} & -\big(A + \Delta A(t)\big)x(t) \\ & + \big(W_0 + \Delta W_0(t)\big) f\big(x(t)\big) \\ & + \big(W_1 + \Delta W_1(t)\big) f\big(x\big(t - h(t)\big)\big) \\ & + \big(W_2 + \Delta W_2(t)\big) \\ & \times \int_{t-\tau(t)}^{t} f\big(x(s)\big)\, ds + u(t), \\ y(t) = {} & C_1 f\big(x(t)\big) + C_2 f\big(x\big(t - h(t)\big)\big),\end{aligned} \tag{1}$$

where n denotes the number of neurons in a neural network, $x(t) = [x_1(t), \dots, x_n(t)]^T \in \mathbb{R}^n$ is the neuron state vector, $y(t) \in \mathbb{R}^n$ is the output vector, $f(x_i(\cdot)) = [f_1(x_i(\cdot)), \dots, f_n(x_i(\cdot))]^T \in \mathbb{R}^n$ denotes the neuron activation function vector, $u(t) \in \mathbb{R}^n$ is the input vector, $A = \mathrm{diag}\{a_1, \dots, a_n\} \in \mathbb{R}^{n\times n}$ is a positive diagonal matrix, $W_i \in \mathbb{R}^{n\times n}$ $(i = 0, 1, 2)$ are the interconnection weight matrices, $C_i \in \mathbb{R}^{n\times n}$ $(i = 1, 2)$ are known constant matrices, and $\Delta A(t)$ and $\Delta W_i(t)$ $(i = 0, 1, 2)$ are the parameter uncertainties of the form

$$\begin{aligned}&\big[\Delta A(t), \Delta W_0(t), \Delta W_1(t), \Delta W_2(t)\big] \\ &\quad = D F(t)[E_a, E_0, E_1, E_2],\end{aligned}$$

where $F(t)$ is the time-varying nonlinear function satisfying

$$F^T(t)F(t) \le I, \quad \forall t \ge 0. \tag{2}$$

The delays $h(t)$ and $\tau(t)$ are time-varying delays satisfying

$$\begin{aligned}&0 \le h(t) \le h_U, \qquad -\infty < \dot{h}(t) \le h_D, \\ &0 \le \tau(t) \le \tau_U,\end{aligned}$$

where h_U, h_D, and τ_U are known positive scalars.

It is assumed that the neuron activation functions satisfy the following condition.

Assumption 1 [42] The neuron activation functions $f_i(\cdot)$, $i = 1, \dots, n$ are continuous, bounded, and satisfy

$$\begin{aligned}&k_i^- \le \frac{f_i(u) - f_i(v)}{u - v} \le k_i^+, \quad u, v \in \mathbb{R}, \\ &u \ne v, \quad i = 1, \dots, n,\end{aligned} \tag{3}$$

where k_i^+ and k_i^- are constants.

Remark 1 In Assumption 1, k_i^+ and k_i^- can be allowed to be positive, negative, or zero. As mentioned in [14], Assumption 1 describes the class of globally Lipschitz continuous and monotone nondecreasing activation when $k_i^- = 0$ and $k_i^+ > 0$. Also, the class of globally Lipschitz continuous and monotone increasing activation functions can be described when $k_i^+ > k_i^- > 0$.

For passivity analysis, the systems (1) can be rewritten as

$$\begin{aligned}\dot{x}(t) = {} & -Ax(t) + W_0 f\big(x(t)\big) + W_1 f\big(x\big(t - h(t)\big)\big) \\ & + W_2 \int_{t-\tau(t)}^{t} f\big(x(s)\big)\, ds + u(t) + Dp(t), \\ p(t) = {} & F(t)q(t), \\ q(t) = {} & -E_a x(t) + E_0 f\big(x(t)\big) + E_1 f\big(x\big(t - h(t)\big)\big) \\ & + E_2 \int_{t-\tau(t)}^{t} f\big(x(s)\big)\, ds, \\ y(t) = {} & C_1 f\big(x(t)\big) + C_2 f\big(x\big(t - h(t)\big)\big).\end{aligned} \tag{4}$$

The objective of this paper is to investigate delay-dependent passivity conditions for system (4). Before deriving our main results, we state the following definition and lemmas.

Definition 1 The system (1) is called passive if there exists a scalar $\gamma \ge 0$ such that

$$-\gamma \int_0^{t_p} u^T(s)u(s)\, ds \le 2 \int_0^{t_p} y^T(s)u(s)\, ds, \tag{5}$$

for all $t_p \ge 0$ and for all solution of (1) with $x(0) = 0$.

Lemma 1 [43] *Let* $\zeta \in \mathbb{R}^n$, $\Phi = \Phi^T \in \mathbb{R}^{n\times n}$, *and* $\Upsilon \in \mathbb{R}^{m\times n}$ *such that* $\mathrm{rank}(\Upsilon) < n$. *The following statements are equivalent*:

(i) $\zeta^T \Phi \zeta < 0, \forall \Upsilon \zeta = 0, \zeta \ne 0$,
(ii) ${\Upsilon^{\perp}}^T \Phi \Upsilon^{\perp} < 0$.

3 Main results

In this section, new passivity criteria for network (4) will be proposed. For the sake of simplicity on matrix representation, $e_i \in \mathbb{R}^{18n\times n}$ $(i = 1, 2, \dots, 18)$ are defined as block entry matrices (for example, $e_2 = [0_n, I_n, 0_{16n\times n}]^T$). The notations of several matrices are defined as

$$\zeta^T(t) = \Big[x^T(t), x^T\big(t-h(t)\big), x^T(t-h_U), \dot{x}^T(t), \dot{x}^T(t-h_U), \int_{t-h(t)}^{t} x^T(s)\,ds, \int_{t-h_U}^{t-h(t)} x^T(s)\,ds, f^T\big(x(t)\big), f^T\big(x\big(t-h(t)\big)\big), f^T\big(x(t-h_U)\big), f^T\big(x(t-\tau_U)\big), \int_{t-h(t)}^{t} f^T\big(x(s)\big)\,ds, \int_{t-h_U}^{t-h(t)} f^T\big(x(s)\big)\,ds, \int_{t-\tau(t)}^{t} f^T\big(x(s)\big)\,ds, \int_{t-\tau_U}^{t-\tau(t)} f^T\big(x(s)\big)\,ds, x^T(t-\tau_U), u^T(t), p^T(t)\Big],$$

$$\mu^T(t) = \big[x^T(t), \dot{x}^T(t), f^T\big(x(t)\big)\big],$$

$$\nu^T(t) = \big[x^T(t), f^T\big(x(t)\big)\big],$$

$$\Upsilon = [-A, 0_{2n\times n}, -I_n, 0_{3n\times n}, W_0, W_1, 0_{4n\times n}, W_2, 0_{2n\times n}, I_n, D],$$

$$\bar{P}_i = \begin{bmatrix} 0 & P_i & 0 \\ P_i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (i = 1, 2),$$

