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Conservation of a resource based fishery through optimal taxation

U.K. Pahari · T.K. Kar

Abstract In the present paper, we study a dynamic reaction model in which (i) the predator is provided with an alternative food in addition to the prey species, (ii) the predator is harvested, and (iii) a tax is imposed to regulate the system. The existence of possible steady states along with their local as well as global stability is discussed for both the exploited and unexploited systems. Boundedness of the system is also discussed. It is seen that the system undergoes a Hopf bifurcation by the addition of alternative prey and the criteria for the Hopf-bifurcation is also discussed. Optimal tax policy is discussed using Pontryagin's maximal principle. Finally, some numerical simulations are given to show the consistency with theoretical analysis.

Keywords Alternative prey · Dynamic reaction · Harvesting · Global stability · Bifurcation · Taxation

U.K. Pahari (✉)
Department of Mathematics, Sree Chaitanya College, Habra, India
e-mail: pahariu@yahoo.in

T.K. Kar
Department of Mathematics, Bengal Engineering and Science University, Shibpur, Howrah, India
e-mail: tkar1117@gmail.com

1 Introduction

Population dynamics has attracted interest from the commercial harvesting industry and from many scientific communities including biology, ecology, and economics. Population ecologists study how births and deaths affect the dynamics of populations and communities, while ecosystem ecologists study how species control the flux of energy and materials through food webs and ecosystems. Although all these processes occur simultaneously in nature, the mathematical frameworks bridging the two disciplines have developed independently. Research in the area of theoretical ecology was initiated by Lotka–Volterra. Since then, many mathematicians and ecologists contributed to the growth of this area of knowledge as reported in the treatises of Paul Colinvaux [19], Freedman [7], Kapur ([11] and [12]), etc.

Harvesting of multi-species fisheries is an important area of study in fishery modelling. The issues and techniques related to this field of study and the problem of combined harvesting of two ecologically independent populations obeying logistic law of growth are discussed in detail by Clark [2]. The effect of constant rate harvesting on the dynamics of predator–prey systems has been investigated by Dai and Tang [5], Myerscough et al. [18], and Xiao and Ruan [21], and they obtained very rich and interesting dynamical behaviors. Zhang et al. [22] have investigated the dynamics of the inshore-offshore fishing model with impulsive diffusion and pulse harvesting at different fixed times.

A model in which the revenue is generated from fishing and the growth of fish depends upon the plankton, which in turn grows logistically is developed by Dhar et al. [4]. They further formulated the model with a delay in digestion of plankton by fish and found the threshold value of a conversional parameter for Hopf-bifurcation. Kar and Matsuda [15], Kar and Pahari [16] discussed the predator–prey model with time delay and analyzed the effects of time delay on model dynamics such as the time delay may change the stability of equilibrium points and even cause a switching of stabilities. The effect of environmental fluctuations and gestation delay on the harvesting population model is investigated by Zhang and Zhang [23].

The role of alternative prey in sustaining predator populations has been widely studied. Many prey–predator models suggest that adding alternative prey to a predator would lower the density of the target prey. However, from some empirical works of Harwood and Obrycki [8], Halt and Lawton [10], and Wootton [20], it is clear that the addition of an alternative prey does not always increase the target population. Thus, there is an apparent conflict between theory and empirical observations.

Taxation plays an important role in the regulation of exploitation of biological resources. In fishery regulation, taxation, license fees, lease of property rights, seasonal harvesting, etc. are usually considered as possible governing instruments. Economists are particularly attracted to taxation, because of its economic flexibility. Harvesting problems with taxation as a control instrument are studied by Kar and Chaudhuri [13], Dubey et al. [6], Kar et al. [14], etc.

