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ORIGINAL PAPER

Triple mode alignment in a canonical model of the blue-sky catastrophe

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Abstract The blue-sky catastrophe (BSC) is a homoclinic bifurcation of a saddle node periodic orbit of codimension one, which has been found to occur in a number of physically relevant dynamics systems. The onset and termination of the BSC in a chaotic system is shown to coincide with the occurrence of triple mode alignment in a canonical model undergoing the BSC when the model is recast as an oscillator system. Typically, such behavior is only seen in hyperchaotic systems of dimension greater than three. Hence, in the case of three dimensional chaotic systems, competitive modes may under some circumstances be used in the prediction of the blue-sky catastrophe. Limitations to this approach are also discussed.

Keywords Blue-sky catastrophe · Onset of chaos · Competitive modes · Volume contraction

1 Introduction

The blue-sky catastrophe (BSC) is a homoclinic bifurcation of a saddle node periodic orbit of codimension one. As discussed in [1], chaotic attractors may also appear or bifurcate via BSC, which occur when an attractor touches the inset of a saddle cycle [2]. Of the seven known bifurcations of a periodic orbit, it was the last to be discovered [3].

Shilnikov and Cymbalyuk [4] studied a continuous and reversible transition between periodic tonic spiking and bursting activities in a neuron model and determined the model exhibits the BSC; such a transition constitutes serves as a biophysically plausible mechanism for the regulation of burst duration. These results were extended by Channell, Cymbalyuk, and Shilnikov [5], where it was shown that as the activation kinetics of the slow potassium current are shifted toward depolarized membrane potential values, the bursting phase accommodates incrementally more spikes into the train. For a further analysis a the neuron model, see Shilnikov [6]. Abraham and Stewart [7] studied the BSC in the Shaw variant of the van der Pol oscillator used as a model of a forced relaxation oscillator. Meanwhile, BSC was shown to occur in a double-well Duffing–van der Pol oscillator by Venkatesan and Lakshmanan [8], in the forced Brusselator by Knudsen et al. [9], in the delayed circle map (a discrete system) by Franciosi [10]. Meca et al. [11] discovered the BSC in a small Prandtl number binary mixture contained in a laterally heated cavity. Here, the BSC appears at the destruction of a stable orbit for certain Rayleigh numbers. Likewise, bursting in the Taylor–Couette flow due to the BSC was observed by Abshagen et al. [12]. McCann and Yodzis [13] find the BSC in a population model, and show that it can serve as one mechanism for population extinction. The relation between population extinction and the BSC was

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further investigated by Schreiber in [14]. The BSC was found in a model of parametrically excited hinged-clamped beams by Chin and Nayfeh [15]. Both Nordstrom Jensen and True [16] along with the more recent work of Gao, Li, and Yue [17] found the BSC in a symmetric wheel-rail system. A fuzzy BSC was reported by Hong and Sun [18].

Due the diverse applications in which nonlinear models exhibit the BSC, it is clear that there would be some utility in predicting the occurrence of the BSC. McRobie [19] reported that the first change in the period one Birkhoff signature is observed to occur close to (and shortly after) the chaotic escape (i.e., the BSC) of the primary resonant attractor. In the present paper, we show that the onset of the blue-sky catastrophe can be predicted by the use of competitive modes [20–26]. In particular, at the onset of the blue-sky catastrophe, we find that three mode frequencies become equal before breaking away from one another. During the blue-sky catastrophe, two of the modes are intermittently competitive, which is what we expect from a chaotic system. Furthermore, the triple mode alignment corresponds to strong volume contraction in the system. In this way, the onset and end of the blue-sky catastrophe can be approximated through competitive modes, which therefore serve as an indicator of such behavior.

Within the context of existing works in this area, it is clear that being able to detect the BSC is useful for applications. Hence, we consider a specific model from the literature which admits the BSC, and determine the features of the mode frequencies near the onset and termination of the bursting behavior. This type of diagnostic could be applied to other nonlinear dynamical systems, which display the BSC, such as those mentioned in [4–18].