$$\Pi_1 = [e_1, e_3, e_6 + e_7, e_{12} + e_{13}, e_{14} + e_{15}],$$

$$\Pi_2 = [e_4, e_5, e_1 - e_3, e_8 - e_{10}, e_8 - e_{11}],$$

$$\Pi_3 = [e_1, e_4, e_8], \qquad \Pi_4 = [e_3, e_5, e_{10}],$$

$$\Pi_5 = [e_1, e_8], \qquad \Pi_6 = [e_2, e_9],$$

$$\Pi_7 = [e_6, e_1 - e_2, e_{12}, e_7, e_2 - e_3, e_{13}],$$

$$\Pi_8 = [e_{14}, e_{15}],$$

$$\Phi_1 = \mathrm{sym}\{\Pi_1 R \Pi_2^T\},$$

$$\Phi_2 = \Pi_3 N \Pi_3^T - \Pi_4 N \Pi_4^T + \sum_{i=1}^{2} \mathrm{sym}\Big\{[e_{2(i+3)}, e_{2i-1}] \times \begin{bmatrix} \Lambda_i - \Delta_i \\ K^+\Delta_i - K^-\Lambda_i \end{bmatrix} e_{3+i}^T\Big\}, + \Pi_5 G \Pi_5^T - (1-h_D)\Pi_6 G \Pi_6^T + h_U^2 \Pi_3 Q_1 \Pi_3^T + h_U\big(e_1 P_1 e_1^T - e_2(P_1 - P_2)e_2^T - e_3 P_2 e_3^T\big) - \Pi_7 \left[\begin{array}{c|c} Q_1 + \bar{P}_1 & S_1 \\ \hline \star & Q_1 + \bar{P}_2 \end{array}\right] \Pi_7^T + \big(h_U^2/2\big)^2 e_4 Q_3 e_4^T - (h_U e_1 - e_6 - e_7) \times Q_3 (h_U e_1 - e_6 - e_7)^T,$$

$$\Phi_3 = [e_1, e_8] M [e_1, e_8]^T - [e_{16}, e_{11}] M [e_{16}, e_{11}]^T + \tau_U^2 e_8 Q_2 e_8^T - \Pi_8 \left[\begin{array}{c|c} Q_2 & S_2 \\ \hline \star & Q_2 \end{array}\right] \Pi_8^T,$$

$$\Omega = \varepsilon\big(-E_a e_1^T + E_0 e_8^T + E_1 e_9^T + E_2 e_{14}^T\big)^T \times \big(-E_a e_1^T + E_0 e_8^T + E_1 e_9^T + E_2 e_{14}^T\big) - \varepsilon e_{18} e_{18}^T,$$

$$\Psi = \sum_{l=1}^{3} \Phi_l + \Omega - \mathrm{sym}\{e_8 C_1^T e_{17}^T\} - \mathrm{sym}\{e_9 C_2^T e_{17}^T\} - \gamma e_{17} e_{17}^T,$$

$$\Theta_{1j} = -\sum_{i=1}^{3} \mathrm{sym}\big\{\big[e_{7+i} - e_i\big(K^- + \big((j-1)/m\big) \times \big(K^+ - K^-\big)\big)\big] H_{3(j-1)+i} \times \big[e_{7+i} - e_i\big(K^- + (j/m)\big(K^+ - K^-\big)\big)\big]^T\big\},$$

$$\Theta_{2j} = -\mathrm{sym}\big\{\big[e_{11} - e_{16}\big(K^- + \big((j-1)/m\big) \times \big(K^+ - K^-\big)\big)\big] \tilde{H}_j \times \big[e_{11} - e_{16}\big(K^- + (j/m)\big(K^+ - K^-\big)\big)\big]^T\big\},$$

$$\Theta_j = \Theta_{1j} + \Theta_{2j} \quad (j = 1, 2, \dots, m). \tag{6}$$

Then the main result is given by the following theorem.

Theorem 1 *For given positive scalars h_U, h_D, τ_U, and a positive integer m, diagonal matrices $K^- = \mathrm{diag}\{k_1^-, \dots, k_n^-\}$ and $K^+ = \mathrm{diag}\{k_1^+, \dots, k_n^+\}$, the system (4) is passive for $0 \le h(t) \le h_U$, $\dot{h}(t) \le h_D$ and $0 \le \tau(t) \le \tau_U$, if there exist positive scalars ε and γ,* *positive diagonal matrices* $\Lambda_a = \mathrm{diag}\{\lambda_{a1}, \dots, \lambda_{an}\}$ $(a = 1, 2)$, $\Delta_a = \mathrm{diag}\{\delta_{a1}, \dots, \delta_{an}\}$ $(a = 1, 2)$, $H_b = \mathrm{diag}\{h_1^b, \dots, h_n^b\}$ $(b = 1, 2, \dots, 3m)$, $\tilde{H}_b = \mathrm{diag}\{\tilde{h}_1^b, \dots, \tilde{h}_n^b\}$ $(b = 1, 2, \dots, m)$, *positive definite matrices* $R \in \mathbb{R}^{5n\times 5n}$, $N \in \mathbb{R}^{3n\times 3n}$, $M \in \mathbb{R}^{2n\times 2n}$, $G \in \mathbb{R}^{2n\times 2n}$, $Q_1 \in \mathbb{R}^{3n\times 3n}$, $Q_2 \in \mathbb{R}^{n\times n}$, $Q_3 \in \mathbb{R}^{n\times n}$, *any symmetric matrices* $P_1 \in \mathbb{R}^{n\times n}$, $P_2 \in \mathbb{R}^{n\times n}$, *and any matrices* $S_1 \in \mathbb{R}^{3n\times 3n}$, $S_2 \in \mathbb{R}^{n\times n}$, *satisfying the following LMIs*:

$$(\Upsilon^\perp)^T(\Psi + \Theta_j)\Upsilon^\perp < 0 \quad (j = 1, 2, \dots, m), \tag{7}$$

$$\left[\begin{array}{c|c} Q_1 + \bar{P}_1 & S_1 \\ \hline \star & Q_1 + \bar{P}_2 \end{array}\right] \ge 0, \tag{8}$$

$$\left[\begin{array}{c|c} Q_2 & S_2 \\ \hline \star & Q_2 \end{array}\right] \ge 0, \tag{9}$$

where Υ, $\bar{P}_1$, $\bar{P}_2$, Ψ, *and* Θ_j *are defined in* (6).

Proof Consider the following Lyapunov–Krasovskii functional candidate as

$$V = V_1 + V_2 + V_3 + V_4 + V_5, \tag{10}$$

where

$$V_1 = \begin{bmatrix} x(t) \\ x(t-h_U) \\ \int_{t-h_U}^t x(s)\,ds \\ \int_{t-h_U}^t f(x(s))\,ds \\ \int_{t-\tau_U}^t f(x(s))\,ds \end{bmatrix}^T R \begin{bmatrix} x(t) \\ x(t-h_U) \\ \int_{t-h_U}^t x(s)\,ds \\ \int_{t-h_U}^t f(x(s))\,ds \\ \int_{t-\tau_U}^t f(x(s))\,ds \end{bmatrix},$$

$$\begin{aligned} V_2 &= \int_{t-h_U}^t \mu^T(s)N\mu(s)\,ds + \int_{t-\tau_U}^t \nu^T(s)M\nu(s)\,ds \\ &\quad + 2\sum_{i=1}^n \Big(\lambda_{1i}\int_0^{x_i(t)} \big(f_i(s) - k_i^- s\big)\,ds \\ &\quad + \delta_{1i}\int_0^{x_i(t)} \big(k_i^+ s - f_i(s)\big)\,ds\Big) \\ &\quad + 2\sum_{i=1}^n \Big(\lambda_{2i}\int_0^{x_i(t-h_U)} \big(f_i(s) - k_i^- s\big)\,ds \\ &\quad + \delta_{2i}\int_0^{x_i(t-h_U)} \big(k_i^+ s - f_i(s)\big)\,ds\Big), \end{aligned}$$

$$V_3 = \int_{t-h(t)}^t \nu^T(s)G\nu(s)\,ds,$$

$$\begin{aligned} V_4 &= h_U\int_{t-h_U}^t\int_s^t \mu^T(u)Q_1\mu(u)\,du\,ds \\ &\quad + \tau_U\int_{t-\tau_U}^t\int_s^t f^T\big(x(u)\big)Q_2 f\big(x(u)\big)\,du\,ds, \end{aligned}$$

$$V_5 = \frac{h_U^2}{2}\int_{t-h_U}^t\int_s^t\int_u^t \dot{x}^T(v)Q_3\dot{x}(v)\,dv\,du\,ds.$$