In the present paper, we consider a prey–predator fishery model by taking an alternative food for the predator where only the predator species is harvested. We also take tax as a control instrument. The main objective of this paper is to find the proper taxation policy, which would give the best possible benefit to the society through harvesting. We first determine the existence of possible steady states of the unexploited system, and study their local as well as global stability. Then we have considered the effect of harvesting. Boundedness of the system is discussed. Criteria for Hopf bifurcation is also developed. The optimal tax policy is studied, and the solution is derived in the equilibrium case by using Pontryagin's maximum principle. Finally, some numerical examples are discussed.

2 The model formulation

We consider the model equations of two interacting species, which are in a prey–predator relationship, and where both species have an independent specific growth rate in the absence of the other. Whenever there is a large catch of the predator, there exists serious implications for production of both the species and, therefore, it is necessary to regulate harvesting on the predator species. The rate equations of growth of two species are given by

$$\frac{dx}{dt} = rx\left(1 - \frac{x}{k}\right) - \frac{axy}{b+x},$$
$$\frac{dy}{dt} = \frac{maxy}{b+x} + dy - cy^2 - h(t), \tag{1}$$

with $x(0) = x_0 > 0$, $y(0) = y_0 > 0$. Here, $x = x(t) =$ density of the prey population at time t, $y = y(t) =$ density of the predator population at time t, $k =$ environmental carrying capacity of the prey, $r =$ average net per-capita growth rate of the prey, i.e. maximum specific growth rate of the prey, $a =$ maximal relative increase of predation, $b =$ half saturation level which is a constant, $d =$ growth rate of the predator due to availability of alternative food sources, $c =$ mortality rate of the predator population, $m =$ conversion factor, $h(t) =$ harvesting rate of the predator at time t.

In the model equation, predator mortality is assumed to be a rate proportional to y^2 rather than y. This non-linear dependency reflects the combined effects of increased predation by the super predator (not considered in the model directly) and the interface or competition among the predators. So, the growth of the predator species in the second equation is limited due to the presence of the term cy^2 and even if the density of the prey is very high.

The term dy represents a growth rate of the predator due to the availability of alternative food sources. It is quite natural that when focal prey is low, the predators increase their feeding on alternative prey. But when the focal prey increases, the predator uses less alternative prey, and as focal prey approaches to its saturation value k, the amount of alternative prey consumed by the predator tends to zero and then only predation of the focal prey occurs. For this reason, we modify the term dy by the factor $dy(1 - \frac{x}{k})$.

The amount of prey consumed by the predator is assumed to follow the Holling-type II [9] functional form. Here, we assume that the per-capita rate of consumption of prey by the predator is $\frac{ax}{b+x}$. This type

of function for consumption is called depensatory because the per-capita consumption rate decreases as prey density increases.

We assume that the predator population is harvested according to the catch-per-unit-effort (CPUE) hypothesis [2], which describes that catch per unit effort is proportional to the stock level. Thus, we consider $h = qEy$ where E is the harvesting effort and q is the catchability coefficient.

To control exploitation of the fishery, regulatory agency imposes a tax τ (> 0) per unit biomass of the landed predator fish. Any subsidy to the fishermen may be interpreted as the negative value of τ.

The net economic revenue to the fishermen (perceived rent) is given by $E[q(p - \tau)y - c]$ where p is the price per unit biomass and c is the cost of unit harvesting effort.

In an open access fishery of a fully dynamic model, the level of fishing effort expands or contracts according as the perceived rent to the fisherman is positive or negative. A model reflecting this dynamic interaction between the perceived rent and effort in a fishery is called a dynamic reaction model. The harvesting effort E is, therefore, a dynamic variable governed by the differential equation

$$\frac{dE}{dt} = \left[(p - \tau)qy - c \right] E.$$

Thus, the final model becomes

$$\frac{dx}{dt} = rx \left(1 - \frac{x}{k} \right) - \frac{axy}{b + x},$$
$$\frac{dy}{dt} = \frac{maxy}{b + x} + dy \left(1 - \frac{x}{k} \right) - cy^2 - qEy, \qquad (2)$$
$$\frac{dE}{dt} = \left[(p - \tau)qy - c \right] E.$$