2 Competitive modes: An overview

For a differentiable vector field $\mathbf{F}(\mathbf{x})$ on $\mathbb{R}^n$ and the related dynamical system $\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x})$, let us write

$$\frac{d}{dt}F_j(\mathbf{x}) = -g_j(\mathbf{x})x_j + h_j(\hat{\mathbf{x}}_j), \quad 1 \le j \le n, \tag{1}$$

where $\hat{\mathbf{x}}_j = (x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_n)$. Then

$$\ddot{x}_j + g_j(\mathbf{x})x_j = h_j(\hat{\mathbf{x}}_j), \quad 1 \le j \le n, \tag{2}$$

is a system of oscillator equations (provided that $g_j > 0$ for two or more j's). We refer to the g_j's as more frequencies for this reason. Let g_k and g_l be any two mode frequencies. Then, if (i) g_k and g_l are positive at some $t^* \ge 0$; (ii) $g_k(t^*) \approx g_l(t^*)$; (iii) at least one of g_k, g_l is nonconstant; (iv) at least one h_k, h_l is a function of system variables, we say that the modes $x_k(t)$ and $x_l(t)$ are competitive at $t = t^*$. Yao, Yu, and Essex [20] conjectured that (i)–(iv) are equivalent to the conditions for a dynamical system to be chaotic. However, there are some cases where a system satisfying (i)–(iv) is nonchaotic (particularly, in a neighborhood of an equilibrium, two modes may become competitive for all time [24]). However, for all chaotic systems studied with the method, (i)–(iv) have held intermittently. This suggests that (i)–(iv) are necessary (assuming such conditions hold intermittently, and not always), though not sufficient, conditions for a system to be chaotic [25, 26]. In chaotic systems, it is standard for two modes to be intermittently competitive, whereas when three modes become competitive we often have hyperchaos; this was demonstrated in the case of quadratic response functions in Choudhury and Van Gorder [26], where known dimension-four hyperchaotic dynamical systems were shown to have three intermittently competitive modes.

3 Two-parameter model for the BSC

As mentioned previously, a number of models exist which demonstrate the BSC. For sake of demonstration, we shall consider the two-parameter model

$$\begin{aligned}
\dot{x} &= \big(2 + a - 10(x^2 + y^2)\big)x + y^2 + 2y + z^2, \\
\dot{y} &= -z^3 - (1 + y)\big(y^2 + 2y + z^2\big) - 4x + ay, \\
\dot{z} &= (1 + y)z^2 + x^2 - b,
\end{aligned} \tag{3}$$

which has been used as one model of the BSC [27, 28]. In particular, when $a = 0.456$ and $b = 0.0357$, the system exhibits the BSC; see Fig. 1. This system is quite distinct from the chaotic or even hyperchoaotic systems considered in [26], since there are cubic (as opposed to only quadratic) response functions.

Note that for these parameter values, the system has volume element expansion or contraction given by

$$\begin{aligned}
\triangle V &= \nabla \cdot (\dot{x}, \dot{y}, \dot{z}) \\
&= \frac{\partial \dot{x}}{\partial x} + \frac{\partial \dot{y}}{\partial y} + \frac{\partial \dot{z}}{\partial z}
\end{aligned}$$

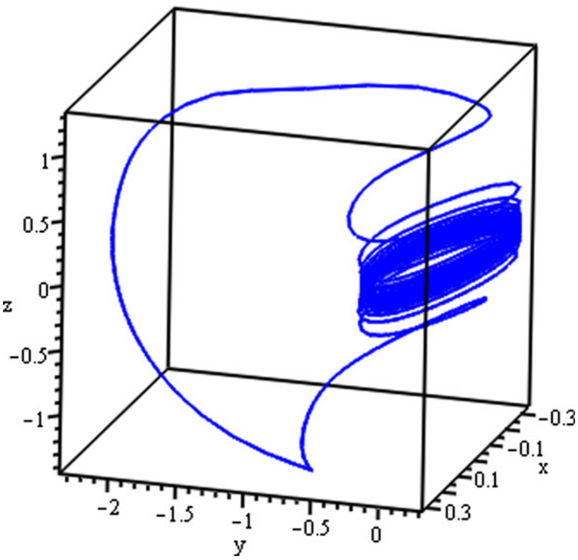

Fig. 1 Phase portrait for the BSC solutions of (3) with $a = 0.456$ and $b = 0.0357$