Time-derivative of V_1, V_2, and V_3 are calculated as

$$\begin{aligned} \dot{V}_1 &= 2\begin{bmatrix} x(t) \\ x(t-h_U) \\ \int_{t-h(t)}^t x(s)\,ds + \int_{t-h_U}^{t-h(t)} x(s)\,ds \\ \int_{t-h(t)}^t f(x(s))\,ds + \int_{t-h_U}^{t-h(t)} f(x(s))\,ds \\ \int_{t-\tau(t)}^t f(x(s))\,ds + \int_{t-\tau_U}^{t-\tau(t)} f(x(s))\,ds \end{bmatrix}^T \\ &\quad \times R\begin{bmatrix} \dot{x}(t) \\ \dot{x}(t-h_U) \\ x(t) - x(t-h_U) \\ f(x(t)) - f(x(t-h_U)) \\ f(x(t)) - f(x(t-\tau_U)) \end{bmatrix} \\ &= \zeta^T(t)\big(\mathrm{sym}\{\Pi_1 R\Pi_2^T\}\big)\zeta(t), \end{aligned} \tag{11}$$

$$\begin{aligned} \dot{V}_2 &= \mu^T(t)N\mu(t) - \mu^T(t-h_U)N\mu(t-h_U) \\ &\quad + \nu^T(t)M\nu(t) - \nu^T(t-\tau_U)M\nu(t-\tau_U) \\ &\quad + 2\big(f\big(x(t)\big) - K^- x(t)\big)^T\Lambda_1\dot{x}(t) \\ &\quad + 2\big(K^+ x(t) - f\big(x(t)\big)\big)\Delta_1\dot{x}(t) \\ &\quad + 2\big(f\big(x(t-h_U)\big) - K^- x(t-h_U)\big)^T \\ &\quad \times \Lambda_2\dot{x}(t-h_U) \\ &\quad + 2\big(K^+ x(t-h_U) - f\big(x(t-h_U)\big)\big) \\ &\quad \times \Delta_2\dot{x}(t-h_U) \\ &= \zeta^T(t)\big(\Pi_3 N\Pi_3^T - \Pi_4 N\Pi_4^T + [e_1, e_8]M[e_1, e_8]^T \\ &\quad - [e_{16}, e_{11}]M[e_{16}, e_{11}]^T \\ &\quad + \mathrm{sym}\{(e_8 - e_1 K^-)\Lambda_1 e_4^T\} \\ &\quad + \mathrm{sym}\{(e_1 K^+ - e_8)\Delta_1 e_4^T\} \\ &\quad + \mathrm{sym}\{(e_{10} - e_3 K^-)\Lambda_2 e_5^T\} \\ &\quad + \mathrm{sym}\{(e_3 K^+ - e_{10})\Delta_2 e_5^T\}\big)\zeta(t), \end{aligned} \tag{12}$$

$$\dot{V}_3 \le \nu(t)^T G\nu(t) - (1-h_D)\nu^T\big(t-h(t)\big)G\nu \times \big(t-h(t)\big) = \zeta^T(t)\big(\Pi_5 G\Pi_5^T - (1-h_D)\Pi_6 G\Pi_6^T\big)\zeta(t). \quad (13)$$

Inspired by the work of [40], by adding the following two zero equalities with any symmetric matrices P_1 and P_2:

$$0 = x^T(t)(h_U P_1)x(t) - x\big(t-h(t)\big)(h_U P_1)x\big(t-h(t)\big) - 2h_U\int_{t-h(t)}^{t} x(s)P_1\dot{x}(s)\,ds,$$
$$0 = x^T\big(t-h(t)\big)(h_U P_2)x\big(t-h(t)\big) - x^T(t-h_U)(h_U P_2)x(t-h_U) - 2h_U\int_{t-h_U}^{t-h(t)} x(s)P_2\dot{x}(s)\,ds \quad (14)$$

into the time-derivative of V_4 and using Jensen's inequality [44], we get

$$\begin{aligned}\dot{V}_4 &= h_U^2\mu(t)^T Q_1\mu(t) + \tau_U^2 f^T\big(x(t)\big)Q_2 f\big(x(t)\big) + x^T(t)(h_U P_1)x(t) - x^T\big(t-h(t)\big)(h_U P_1)x\big(t-h(t)\big) \\ &\quad + x^T\big(t-h(t)\big)(h_U P_2)x\big(t-h(t)\big) - x^T(t-h_U)(h_U P_2)x(t-h_U) \\ &\quad - h_U\int_{t-h(t)}^{t}\mu^T(s)\times\Bigg(Q_1 + \underbrace{\begin{bmatrix}0_n & P_1 & 0_n\\ P_1 & 0_n & 0_n\\ 0_n & 0_n & 0_n\end{bmatrix}}_{\bar{P}_1}\Bigg)\mu(s)\,ds \\ &\quad - h_U\int_{t-h_U}^{t-h(t)}\mu^T(s)\times\Bigg(Q_1 + \underbrace{\begin{bmatrix}0_n & P_2 & 0_n\\ P_2 & 0_n & 0_n\\ 0_n & 0_n & 0_n\end{bmatrix}}_{\bar{P}_2}\Bigg)\mu(s)\,ds \\ &\quad - \tau_U\int_{t-\tau(t)}^{t} f^T\big(x(s)\big)Q_2 f\big(x(s)\big)\,ds - \tau_U\int_{t-\tau_U}^{t-\tau(t)} f^T\big(x(s)\big)Q_2 f\big(x(s)\big)\,ds. \end{aligned} \quad (15)$$

If the inequality (8) hold, then the two inequalities, $Q_1 + \bar{P}_1 \ge 0$ and $Q_1 + \bar{P}_2 \ge 0$, are satisfied. Thus, $\dot{V}_4$ can be estimated as

$$\begin{aligned}\dot{V}_4 &\le h_U^2\mu(t)^T Q_1\mu(t) + \tau_U^2 f^T\big(x(t)\big)Q_2 f\big(x(t)\big) + x^T(t)(h_U P_1)x(t) - x^T\big(t-h(t)\big)(h_U P_1)x\big(t-h(t)\big) \\ &\quad + x^T\big(t-h(t)\big)(h_U P_2)x\big(t-h(t)\big) - x^T(t-h_U)(h_U P_2)x(t-h_U) \\ &\quad - \begin{bmatrix}\int_{t-h(t)}^{t}\mu(s)\,ds\\ \int_{t-h_U}^{t-h(t)}\mu(s)\,ds\end{bmatrix}^T \left[\begin{array}{c|c}\frac{1}{1-\alpha(t)}(Q_1+\bar{P}_1) & 0_{3n}\\ \hline 0_{3n} & \frac{1}{\alpha(t)}(Q_1+\bar{P}_2)\end{array}\right]\begin{bmatrix}\int_{t-h(t)}^{t}\mu(s)\,ds\\ \int_{t-h_U}^{t-h(t)}\mu(s)\,ds\end{bmatrix} \\ &\quad - \begin{bmatrix}\int_{t-\tau(t)}^{t}f(x(s))\,ds\\ \int_{t-\tau_U}^{t-\tau(t)}f(x(s))\,ds\end{bmatrix}^T \left[\begin{array}{c|c}\frac{1}{1-\beta(t)}Q_2 & 0_n\\ \hline 0_n & \frac{1}{\beta(t)}Q_2\end{array}\right]\begin{bmatrix}\int_{t-\tau(t)}^{t}f(x(s))\,ds\\ \int_{t-\tau_U}^{t-\tau(t)}f(x(s))\,ds\end{bmatrix}, \end{aligned} \quad (16)$$

where $\alpha(t) = 1 - h(t)h_U^{-1}$ and $\beta(t) = 1 - \tau(t)\tau_U^{-1}$, which satisfy $0 < \alpha(t) < 1$ and $0 < \beta(t) < 1$ when $0 < h(t) < h_U$ and $0 < \tau(t) < \tau_U$, respectively. Then, by reciprocally convex approach [41], if the LMIs (8) and (9) satisfy, then the following inequalities hold for any matrices S_1 and S_2