3 The case of unexploited fishery

In this case $E = 0$ and the model (2) reduces to

$$\frac{dx}{dt} = rx \left(1 - \frac{x}{k} \right) - \frac{axy}{b + x},$$
$$\frac{dy}{dt} = \frac{maxy}{b + x} + dy \left(1 - \frac{x}{k} \right) - cy^2. \qquad (3)$$

Model (3) has four non-negative equilibria $E_0(0, 0)$, $E_1(k, 0)$, $E_2(0, \frac{d}{c})$, and $E_3(\bar{x}, \bar{y})$ where $\bar{y} = \frac{r}{a}(b + \bar{x})(1 - \frac{\bar{x}}{k})$ and $\bar{x}$ is the unique real positive root of the cubic equation
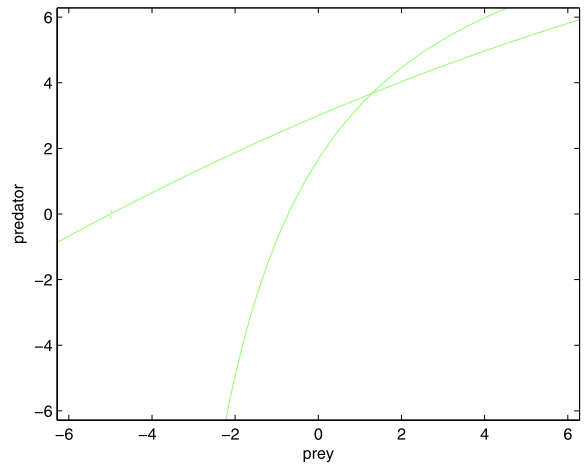

Fig. 1 This figure is given for $r = 9$, $a = 15$, $b = 5$, $k = 50$, $m = 0.8$, $c = 1.2$, $d = 2$

$$x^3 + \left(2b - k - \frac{ad}{rc} \right) x^2$$
$$+ \left(b^2 - 2bk - \frac{abd}{rc} + \frac{adk}{rc} + \frac{ma^2k}{rc} \right) x$$
$$+ \left(\frac{abdk}{rc} - b^2k \right) = 0.$$

This cubic equation has at least one real positive root if $\frac{d}{c} < \frac{br}{a}$. So, the system (3) has a unique interior equilibrium point $E_3(\bar{x}, \bar{y})$ if and only if $\frac{d}{c} < \frac{br}{a}$.

The following figure (see Fig. 1) indicates that for some parameter values, the system (3) has at least one interior equilibrium point.

To analyze the behavior of the system (3), firstly, we discuss the local behavior of the equilibria of the system (3). The variational matrix of the system (3) takes the form

$$J(x, y)$$
$$= \begin{pmatrix} r(1 - \frac{2x}{k}) - \frac{aby}{(b+x)^2} & -\frac{ax}{b+x} \\ \frac{maby}{(b+x)^2} - \frac{dy}{k} & \frac{max}{b+x} + d(1 - \frac{x}{k}) - 2cy \end{pmatrix}.$$

At E_0,

$$J_0 = \begin{pmatrix} r & 0 \\ 0 & d \end{pmatrix}.$$

Therefore, $\text{tr}(J_0) = r + d > 0$ and $\det(J_0) = rd > 0$. Hence, the equilibrium point E_0 is unstable.

At E_1,

$$J_1 = \begin{pmatrix} -r & -\frac{ak}{b+k} \\ 0 & \frac{mak}{b+k} \end{pmatrix}.$$

Therefore, $\det(J_1) = -\frac{makr}{b+k} < 0$. Hence, E_1 is a saddle point.

Let us find out the saddle manifold. We consider one orbit of the system (3) along x-axis. As $0 < x < k$, $\dot{x}$ is positive. Thus, the orbit along x-axis with $0 < x < k$ goes to E_1. Similarly, when $x > k$, the orbit also goes to E_1. Hence, the x-axis is the stable manifold.