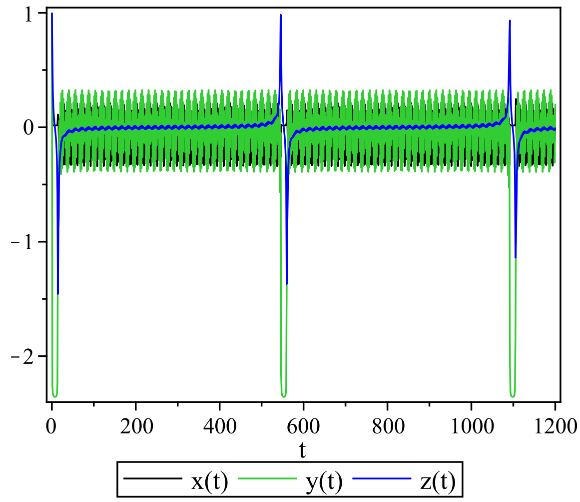

Fig. 2 Time series on $t \in [0, 1200]$ for the BSC solutions of (3) with $a = 0.456$ and $b = 0.0357$

$$= \frac{6a+4}{3} - \bigl(30x^2 + 12(y+1/6)^2 + (y-z+1)^2\bigr). \tag{4}$$

From (3), we compute

$$g_1 = -300x^4 + \bigl(80 + 40a - 400y^2\bigr)x^2 + 8y + \bigl(30\bigl(z^2+y^2\bigr) - 2z - 20y\bigr)x - \bigl(2 + a - 10y^2\bigr)^2 + 20y\bigl(ay - z^2 - (1+y)\bigl(y^2+2y+z^2\bigr)\bigr) + 8, \tag{5}$$

$$g_2 = -3y^4 - 15y^3 + \bigl(4a - 4z^2 - 26\bigr)y^2 + 3z^4 + \bigl(9a - 52x - 12z^2 - z^3 - 14\bigr)y - 4z^3 - 8z^2 - 24x + \bigl(a - 2 - z^2\bigr)\bigl(z^2 - a\bigr)n + 4 + 2a + 2z\bigl(z^2 + x^2 - b\bigr), \tag{6}$$

$$g_3 = z^4 + (1+y)z^3 - 2(1+y)^2z^2 - \bigl(ay - (1+y)\bigl(y^2+2y\bigr) - 2x\bigr)z - 2(1+y)\bigl(x^2\bigr) - b. \tag{7}$$

We know that necessary conditions for mode competitiveness are $g_1 = g_2$, $g_1 = g_3$, or $g_2 = g_3$ for some $t \geq 0$. Without loss of generality, assume that this occurs at some $t = t^*$. Then we may define the new variables $\tau = t - t^*$ so that the τ-system is competitive at $\tau = 0$. Hence, it often suffices to detect initial data $(x(0), y(0), z(0)) = (x_0, y_0, z_0)$ for which the modes become competitive. If a and b are kept arbitrary for the moment, note that forcing one competitiveness condition results in a restriction $b = b^*(x_0, y_0, z_0, a)$. Then forcing another competitiveness condition can yield a restriction on a, say $a = a^*(x_0, y_0, z_0)$. If these conditions hold jointly, we can have triple-mode competitiveness for $b = b^*(x_0, y_0, z_0, a^*(x_0, y_0, z_0))$. Yet, from the above, we know that $a = 0.456$ and $b = 0.0357$ give the BSC, so we can have triple-mode competitiveness when $a^*(x_0, y_0, z_0) = 0.456$ and $b^*(x_0, y_0, z_0, 0.456) = 0.0357$, provided such (x_0, y_0, z_0) exist.