$$\begin{aligned}&\left[\begin{array}{c|c}-\sqrt{\frac{\alpha(t)}{1-\alpha(t)}}I_n & 0_{3n}\\ \hline 0_{3n} & \sqrt{\frac{1-\alpha(t)}{\alpha(t)}}I_n\end{array}\right]\left[\begin{array}{c|c}Q_1+\bar{P}_1 & S_1\\ \hline \star & Q_1+\bar{P}_2\end{array}\right]\left[\begin{array}{c|c}-\sqrt{\frac{\alpha(t)}{1-\alpha(t)}}I_n & 0_{3n}\\ \hline 0_{3n} & \sqrt{\frac{1-\alpha(t)}{\alpha(t)}}I_n\end{array}\right] > 0_{6n}, \\ &\left[\begin{array}{c|c}-\sqrt{\frac{\beta(t)}{1-\beta(t)}}I_n & 0_n\\ \hline 0_n & \sqrt{\frac{1-\beta(t)}{\beta(t)}}I_n\end{array}\right]\left[\begin{array}{c|c}Q_2 & S_2\\ \hline \star & Q_2\end{array}\right]\left[\begin{array}{c|c}-\sqrt{\frac{\beta(t)}{1-\beta(t)}}I_n & 0_n\\ \hline 0_n & \sqrt{\frac{1-\beta(t)}{\beta(t)}}I_n\end{array}\right] > 0_{2n}.\end{aligned} \quad (17)$$

Also, when $h(t)=0$, $h(t)=h_M$ and $\tau(t)=0$, $\tau(t)=\tau_U$, respectively, we get

$$\int_{t-h(t)}^{t}\mu_i(s)\,ds=\int_{t-0}^{t}\mu_i(s)\,ds=0_{3n\times 1},\qquad \int_{t-h_U}^{t-h(t)}\mu_i(s)\,ds=\int_{t-h_U}^{t-h_U}\mu_i(s)\,ds=0_{3n\times 1}$$

and

$$\int_{t-\tau(t)}^{t}f\big(x(s)\big)\,ds=\int_{t-0}^{t}f\big(x(s)\big)\,ds=0_{n\times 1},\qquad \int_{t-\tau_U}^{t-\tau(t)}f\big(x(s)\big)\,ds=\int_{t-\tau_U}^{t-\tau_U}f\big(x(s)\big)\,ds=0_{n\times 1},\tag{18}$$

respectively.

Thus, from (17) and (18), the following inequality still holds:

$$\begin{aligned}
&-\begin{bmatrix}\int_{t-h(t)}^{t}\mu(s)\,ds\\ \int_{t-h_U}^{t-h(t)}\mu(s)\,ds\end{bmatrix}^T\left[\begin{array}{c|c}\frac{1}{1-\alpha(t)}(Q_1+\bar{P}_1) & 0_{3n}\\ \hline 0_{3n} & \frac{1}{\alpha(t)}(Q_1+\bar{P}_2)\end{array}\right]\begin{bmatrix}\int_{t-h(t)}^{t}\mu(s)\,ds\\ \int_{t-h_U}^{t-h(t)}\mu(s)\,ds\end{bmatrix}\\
&-\begin{bmatrix}\int_{t-\tau(t)}^{t}f(x(s))\,ds\\ \int_{t-\tau_U}^{t-\tau(t)}f(x(s))\,ds\end{bmatrix}^T\left[\begin{array}{c|c}\frac{1}{1-\beta(t)}Q_2 & 0_n\\ \hline 0_n & \frac{1}{\beta(t)}Q_2\end{array}\right]\begin{bmatrix}\int_{t-\tau(t)}^{t}f(x(s))\,ds\\ \int_{t-\tau_U}^{t-\tau(t)}f(x(s))\,ds\end{bmatrix}\\
&\le -\begin{bmatrix}\int_{t-h(t)}^{t}x(s)\,ds\\ x(t)-x(t-h(t))\\ \int_{t-h(t)}^{t}f(x(s))\,ds\\ \int_{t-h_U}^{t-h(t)}x(s)\,ds\\ x(t-h(t))-x(t-h_U)\\ \int_{t-h_U}^{t-h(t)}f(x(s))\,ds\end{bmatrix}^T\left[\begin{array}{c|c}Q_1+\bar{P}_1 & S_1\\ \hline \star & Q_1+\bar{P}_2\end{array}\right]\begin{bmatrix}\int_{t-h(t)}^{t}x(s)\,ds\\ x(t)-x(t-h(t))\\ \int_{t-h(t)}^{t}f(x(s))\,ds\\ \int_{t-h_U}^{t-h(t)}x(s)\,ds\\ x(t-h(t))-x(t-h_U)\\ \int_{t-h_U}^{t-h(t)}f(x(s))\,ds\end{bmatrix}\\
&-\begin{bmatrix}\int_{t-\tau(t)}^{t}f(x(s))\,ds\\ \int_{t-\tau_U}^{t-\tau(t)}f(x(s))\,ds\end{bmatrix}^T\left[\begin{array}{c|c}Q_2 & S_2\\ \hline \star & Q_2\end{array}\right]\begin{bmatrix}\int_{t-\tau(t)}^{t}f(x(s))\,ds\\ \int_{t-\tau_U}^{t-\tau(t)}f(x(s))\,ds\end{bmatrix}.
\end{aligned}\tag{19}$$

Here, if the inequality (8) holds, then an upper bound of the $\dot{V}_4$ can be rebounded as

$$\begin{aligned}
\dot{V}_4\le \zeta^T(t)\Bigg(&h_U^2\Pi_3Q_1\Pi_3^T+\tau_U^2e_8Q_2e_8^T\\
&+h_U\big(e_1P_1e_1^T-e_2(P_1-P_2)e_2^T-e_3P_2e_3^T\big)\\
&-\Pi_7\left[\begin{array}{c|c}Q_1+\bar{P}_1 & S_1\\ \hline \star & Q_1+\bar{P}_2\end{array}\right]\Pi_7^T\\
&-\Pi_8\left[\begin{array}{c|c}Q_2 & S_2\\ \hline \star & Q_2\end{array}\right]\Pi_8^T\Bigg)\zeta(t).
\end{aligned}\tag{20}$$

Lastly, an upper bound of $\dot{V}_5$ can be obtained as

$$\begin{aligned}
\dot{V}_5&=\left(\frac{h_U^2}{2}\right)^2\dot{x}^T(t)Q_3\dot{x}(t)-\frac{h_U^2}{2}\int_{t-h_U}^{t}\int_{s}^{t}\dot{x}^T(u)Q_3\dot{x}(u)\,du\,ds\\
&\le\left(\frac{h_U^2}{2}\right)^2\dot{x}^T(t)Q_3\dot{x}(t)-\left(h_Ux(t)-\int_{t-h(t)}^{t}x(s)\,ds-\int_{t-h_U}^{t-h(t)}x(s)\,ds\right)^TQ_3\\
&\quad\times\left(h_Ux(t)-\int_{t-h(t)}^{t}x(s)\,ds-\int_{t-h_U}^{t-h(t)}x(s)\,ds\right)\\
&=\zeta^T(t)\left(\left(\frac{h_U^2}{2}\right)^2e_4Q_3e_4^T-(h_Ue_1-e_6-e_7)\right.\\
&\quad\left.\times Q_3(h_Ue_1-e_6-e_7)^T\right)\zeta(t),
\end{aligned}\tag{21}$$

where Lemma 2 in [36] was utilized in above inequality.