The variational matrix of the system (3) at E_2 is

$$J_2 = \begin{pmatrix} r - \frac{ad}{bc} & 0 \\ \frac{mad}{bc} - \frac{d^2}{ck} & -d \end{pmatrix}.$$

Now, $\mathrm{tr}(J_2) = r - (\frac{ad}{bc} + d)$ and $\det(J_2) = -d(r - \frac{ad}{bc})$.

Thus, we see that if $\frac{d}{c} > \frac{br}{a}$, then $\mathrm{tr}(J_2) < 0$ and $\det(J_2) > 0$.

Thus, if $\frac{d}{c} > \frac{br}{a}$, then the equilibrium point E_2 is locally asymptotically stable.

The variational matrix of the system (3) at $E_3(\bar{x}, \bar{y})$ is

$$\bar{J}(\bar{x}, \bar{y}) = \begin{pmatrix} \frac{r\bar{x}(1 - \frac{b}{k} - \frac{2\bar{x}}{k})}{b + \bar{x}} & -\frac{a\bar{x}}{b+\bar{x}} \\ \bar{y}(\frac{mab}{(b+\bar{x})^2} - \frac{d}{k}) & -c\bar{y} \end{pmatrix}.$$

We see that

$$\mathrm{tr}(\bar{J}) = \frac{r\bar{x}(1 - \frac{b}{k} - \frac{2\bar{x}}{k})}{b + \bar{x}} - c\bar{y},$$

$$\det(\bar{J}) = \frac{a\bar{x}\bar{y}}{b + \bar{x}}\left[\frac{mab}{(b+\bar{x})^2} - \frac{d}{k} - \frac{rc}{a}\left(1 - \frac{b}{k} - \frac{2\bar{x}}{k}\right)\right].$$

Now $\mathrm{tr}(\bar{J}) < 0$ if $1 - \frac{b}{k} - \frac{2\bar{x}}{k} < \frac{c\bar{y}(b+\bar{x})}{r\bar{x}}$ and $\det(\bar{J}) > 0$ if $1 - \frac{b}{k} - \frac{2\bar{x}}{k} < \frac{a}{rc}(\frac{mab}{(b+\bar{x})^2} - \frac{d}{k})$. Thus, if

$$1 - \frac{b}{k} - \frac{2\bar{x}}{k}$$
$$< \min\left\{\frac{c\bar{y}(b+\bar{x})}{r\bar{x}}, \frac{a}{rc}\left(\frac{mab}{(b+\bar{x})^2} - \frac{d}{k}\right)\right\},$$

then $E_3(\bar{x}, \bar{y})$ is locally asymptotically stable.

Now we shall discuss the global stability of the interior equilibrium point $E_3(\bar{x}, \bar{y})$.

Theorem 1 *If $c > \max\{\frac{ad}{br}, \frac{a}{2b}\}$, then $E_3(\bar{x}, \bar{y})$ is globally asymptotically stable.*

Proof Let us define $H(x, y) = \frac{1}{xy}$. Clearly, $H > 0$ if $x > 0$ and $y > 0$. Let

$$f_1(x, y) = rx\left(1 - \frac{x}{k}\right) - \frac{axy}{b+x}$$

and

$$f_2(x, y) = \frac{maxy}{b+x} + dy\left(1 - \frac{x}{k}\right) - cy^2.$$

Now

$$\nabla(x, y) = \frac{\partial}{\partial x}(f_1 H) + \frac{\partial}{\partial y}(f_2 H)$$

$$= \frac{\partial}{\partial x}\left[r\left(\frac{1}{y} - \frac{x}{ky}\right) - \frac{a}{b+x}\right]$$

$$\quad + \frac{\partial}{\partial y}\left[\frac{ma}{b+x} + d\left(\frac{1}{x} - \frac{1}{k}\right) - \frac{cy}{x}\right]$$

$$= -\frac{r}{ky} + \frac{a}{(b+x)^2} - \frac{c}{x}$$

$$< 0 \quad \text{if } \frac{a}{(b+x)^2} - \frac{c}{x} < 0,$$

i.e. if $c(b^2 + x^2) + x(2bc - a) > 0$, i.e. if $c > \frac{a}{2b}$.