Now, in Fig. 2, we plot the time series over $t \in [0, 1200]$ for the solutions and in Fig. 3 we plot the mode frequencies over $t \in [0, 1200]$, both for $a = 0.456$ and $b = 0.0357$. The following pattern emerges:

- when the system begins a bursting pattern, three modes become competitive for an instant;
- when the system is bursting, two modes are intermittently competitive;
- when the system stops a bursting period, three modes become competitive for an instant;
- when the system is on a periodic orbit connecting two bursts, the modes decouple;

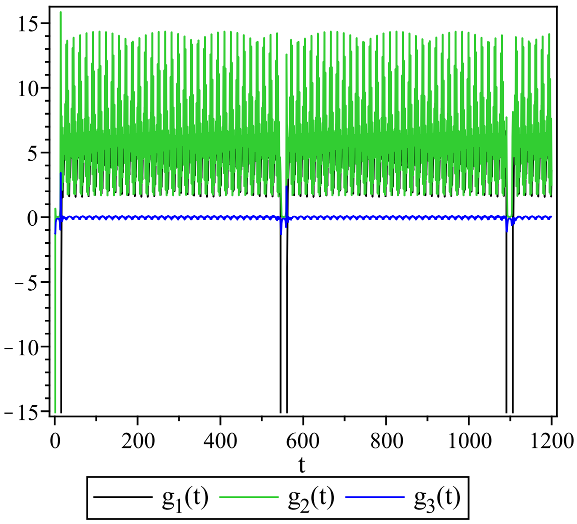

Fig. 3 Mode frequencies g_1, g_2, g_3 on $t \in [0, 1200]$ for the BSC solutions of (3) with $a = 0.456$ and $b = 0.0357$. We find that three mode frequencies become intermittently competitive; this corresponds precisely to the change in qualitative behavior observed in solutions to (3) due to the BSC

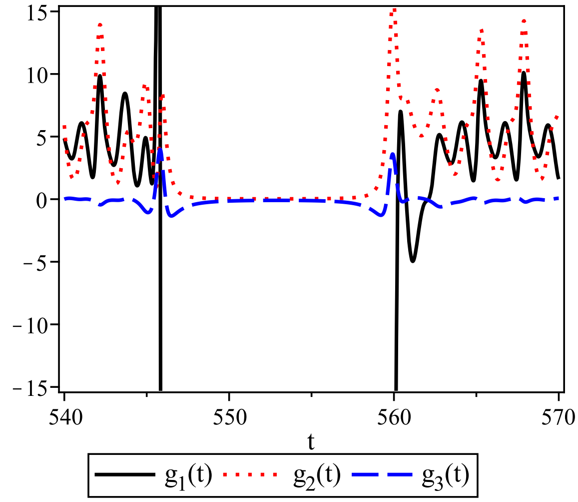

Fig. 5 Mode frequencies g_1, g_2, g_3 on $t \in [540, 570]$ for the BSC solutions of (3) with $a = 0.456$ and $b = 0.0357$. Three modes become competitive only at the start and conclusion of the burst, the modes decouple while on the periodic orbit, and during the bursting phase, two modes remain competitive

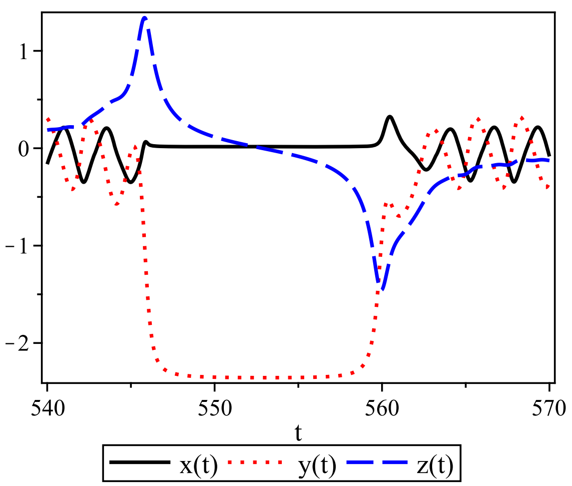

Fig. 4 Time series on $t \in [540, 570]$ for the BSC solutions of (3) with $a = 0.456$ and $b = 0.0357$. Here, we take a closer look at an orbit