Since the inequality $p^T(t)p(t) \le q^T(t)q(t)$ holds from (2) and (4), there exists a positive scalar ε satisfying the following inequality:

$$0 \le \varepsilon\big(q^T(t)q(t) - p^T(t)p(t)\big) = \zeta^T(t)\Omega\zeta(t). \quad (22)$$

Let us choose $v = 0$ from (3) and divide its range of (3) into m interval. It should be noted that subinterval of the range of (3) can be described as

$$\begin{cases} \text{Range }(1), & k_i^- \le \frac{f_i(u)}{u} \le k_i^- + \frac{1}{m}(k_i^+ - k_i^-), \\ \text{Range }(2), & k_i^- + \frac{1}{m}(k_i^+ - k_i^-) \le \frac{f_i(u)}{u} \le k_i^- + \frac{2}{m}(k_i^+ - k_i^-), \\ \quad\vdots & \quad\vdots \\ \text{Range }(m-1), & k_i^- + \frac{m-2}{m}(k_i^+ - k_i^-) \le \frac{f_i(u)}{u} \le k_i^- + \frac{m-1}{m}(k_i^+ - k_i^-), \\ \text{Range }(m), & k_i^- + \frac{m-1}{m}(k_i^+ - k_i^-) \le \frac{f_i(u)}{u} \le k_i^+, \end{cases}$$

where m is positive integer, and each condition is equivalent to

$$\begin{cases} \text{Range }(1), & [f_i(u) - k_i^- u][f_i(u) - (k_i^- + \frac{1}{m}(k_i^+ - k_i^-))u] < 0, \\ \text{Range }(2), & [f_i(u) - (k_i^- + \frac{1}{m}(k_i^+ - k_i^-))u][f_i(u) - (k_i^- + \frac{2}{m}(k_i^+ - k_i^-))u] < 0, \\ \quad\vdots & \quad\vdots \\ \text{Range }(m-1), & [f_i(u) - (k_i^- + \frac{m-2}{m}(k_i^+ - k_i^-))u][f_i(u) - (k_i^- + \frac{m-1}{m}(k_i^+ - k_i^-))u] < 0, \\ \text{Range }(m), & [f_i(u) - (k_i^- + \frac{m-1}{m}(k_i^+ - k_i^-))u][f_i(u) - k_i^+ u] < 0. \end{cases}$$

From the conditions just above, the following inequalities hold for any positive diagonal matrices: $H_{3(j-1)+l} = \mathrm{diag}\{h_1^{3(j-1)+l}, \dots, h_n^{3(j-1)+l}\}$ and $\tilde{H}_j = \mathrm{diag}\{\tilde{h}_1^j, \dots, \tilde{h}_n^j\}$, where $j = 1, 2, \dots, m$ and $l = 1, 2, 3$.

Case (j) for Range (j) with $j = 1, 2, \dots, m$:

$$\begin{aligned} 0 \le{}& -2\sum_{i=1}^{n} h_i^{3(j-1)+1}\Big[f_i\big(x_i(t)\big) - \Big(k_i^- + \frac{j-1}{m}(k_i^+ - k_i^-)\Big)x_i(t)\Big] \\ &\times \Big[f_i\big(x_i(t)\big) - \Big(k_i^- + \frac{j}{m}(k_i^+ - k_i^-)\Big)x_i(t)\Big] \\ &-2\sum_{i=1}^{n} h_i^{3(j-1)+2}\Big[f_i\big(x_i(t-h(t))\big) - \Big(k_i^- + \frac{j-1}{m}(k_i^+ - k_i^-)\Big)x_i\big(t-h(t)\big)\Big] \\ &\times \Big[f_i\big(x_i(t-h(t))\big) - \Big(k_i^- + \frac{j}{m}(k_i^+ - k_i^-)\Big)x_i\big(t-h(t)\big)\Big] \\ &-2\sum_{i=1}^{n} h_i^{3(j-1)+3}\Big[f_i\big(x_i(t-h_U)\big) - \Big(k_i^- + \frac{j-1}{m}(k_i^+ - k_i^-)\Big)x_i(t-h_U)\Big] \\ &\times \Big[f_i\big(x_i(t-h_U)\big) - \Big(k_i^- + \frac{j}{m}(k_i^+ - k_i^-)\Big)x_i(t-h_U)\Big] \\ &-2\sum_{i=1}^{n} \tilde{h}_i^{j}\Big[f_i\big(x_i(t-\tau_U)\big) - \Big(k_i^- + \frac{j-1}{m}(k_i^+ - k_i^-)\Big)x_i(t-\tau_U)\Big] \\ &\times \Big[f_i\big(x_i(t-\tau_U)\big) - \Big(k_i^- + \frac{j}{m}(k_i^+ - k_i^-)\Big)x_i(t-\tau_U)\Big] \\ ={}& \zeta^T(t)\Theta_j\zeta(t). \end{aligned} \quad (23)$$

From (11)–(23) and by applying S-procedure [45], an upper bound of $\dot{V} - 2y^T(t)u(t) - \gamma u^T(t)u(t)$ can be

$$\dot{V} - 2y^T(t)u(t) - \gamma u^T(t)u(t) \le \zeta^T(t)(\Psi + \Theta_j)\zeta(t) \quad (j = 1, 2, \ldots, m). \tag{24}$$

By Lemma 1, $\zeta^T(t)(\Psi + \Theta_j)\zeta(t)$ with $\Upsilon\zeta(t) = 0_{n\times 1}$ is equivalent to $(\Upsilon^{\perp})^T(\Psi + \Theta_j)\Upsilon^{\perp} < 0$. Therefore, if LMIs (7), (8), and (9) hold, then $(\Upsilon^{\perp})^T(\Psi + \Theta_j)\Upsilon^{\perp} < 0$ holds, which means

$$\dot{V} - 2y^T(t)u(t) - \gamma u^T(t)u(t) < 0. \tag{25}$$

By integrating (25) with respect to t over the time period from 0 to t_p, we have

$$V\big(x(t_p)\big) - V\big(x(0)\big) - \gamma \int_0^{t_p} u^T(s)u(s)\,ds \le 2\int_0^{t_p} y^T(s)u(s)\,ds, \tag{26}$$

for $x(0) = 0$. Since $V(x(0)) = 0$, the inequality (5) in Definition 1 holds. This implies that the neural networks (1) is passive in the sense of Definition 1. This completes our proof. □

Remark 2 Unlike the method of [38], the utilized augmented vector $\zeta(t)$ includes the state vector such as $f(x(t-\tau_U))$ and $x(t-\tau_U)$. These state vectors have not been utilized as an element of augmented vector $\zeta(t)$ in any other literature, which is the main difference between Theorem 1 and the methods in other literature. Correspondingly, in (23), the terms such as $-2\sum_{i=1}^n \tilde{h}_i^j[f_i(x_i(t-\tau_U)) - (k_i^- + \frac{j-1}{m}(k_i^+ - k_i^-))x_i(t-\tau_U)][f_i(x_i(t-\tau_U)) - (k_i^- + \frac{j}{m}(k_i^+ - k_i^-))x_i(t-\tau_U)]$ are utilized for the first time.

Remark 3 Recently, the reciprocally convex optimization technique to reduce the conservatism of stability for systems with time-varying delays was proposed in [41]. Motivated by this work, in (15)–(19), the proposed method of [41] was utilized in obtaining upper bounds of the terms such as

$$-h_U\int_{t-h(t)}^{t} \mu^T(s)(Q_1 + P_1)\mu(s)\,ds - h_U\int_{t-h_U}^{t-h(t)} \mu^T(s)(Q_1 + P_2)\mu(s)\,ds \tag{27}$$

and

$$-\tau_U\int_{t-\tau(t)}^{t} f^T\big(x(s)\big)Q_2 f\big(x(s)\big)\,ds - \tau_U\int_{t-\tau_U}^{t-\tau(t)} f^T\big(x(s)\big)Q_2 f\big(x(s)\big)\,ds. \tag{28}$$

Remark 4 In (14), two zero equalities are proposed inspired by the work of [40] and utilized in Theorem 1 to reduce the conservatism of the stability condition. As presented in (14), the terms $x^T(t)(h_U P_1)x(t) - x^T(t-h(t))(h_U P_1)x(t-h(t))$ and $x^T(t-h(t)) \times (h_U P_2)x(t-h(t)) - x^T(t-h_U)(h_U P_2)x(t-h_U)$ provide the enhanced feasible region of the passivity criterion. Furthermore, as shown in (15), the two integral terms such as $-2h_U\int_{t-h(t)}^{t} \dot{x}^T(s)P_1x(s)\,ds$ and $-2h_U\int_{t-h_U}^{t-h(t)} \dot{x}^T(s)P_2x(s)\,ds$ presented in (14) are merged into the integral terms $-h_U\int_{t-h(t)}^{t}\mu^T(s) \times Q_1\mu(s)\,ds$ and $-h_U\int_{t-h_U}^{t-h(t)}\mu^T(s)Q_1\mu(s)\,ds$, which cause the conservatism of the passivity criterion.