Again if $c > \frac{ad}{br}$, then the equilibrium point $E_3(\bar{x}, \bar{y})$ exists.

Hence, if

$$c > \max\left\{\frac{ad}{br}, \frac{a}{2b}\right\}$$

then

$$\nabla(x, y) < 0.$$

Since E_3 is locally asymptotically stable, from the Bendixin–Dulac criterion (Conway and Smoller) [3], we may conclude that E_3 is globally asymptotically stable in $\Re_+^2$ if $c > \max\{\frac{ad}{br}, \frac{a}{2b}\}$.

This completes the proof. $\qquad\square$

4 Effect of harvesting

The system (2) under investigation has six equilibria:
(i) $P_0(0, 0, 0)$, (ii) $P_1(k, 0, 0)$, (iii) $P_2(0, d/c, 0)$, (iv) $P_3(0, \hat{y}, \hat{E})$, where $\hat{y} = \frac{c}{(p-\tau)q}$, $\hat{E} = \frac{1}{q}(d - c\hat{y})$, (v) $P_4(\bar{x}, \bar{y}, 0)$, where $\bar{y} = \frac{r}{ak}(k - \bar{x})(b + \bar{x})$ and $\bar{x}$ is the unique positive root of the equation

$$x^3 + \left(2b - k - \frac{ad}{cr}\right)x^2$$

$$+ \left\{(k - b)\left(\frac{ad}{cr} - b\right) + k\left(\frac{a^2 m}{cr} - b\right)\right\}x$$

$$+ bk\left(\frac{ad}{cr} - b\right) = 0$$

and $P_5(x^*, y^*, E^*)$, where

$$y^* = \frac{c}{(p-\tau)q},$$

$$x^* = \frac{k-b}{2} + \sqrt{\left(\frac{k+b}{2}\right)^2 - \frac{aky^*}{r}},$$

$$E^* = \frac{1}{q}\left\{\frac{max^*}{b+x^*} + d\left(1 - \frac{x^*}{k}\right) - cy^*\right\}.$$

We now study the different conditions under which these steady states exist.

The equilibria $P_0(0, 0, 0)$, $P_1(k, 0, 0)$, and $P_2(0, d/c, 0)$ always exist. In the case of taxation, it is natural to assume that $p > \tau > 0$.

Hence, $\hat{y} > 0$ and $\hat{E} > 0$ if $d > \frac{c^2}{(p-\tau)q}$.

Therefore, the equilibrium point $P_3(0, \hat{y}, \hat{E})$ exists if $p > \tau > 0$ and $d > \frac{c^2}{(p-\tau)q}$.

The equilibrium point $P_4(\bar{x}, \bar{y}, 0)$ exists if $\frac{d}{c} < \frac{br}{a}$.

This means that the ratio of the rate of growth due to alternative prey and mortality of predator is always less than the ratio of the product of specific growth and the half-saturation level of the prey to its maximum capture rate due to predation.

Before studying the stability of the model, we show that the solutions of the system are bounded in a finite region initiating at $(x(0), y(0), E(0))$.