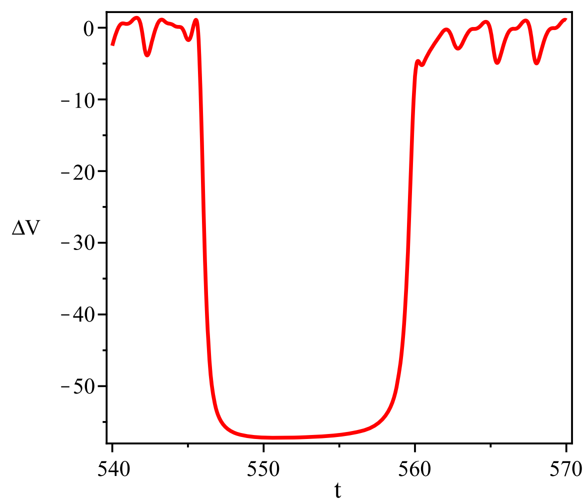

Fig. 6 Plot of $\triangle V$ versus $t \in [540, 570]$. We see volume expansion and contraction during the bursting regime, and strong volume contraction in the intermediate phase. Note that the onset of strong volume contraction corresponds to the onset of triple mode alignment

– the onset of strong volume contraction or expansion corresponds to the onset of triple mode alignment (Fig. 6).

This pattern continues as t grows. We isolate one region of this interesting behavior in Figs. 4 and 5, which allows us to see each of these four behaviors in the mode frequencies. In Fig. 4, we plot the corresponding time-series for the solution to (3), while in Fig. 5 we plot the mode frequencies.

Note that near $t = 546$ (close to the point at which three modes become competitive), from (4) we have $\triangle V = -22.7458$, hence there is strong volume contraction. Ahead of this, at $t = 540$, we have $\triangle V = -2.5117$, or much more weak volume contraction. At

$t = 550$, in the intermediate phase between bursts, we have $\triangle V = -56.8017$, much stronger volume contraction. To better view the influence of the BSC on volume contraction, we plot the quantity $\triangle V$ over t for $a = 0.456$ and $b = 0.0357$ in Fig. 6. Note that the competitiveness of three modes corresponds directly to the onset of strong volume contraction or expansion.

For $(a, b) = (0.456, 0.0357)$, we have observed that the solutions to (3) undergo BSC when three modes become nearly competitive, hence signifying a correspondence between competitive modes and the occurrence of BSC. However, there is a dual use for the method of competitive modes. As discussed previously, if the competitive modes conditions are forced upon a system, then in some situations we may arrive at parameter regimes for which the starting system exhibits chaotic behavior. In the case where we desire all three modes to be competitive, i.e., $g_i = g_j$ for all pairs (i, j), one simple way to deduce the proper parameter space is to take the differences $\delta_{i,j}(a, b) = g_i(a, b) - g_j(a, b)$ and construct the function $\chi(a, b) = \delta_{1,2}^2 + \delta_{1,3}^2 + \delta_{2,3}^2$ for any set value of the state vector (x, y, z) in parameter space. By construction, $\chi(a, b) \geq 0$ for all $(a, b) \in \mathbb{R}^2$. Take the state vector to be fixed, say $(x, y, z) = (x_0, y_0, z_0) = \mathbf{v}_0$, and define the zero-locus $\ell(\mathbf{v}_0) = \{(a, b) \in \mathbb{R}^2 | \chi(a, b) = 0\}$. Certainly, for some $\mathbf{v}_0 \in \mathbb{R}^3$, $\ell(\mathbf{v}_0)$ may be empty. However, for nonempty $\ell(\mathbf{v}_0)$, the set $\ell(\mathbf{v}_0)$ defines the set of points (a, b) in parameter space that allow three simultaneously competitive modes. However, for any choice of $\mathbf{v}_0$, we may minimize the quadratic function $\chi(a, b)$, to obtain (a, b) for which the modes are nearly competitive (this is often sufficient). We find that we either obtain (i) solutions converging rapidly to equilibria (it so happens that, if modes remain competitive as opposed to being intermittently competitive, the solutions are stable) or (ii) exhibit chaotic behavior. In order to ensure the latter case, we may also consider selecting parameters which make $|\triangle V|$ small, since the chaotic solutions observed correspond to expansions and contractions in the volume elements. So, one may attempt to minimize $\chi(a, b)$ and $(\triangle V)^2$ over the available parameters, for appropriate $\mathbf{v}_0$.