Remark 5 Inspired by the fact that the stability and performance of neural networks are related to the choice of activation functions [46], the range of the term, $k_i^- \le \frac{f_i(u)}{u} \le k_i^+$, is divided into m subintervals such as $k_i^- \le \frac{f_i(u)}{u} \le k_i^- + \frac{1}{m}(k_i^+ - k_i^-)$, $k_i^- + \frac{1}{m}(k_i^+ - k_i^-) \le \frac{f_i(u)}{u} \le k_i^- + \frac{2}{m}(k_i^+ - k_i^-), \ldots, k_i^- + \frac{m-2}{m}(k_i^+ - k_i^-) \le \frac{f_i(u)}{u} \le k_i^- + \frac{m-1}{m}(k_i^+ - k_i^-)$, and $k_i^- + \frac{m-1}{m}(k_i^+ - k_i^-) \le \frac{f_i(u)}{u} \le k_i^+$. By choosing u as $x(t)$, $x(t-h(t))$, $x(t-h_U)$, and $x(t-\tau_U)$, the inequalities (23) are utilized in Theorem 1. This idea has not been proposed in passivity analysis for uncertain neural networks with mixed time-varying delays. The advantage of this approach is that the feasible region of passivity criterion can be enhanced as the partitioning number m increases. It should be pointed unlike the delay-partitioning approach, the augmented vector are not changed. However, as m increases, the number of decision variables becomes larger. Through three numerical examples, it will be shown the feasible region of passivity criterion introduced in Theorem 1 can be significantly enhanced as the partitioning number m increases.

As a special case, the networks (4) without distributed delays can be rewritten as

$$\dot{x}(t) = -\big(A + \Delta A(t)\big)x(t) + \big(W_0 + \Delta W_0(t)\big) f\big(x(t)\big) + \big(W_1 + \Delta W_1(t)\big) f\big(x\big(t - h(t)\big)\big) + u(t),$$
$$y(t) = C_1 f\big(x(t)\big) + C_2 f\big(x\big(t - h(t)\big)\big). \tag{29}$$

We can obtain a passivity criterion of the network (29) by the similar method of the proof of Theorem 1. This result will be introduced in Theorem 2. For the sake of simplicity on matrix representation, $e_i \in \mathbb{R}^{14n\times n}$ $(i = 1, \dots, 14)$ are defined as block entry matrices (for example, $e_2 = [0_n, I_n, 0_{12n\times n}]^T$). Before introducing this, the notations of several matrices are defined as:

$$\hat{\zeta}^T(t) = \Big[x^T(t), x^T\big(t - h(t)\big), x^T(t - h_U), \dot{x}^T(t), \dot{x}^T(t - h_U), \int_{t-h(t)}^{t} x^T(s)\,ds, \int_{t-h_U}^{t-h(t)} x^T(s)\,ds, f^T\big(x(t)\big), f^T\big(x\big(t - h(t)\big)\big), f^T\big(x(t - h_U)\big), \int_{t-h(t)}^{t} f^T\big(x(s)\big)\,ds, \int_{t-h_U}^{t-h(t)} f^T\big(x(s)\big)\,ds, u^T(t), p^T(t)\Big],$$
$$\mu^T(t) = \big[x^T(t), \dot{x}^T(t), f^T\big(x(t)\big)\big],$$
$$\nu^T(t) = \big[x^T(t), f^T\big(x(t)\big)\big],$$
$$\hat{\Upsilon} = [-A, 0_{2n\times n}, -I_n, 0_{3n\times n}, W_0, W_1, 0_{3n\times n}, I_n, D], \tag{30}$$

$$\hat{\Pi}_1 = [e_1, e_3, e_6 + e_7, e_{11} + e_{12}],$$
$$\hat{\Pi}_2 = [e_4, e_5, e_1 - e_3, e_8 - e_{10}],$$
$$\hat{\Pi}_7 = [e_6, e_1 - e_2, e_{11}, e_7, e_2 - e_3, e_{12}],$$
$$\hat{\Phi}_1 = \mathrm{sym}\{\hat{\Pi}_1 R \hat{\Pi}_2^T\},$$
$$\hat{\Omega} = \varepsilon\big(-E_a e_1^T + E_0 e_8^T + E_1 e_9^T\big)^T \times \big(-E_a e_1^T + E_0 e_8^T + E_1 e_9^T\big) - \varepsilon e_{14} e_{14}^T,$$
$$\hat{\Psi} = \hat{\Phi}_1 + \Phi_2 + \hat{\Omega} - \mathrm{sym}\{e_8 C_1^T e_{13}^T\} - \mathrm{sym}\{e_9 C_2^T e_{13}^T\} - \gamma e_{13} e_{13}^T,$$

where $\bar{P}_1$, $\bar{P}_2$, Π_a $(a = 3, \dots, 6)$ and Φ_2 are defined in (6).

Now, we have the following theorem.

Theorem 2 *For given positive scalars h_U, h_D and a positive integer m, diagonal matrices $K^- = \mathrm{diag}\{k_1^-, \dots, k_n^-\}$ and $K^+ = \mathrm{diag}\{k_1^+, \dots, k_n^+\}$, the system (4) is passive for $0 \le h(t) \le h_U$ and $\dot{h}(t) \le h_D$, if there exist positive scalars ε and γ, positive diagonal matrices $\Lambda_a = \mathrm{diag}\{\lambda_{a1}, \dots, \lambda_{an}\}$ $(a = 1, 2)$, $\Delta_a = \mathrm{diag}\{\delta_{a1}, \dots, \delta_{an}\}$ $(a = 1, 2)$, $H_b = \mathrm{diag}\{h_1^b, \dots, h_n^b\}$ $(b = 1, 2, \dots, 3m)$, positive definite matrices $R \in \mathbb{R}^{4n\times 4n}$, $N \in \mathbb{R}^{3n\times 3n}$, $M \in \mathbb{R}^{2n\times 2n}$, $G \in \mathbb{R}^{2n\times 2n}$, $Q_1 \in \mathbb{R}^{3n\times 3n}$, $Q_3 \in \mathbb{R}^{n\times n}$ and any symmetric matrix $P_a \in \mathbb{R}^{n\times n}$ $(a = 1, 2)$, and any matrix $S_1 \in \mathbb{R}^{3n\times 3n}$ satisfying the following LMIs with (8):*

$$\big(\hat{\Upsilon}^\perp\big)^T (\hat{\Psi} + \Theta_{1j}) \hat{\Upsilon}^\perp < 0 \quad (j = 1, 2, \dots, m), \tag{31}$$

where $\hat{\Upsilon}$ and $\hat{\Psi}$ are defined in (30), and $\bar{P}_a$ $(a = 1, 2)$ and Θ_{1j} are in (6).