5 Boundedness

Theorem 2 *All the solutions of the system* (2) *which start in* $\Re_+^3$ *are uniformly bounded.*

Proof Let $(x(t), y(t), E(t))$ be any solution of the system with positive initial conditions. We define the function

$$W = x + \frac{y}{m} + \frac{E}{m(p-\tau)}.$$

Therefore, the time derivative is found to be

$$\frac{dW}{dt} = \frac{dx}{dt} + \frac{1}{m}\frac{dy}{dt} + \frac{1}{m(p-\tau)}\frac{dE}{dt}$$

$$= rx - \frac{r}{k}x^2 + \frac{d}{m}y - \frac{d}{mk}xy$$

$$- \frac{c}{m}y^2 - \frac{cE}{m(p-\tau)}.$$

Now, for each $\mu > 0$, we have

$$\frac{dW}{dt} + \mu W$$

$$= rx - \frac{r}{k}x^2 + \frac{d}{m}y - \frac{d}{mk}xy - \frac{c}{m}y^2$$

$$- \frac{cE}{m(p-\tau)} + \mu x + \frac{\mu y}{m} + \frac{\mu E}{m(p-\tau)}$$

$$\leq rx - \frac{r}{k}x^2 + \frac{d}{m}y - \frac{c}{m}y^2$$

$$- \frac{cE}{m(p-\tau)} + \mu x + \frac{\mu y}{m} + \frac{\mu E}{m(p-\tau)}.$$

Taking $c = \mu$, we have

$$\frac{dW}{dt} + \mu W \leq rx - \frac{r}{k}x^2 + \frac{d}{m}y - \frac{c}{m}y^2 + \mu x + \frac{\mu}{m}y$$

$$\leq \frac{k}{4r}(r+\mu)^2 + \frac{1}{4mc}(d+\mu)^2.$$

Thus, we get

$$\frac{dW}{dt} + \mu W \leq V,$$

where

$$V = \frac{k}{4r}(r+\mu)^2 + \frac{1}{4mc}(d+\mu)^2.$$

Applying the theory of differential inequality (Birkoff and Rota) [1], we obtain

$$0 \leq W(x, y, E) \leq \frac{V}{\mu}\left(1 - e^{-\mu t}\right)$$

$$+ W\left(x(0), y(0), E(0)\right)e^{-\mu t}$$

which upon letting $t \to \infty$, yields

$$0 \leq W \leq \frac{V}{\mu}.$$

Thus, all the solutions of the system (2) that starts in $\Re_3^+$ are confined to the region $B = \{(x, y, E) \in \Re_3^+ : 0 \leq W \leq \frac{V}{\mu} + \varepsilon\}$, for any $\varepsilon > 0$. □

6 Local stability analysis

The variational matrix of the system (2) is

$$M(x, y, E) = \begin{bmatrix} r(1 - \frac{2x}{k}) - \frac{aby}{(b+x)^2} & -\frac{ax}{b+x} & 0 \\ \frac{maby}{(b+x)^2} - \frac{dy}{k} & \frac{max}{b+x} + d(1 - \frac{x}{k}) - 2cy - qE & -qy \\ 0 & (p-\tau)qE & (p-\tau)qy - c \end{bmatrix}.$$

The eigenvalues of the variational matrix $M(0, 0, 0)$ are $r, d, -c$. So, the equilibrium point $P_0(0, 0, 0)$ is unstable.

The eigenvalues of the variational matrix $M(k, 0, 0)$ are $-r, -c, \frac{mak}{b+k}$. So, the equilibrium point $P_1(k, 0, 0)$ is also unstable.

For the equilibrium point $P_2(0, d/c, 0)$, the eigenvalues are $-d, r - \frac{ad}{bc}, (p - \tau)\frac{qd}{c} - c$. Therefore, $P_2(0, d/c, 0)$ is a stable node if $\frac{d}{c} > \frac{br}{a}$ and $\tau > p - \frac{c^2}{qd}$.

For the equilibrium point $P_3(0, \hat{y}, \hat{E})$, one eigenvalue of the variational matrix is $r - \frac{ac}{(p-\tau)qb}$, which is negative if $\tau > p - \frac{ac}{brq}$.