We should remark that such a minimization does not always yield one specific form of chaos. For instance, let us fix $\mathbf{v}_0 = (0.06, 1, 1)$. Taking $(a, b) = (8.0549, 1.5866)$, we find that $\chi(8.0549, 1.5866) = 0.00503$ and $|\triangle V| = 0.0018$, both of which are sufficiently small. With these values, the model (3) admits

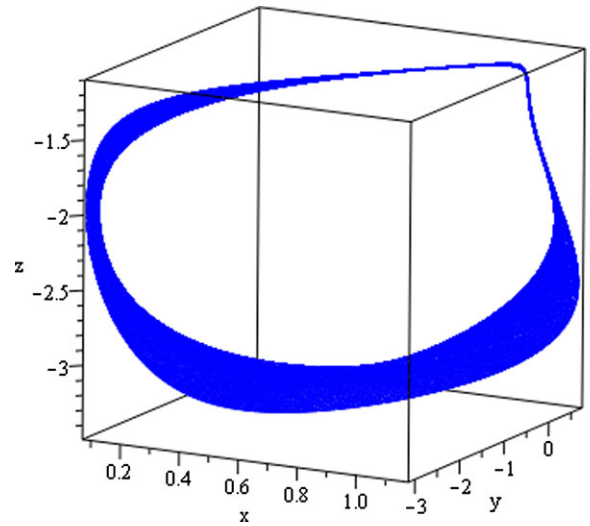

Fig. 7 Phase portrait showing dynamics for (3) with $a = 8.0549$ and $b = 1.5866$

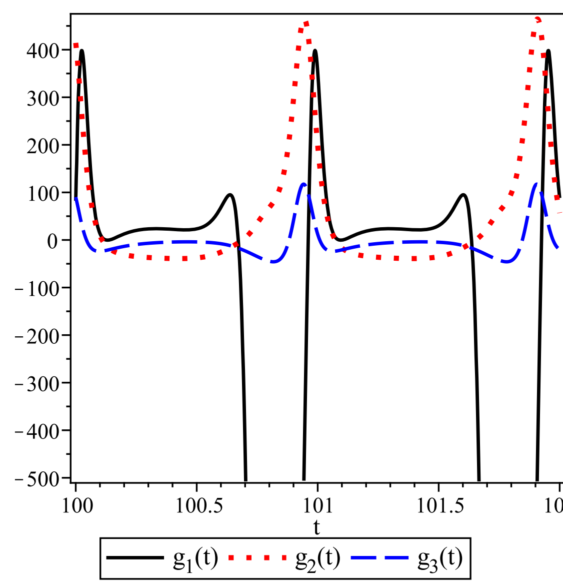

Fig. 8 Mode frequencies g_1, g_2, g_3 on $t \in [540, 570]$ for the solution of (3) with $a = 8.0549$ and $b = 1.5866$. Observe that condition (i) is violated when three mode frequencies agree: $g_1 = g_2 = g_3$, yet all three are negative. Still, there are two mode frequencies equal and positive at other intermittent times

the dynamics seen in Fig. 7. As seen in Fig. 8, the all three mode frequencies become competitive intermittently, though this time at negative values, which has a qualitatively different meaning than given in (i)–(iv) (since in this case the condition (i) fails.

Thus, while the method outlined seems useful for predicting chaos, discerning the actual nature of the inherent chaos seems beyond the scope of the method. Therefore, in order to determine if BSC (or any other chaotic dynamic) is present, one would need to narrow the permissible parameter regime with the method, and then attempt to map the chaotic subregimes within such a set.

4 Conclusions

In summary, we have studied a model of the blue-sky catastrophe in the framework of competitive modes. We find that at the start and finish of a bursting period, all three of the mode frequencies agree, hence three modes are competitive. In other words, when recast as an oscillator system, the three oscillators come into resonance during the transition to and from bursting. This is a rare behavior not commonly observed in 3D chaotic models. More often, three mode frequencies come into alignment when there is hyperchaos (present in some dynamical systems of dimension not less than four).

An area related to competitive modes would be synchronization and antisynchronization in chaotic models, since the competitive modes assumption implicitly involves synchronization of oscillator equations. The process of three modes falling into and out of synchronization has in the present paper been shown to correspond with the emergence of the blue-sky catastrophe. A detailed study of synchronization or antisynchronization problems for a two-parameter model giving the blue-sky catastrophe could be an interesting area of work. Some recent work in related areas can be found in [29–38].

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