Proof Consider the following Lyapunov–Krasovskii functional candidate as

$$V = V_1 + V_2 + V_3 + V_4 + V_5, \tag{32}$$

where

$$V_1 = \begin{bmatrix} x(t) \\ x(t - h_U) \\ \int_{t-h_U}^{t} x(s)\,ds \\ \int_{t-h_U}^{t} f(x(s))\,ds \end{bmatrix}^T R \begin{bmatrix} x(t) \\ x(t - h_U) \\ \int_{t-h_U}^{t} x(s)\,ds \\ \int_{t-h_U}^{t} f(x(s))\,ds \end{bmatrix},$$

$$V_2 = \int_{t-h_U}^{t} \mu^T(s) N \mu(s)\,ds + 2\sum_{i=1}^{n} \Big(\lambda_{1i} \int_0^{x_i(t)} \big(f_i(s) - k_i^- s\big)\,ds + \delta_{1i} \int_0^{x_i(t)} \big(k_i^+ s - f_i(s)\big)\,ds\Big) + 2\sum_{i=1}^{n} \Big(\lambda_{2i} \int_0^{x_i(t-h_U)} \big(f_i(s) - k_i^- s\big)\,ds + \delta_{2i} \int_0^{x_i(t-h_U)} \big(k_i^+ s - f_i(s)\big)\,ds\Big),$$

$$V_3 = \int_{t-h(t)}^{t} \nu^T(s) G \nu(s)\,ds,$$

$$V_4 = h_U \int_{t-h_U}^{t} \int_s^{t} \mu^T(u) Q_1 \mu(u)\,du\,ds,$$

Table 1 Upper bounds of time-delay h_U and τ_U with different h_D (Example 1)

Methods	h_D	0.1	0.3	0.5	0.7	0.9
Chen et al. [31]	$h_U = \tau_U$	0.5005	0.4295	0.4282	0.4275	0.4270
Chen et al. [32]	$h_U = \tau_U$	0.5060	0.4746	0.4742	0.4740	0.4740
Theorem 1 ($m = 1$)	$h_U = \tau_U$	0.5182	0.5120	0.5103	0.5083	0.5059
Theorem 1 ($m = 2$)	$h_U = \tau_U$	0.5209	0.5172	0.5160	0.5149	0.5145
Theorem 1 ($m = 3$)	$h_U = \tau_U$	0.5225	0.5197	0.5188	0.5185	0.5185

Table 2 Upper bounds of time-delay h_U with fixed τ_U and different h_D (Example 1)

Methods	h_D	0.1	0.3	0.5	0.7	0.9
	τ_U	0.5070	0.4800	0.4800	0.4800	0.4800
Li et al. [38] ($\alpha = 0.1$)	h_U	0.5605	0.5185	0.5134	0.5100	0.5098
Li et al. [38] ($\alpha = 0.9$)	h_U	0.5604	0.5182	0.5132	0.5096	0.5095
Theorem 1 ($m = 1$)	h_U	0.6737	0.6381	0.6275	0.6158	0.6021
Theorem 1 ($m = 2$)	h_U	0.7006	0.6711	0.6629	0.6555	0.6523
Theorem 1 ($m = 3$)	h_U	0.7174	0.6888	0.6820	0.6787	0.6784

$$V_5 = \frac{h_U^2}{2}\int_{t-h_U}^{t}\int_{s}^{t}\int_{u}^{t}\dot{x}^T(v)Q_3\dot{x}(v)\,dv\,du\,ds.$$

The other procedure of proof is straightforward from the proof of Theorem 1, so it is omitted. □

4 Numerical examples

In this section, three numerical examples will be shown to illustrate the effectiveness of the proposed criteria. In examples, MATLAB, YALMIP 3.0, and SeDuMi 1.3 are used to solve LMI problems.

Example 1 Consider the neural networks (1) with

$$A = \begin{bmatrix} 2.3 & 0 \\ 0 & 2.5 \end{bmatrix}, \qquad W_0 = \begin{bmatrix} 0.3 & 0.2 \\ 0.4 & 0.1 \end{bmatrix},$$

$$W_1 = \begin{bmatrix} 0.5 & 0.7 \\ 0.7 & 0.4 \end{bmatrix}, \qquad W_2 = \begin{bmatrix} 0.5 & -0.3 \\ 0.2 & 1.2 \end{bmatrix},$$

$$C_1 = I_2, \qquad C_2 = 0_2, \qquad D = 0.2I_2,$$

$$E_a = E_0 = E_1 = E_2 = I_2,$$

$$K^- = 0_2, \qquad K^+ = I_2$$

$$\text{with } f(x) = \frac{1}{2}\big(|x+1| - |x-1|\big).$$

Tables 1 and 2 show the results of the upper bound of time-delay for the above system. It can be seen that Theorem 1 in this paper provides larger delay bound than the previous results. Furthermore, when the partitioning number m increases, the maximum delay bounds get larger. This indicates that the presented passivity conditions relieve the constraint of the passivity caused by time-delay. To confirm one of the obtained results in Table 2 ($h_D = 0.1$, $h_U = 0.7174$, $\tau_U = 0.5070$), a simulation result when $x(0) = [-1, -0.5]^T$, $h(t) = 0.7174\sin^2(0.13t)$, $\tau(t) = 0.5070\sin^2(t)$, $u(t) = 0.1\sin(2\pi t)$, $\Delta A(t) = \Delta W_0(t) = \Delta W_1(t) = \Delta W_2(t) = 0.2\,\mathrm{diag}\{\sin(t), \sin(t)\}$ are given in Fig. 1. Figure 1 shows that the system (1) with above parameters is passive in the sense of Definition 1.

Example 2 Consider the neural networks (29) with

$$A = \begin{bmatrix} 2.2 & 0 \\ 0 & 1.5 \end{bmatrix}, \qquad W_0 = \begin{bmatrix} 1 & 0.6 \\ 0.1 & 0.3 \end{bmatrix},$$

$$W_1 = \begin{bmatrix} 1 & -0.1 \\ 0.1 & 0.2 \end{bmatrix},$$

$$C_1 = I_2, \qquad C_2 = 0_2, \qquad D = 0.1I_2,$$

$$E_a = 0.1I_2, \qquad E_0 = 0.2I_2, \qquad E_1 = 0.3I_2,$$

$$K^- = 0_2, \qquad K^+ = I_2$$

$$\text{with } f(x) = \frac{1}{2}\big(|x+1| - |x-1|\big).$$

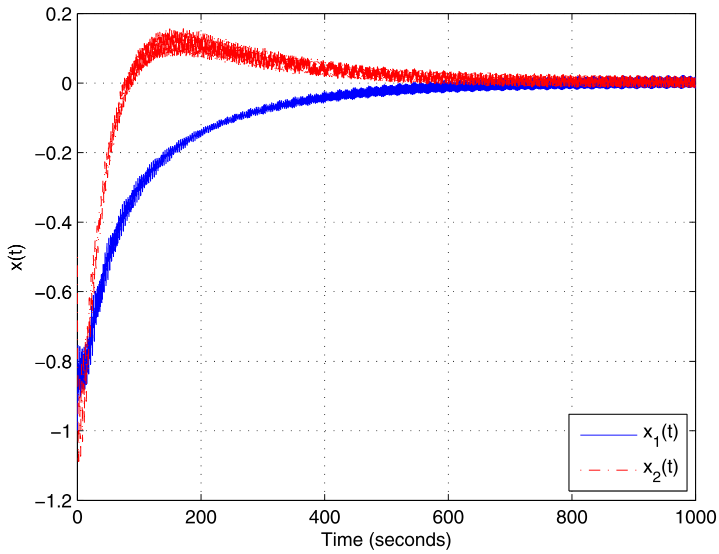

Fig. 1 State trajectories with $h_D = 0.1$, $h_U = 0.7174$ and $\tau_U = 0.5070$ (Example 1)

Table 3 Upper bounds of time-delay h_U with different h_D (Example 2)