The other two eigenvalues are the roots of the quadratic equation

$$\lambda^2 + \frac{c^2}{(p-\tau)q}\lambda + cd - \frac{c^3}{(p-\tau)q} = 0,$$

which has (a) sum of the roots $= -\frac{c^2}{(p-\tau)q}$, which is always negative and (b) product of roots $= cd - \frac{c^3}{(p-\tau)q}$. Hence, the roots of the quadratic equation are real and negative or complex conjugate with negative real part if $\tau < p - \frac{c^2}{dq}$.

Therefore, the equilibrium point P_3 is locally asymptotically stable if

$$p - \frac{ac}{brq} < \tau < p - \frac{c^2}{dq}.$$

So, it is observed that even in the absence of prey x, the predator may exists in its equilibrium level and this happened due to alternative prey.

For the equilibrium point $P_4(\bar{x}, \bar{y}, 0)$, one of the eigenvalues of the corresponding variational matrix is $(p - \tau)q\bar{y} - c$, which is negative if $\tau > p - \frac{c}{q\bar{y}}$.

The other two eigenvalues are the roots of the quadratic equation

$$\lambda^2 + u\lambda + v = 0$$

where

$$u = c\bar{y} - \frac{r\bar{x}}{b+\bar{x}}\left(1 - \frac{b}{k} - \frac{2\bar{x}}{k}\right)$$

and

$$v = \frac{a\bar{x}\bar{y}}{b+\bar{x}}\left\{\frac{mab}{(b+\bar{x})^2} - \frac{d}{k} - \frac{rc}{a}\left(1 - \frac{b}{k} - \frac{2\bar{x}}{k}\right)\right\}.$$

The sign of real part of the eigenvalues are determined by u. Now, $u > 0$ if $k < b + 2\bar{x}$; the equilibrium point $P_4(\bar{x}, \bar{y}, 0)$ is locally asymptotically stable if

$$k < b + 2\bar{x} \quad \text{and} \quad \tau > p - \frac{c}{q\bar{y}}.$$

At $P_5(x^*, y^*, E^*)$, we have

$$M(x^*, y^*, E^*)$$
$$= \begin{bmatrix} \frac{rx^*}{b+x^*}(1 - \frac{b}{k} - \frac{2x^*}{k}) & -\frac{ax^*}{b+x^*} & 0 \\ \frac{mbr}{b+x^*}(1 - \frac{x^*}{k}) - \frac{dy^*}{k} & -cy^* & -qy^* \\ 0 & (p-\tau)qE^* & 0 \end{bmatrix}.$$

The characteristic equation corresponding to $M(x^*, y^*, E^*)$ is

$$\lambda^3 + m_1\lambda^2 + m_2\lambda + m_3 = 0 \qquad (4)$$

where

$$m_1 = cy^* - \frac{rx^*}{b+x^*}\left(1 - \frac{b}{k} - \frac{2x^*}{k}\right),$$

$$m_2 = cqE^* - \frac{crx^*y^*}{b+x^*}\left(1 - \frac{b}{k} - \frac{2x^*}{k}\right)$$
$$+ \frac{ax^*}{b+x^*}\left\{\frac{mbr}{b+x^*}\left(1 - \frac{x^*}{k}\right) - \frac{dy^*}{k}\right\},$$

$$m_3 = -\frac{cqrx^*E^*}{b+x^*}\left(1 - \frac{b}{k} - \frac{2x^*}{k}\right).$$

The Routh–Hurwitz criterion gives a set of necessary and sufficient conditions so that all the roots of the characteristic equation have negative real parts. For the above cubic equation, these criteria are $m_1 > 0, m_3 > 0$, and $m_1m_2 - m_3 > 0$.