Methods	0.3	0.5	0.7	0.9	Unknown
Chen et al. [31]	0.4197	0.4145	0.4117	0.4082	0.3994
Chen et al. [32]	0.5624	0.5580	0.5565	0.5523	0.5420
Fu et al. [34]	0.5763	0.5679	0.5566	0.5273	0.5129
Kwon et al. [36]	0.8171	0.7581	0.7029	0.6380	0.6059
Theorem 2 ($m = 1$)	1.0213	0.9532	0.8910	0.8144	0.7767
Theorem 2 ($m = 2$)	1.1301	1.0868	1.0460	1.0008	0.9888
Theorem 2 ($m = 3$)	1.1921	1.1590	1.1297	1.1081	1.1008

In Table 3, the results of the upper bound of time-delay for guaranteeing passivity are compared with the previous results. From the results of Table 3, it can be seen that the maximum delay bounds for guaranteeing the passivity of the above neural networks are larger than those of other literature listed in Table 3. To confirm one of the obtained result in Table 3 ($h_D = 0.5$, $h_U = 1.1590$), a simulation result when $x(0) = [1, -1]^T$, $h(t) = 1.1590 \sin^2(0.43t)$, $u(t) = 1$, $\Delta A(t) = 0.01\,\mathrm{diag}\{\sin(t), \sin(t)\}$, $\Delta W_0(t) = 0.02\,\mathrm{diag}\{\sin(t), \sin(t)\}$, $\Delta W_1(t) = 0.03\,\mathrm{diag}\{\sin(t), \sin(t)\}$ are shown in Fig. 2. From Fig. 2, it can be confirmed that the neural networks (29) with above parameters when $0 \le h(t) \le 1.1590$ and $\dot{h}(t) \le 0.5$ is passive in the sense of Definition 1.

Example 3 Consider the neural networks (29) with

$$A = \begin{bmatrix} 2.2 & 0 \\ 0 & 1.8 \end{bmatrix}, \qquad W_0 = \begin{bmatrix} 1.2 & 1 \\ -0.2 & 0.3 \end{bmatrix},$$

$$W_1 = \begin{bmatrix} 0.8 & 0.4 \\ -0.2 & 0.1 \end{bmatrix},$$

$$C_1 = I_2, \qquad C_2 = 0_2, \qquad D = E_a = E_1 = E_2 = 0_2,$$

$$K^- = 0_2, \qquad K^+ = I_2$$

with $f(x) = \frac{1}{2}\big(|x+1| - |x-1|\big)$.

When $h_D = 0.5$, the obtained upper bounds of time-delay for guaranteeing the passivity of the above neural networks in [33] and [35] were 0.7230 and 1.3752, respectively. By applying Theorem 2 with $m = 2$, it can be obtained that the upper bound of time-delay is 35.3121, which is much larger delay bound than one in [33] and [35]. When h_D is unknown, the upper bound of time-delay obtained 0.6791 and 1.3027 in [33] and [35], respectively. However, by using Theorem 2 with $m = 2$, one can obtain the upper bound of time-delay is 3.9715. Moreover, by utilizing Theorem 2 with $m = 1$ and $m = 3$, the upper bounds

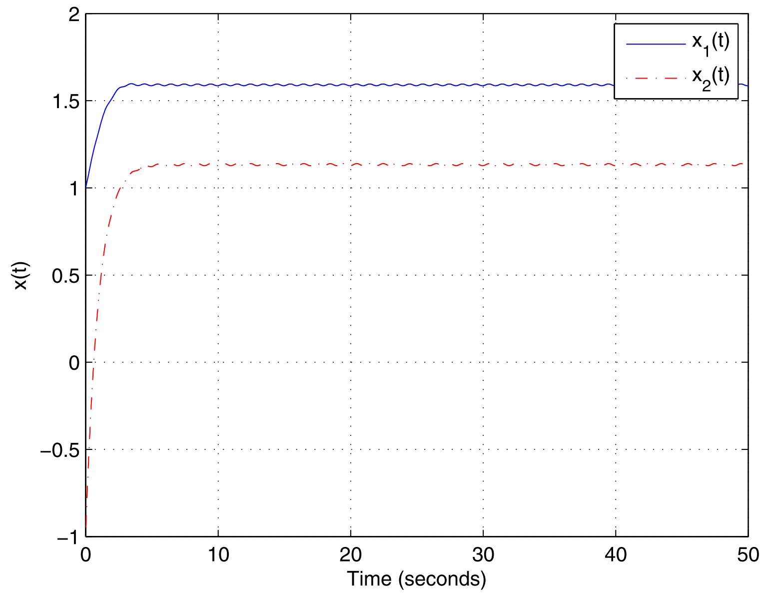

Fig. 2 State trajectories with $h_D = 0.5$ and $h_U = 1.1590$ (Example 2)

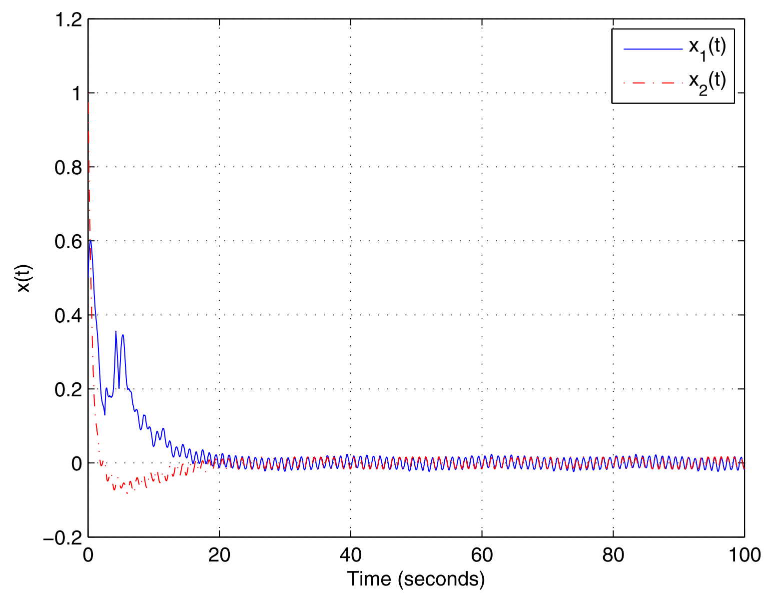

Fig. 3 State trajectories with unknown h_D and $h_U = 4.7368$ (Example 3)

of time-delay with different h_D listed in Table 4. From the results of Table 4, it can be confirmed that Theorem 2 gives larger delay bounds than those obtained by the method of [33] and [35]. To confirm one of the obtained result in Table 4 (h_D is unknown, $h_U = 4.7368$), a simulation result when $x(0) = [0.5, 1]^T$, $h(t) = 4.7368|\sin(t)|$, $u(t) = 0.1\sin(2\pi t)$ are given in Fig. 3. From Fig. 3, it can be also verified that the neural networks (29) with above parameters when $h_U = 4.7368$ and h_D is unknown is passive in the sense of Definition 1.

5 Conclusion

In this paper, the improved passivity criteria for uncertain neural networks with both discrete and distributed time-varying delays have been proposed. In order to

Table 4 Upper bounds of time-delay h_U with different h_D (Example 3)

Methods	0.5	0.7	0.9	Unknown
Xu et al. [33]	0.7230	0.6814	0.6791	0.6791
Zeng et al. [35]	1.3752	1.3036	1.3027	1.3027
Theorem 2 ($m=1$)	2.5363	2.4530	2.3539	2.2725
Theorem 2 ($m=2$)	35.3121	4.2216	4.0058	3.9715
Theorem 2 ($m=3$)	35.3121	9.0128	4.7568	4.7368

drive less conservative results, the suitable Lyapunov–Krasovskii functional and decomposed conditions of activation function divided by states are utilized to enhance the feasible region of passivity criteria. Three numerical examples have been illustrated to show the effectiveness of the proposed methods. Future works will focus on passivity analysis and passification of various neural networks such as fuzzy neural networks, static neural networks, and so on. Furthermore, some new passivity analysis for discrete-time neural network with time-varying delays will be investigated in the near future.

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