We find that if

$$1 - \frac{b}{k} - \frac{2x^*}{k} < 0, \quad \text{then } m_1 > 0, \ m_3 > 0.$$

Now,

$$m_1m_2 - m_3 = m_1\left[cy^*(m_1 - cy^*) + \frac{ax^*}{b+x^*}\left\{\frac{mbr}{b+x^*}\right.\right.$$
$$\left.\left. \times \left(1 - \frac{x^*}{k}\right) - \frac{dy^*}{k}\right\}\right] + c^2qE^*y^*.$$

Hence, $m_1m_2 - m_3 > 0$ if

$$1 - \frac{b}{k} - \frac{2x^*}{k} < 0 \quad \text{and} \quad \frac{mbr}{b+x^*}\left(1 - \frac{x^*}{k}\right) > \frac{dy^*}{k},$$

i.e. if

$$\frac{d(b+x^*)^2}{mab} < k < b + 2x^*. \qquad (5)$$

Therefore, by the Routh–Hurwitz criterion, we say that (5) is the sufficient condition for local asymptotic

stability of the non-trivial steady state $P_5(x^*, y^*, E^*)$. Thus, we have the following theorem.

Theorem 3 *The interior equilibrium point $P_5(x^*, y^*, E^*)$ is locally asymptotically stable if*

$$\frac{d(b+x^*)^2}{mab} < k < b + 2x^*.$$

From the point of view of ecological managers, it may be desirable to have an equilibrium point which is globally stable in order to plan harvesting strategy and keep sustainable ecological development.

7 Global stability analysis

Let us consider the following Lyapunov function:

$$V(x, y, E) = k_1\left(x - x^* - x^* \ln \frac{x}{x^*}\right)$$
$$+ k_2\left(y - y^* - y^* \ln \frac{y}{y^*}\right)$$
$$+ k_3\left(E - E^* - E^* \ln \frac{E}{E^*}\right)$$

on $G = \{(x, y, E) : x > 0, y > 0, E > 0\}$, where k_1, k_2, k_3 are positive constants to be determined in the subsequent steps. It can be easily verified that the function V is zero at the equilibrium (x^*, y^*, E^*) and positive on G.

The time derivative of V along the trajectories of (2) is

$$\frac{dV}{dt} = k_1\left(\frac{x - x^*}{x}\right)\frac{dx}{dt} + k_2\left(\frac{y - y^*}{y}\right)\frac{dy}{dt}$$
$$+ k_3\left(\frac{E - E^*}{E}\right)\frac{dE}{dt}$$
$$= k_1(x - x^*)\left[r\left(1 - \frac{x}{k}\right) - \frac{ay}{b + x}\right]$$
$$+ k_2(y - y^*)\left[\frac{max}{b + x} + d\left(1 - \frac{x}{k}\right) - cy - qE\right]$$
$$+ k_3(E - E^*)[(p - \tau)qy - c].$$

A little manipulation yields

$$\frac{dV}{dt} = -k_1\left[\frac{r}{k} - \frac{ay^*}{(b + x^*)(b + x)}\right](x - x^*)^2$$
$$- k_2 c(y - y^*)^2 + \left[\frac{k_2 mb}{b + x^*} - k_1\right]$$

$$\times \frac{a(x - x^*)(y - y^*)}{b + x} - \frac{k_2 d}{k}(x - x^*)(y - y^*)$$
$$+ [k_3(p - \tau) - k_2]q(y - y^*)(E - E^*).$$

If we choose

$$\frac{b + x^*}{mb}k_1 = k_2 = k_3(p - \tau)$$

then we have

$$\frac{dV}{dt} = -k_1\left[\frac{r}{k} - \frac{ay^*}{(b + x^*)(b + x)} + \frac{k_2 d}{2kk_1}\right](x - x^*)^2$$
$$- k_2\left[c - \frac{d}{2k}\right](y - y^*)^2$$
$$- \frac{k_2 d}{2k}\left[(x - x^*)^2 + (y - y^*)^2\right]$$

clearly, if

$$c > \frac{d}{2k} \quad \text{and} \quad \left[\frac{r}{k} - \frac{ay^*}{(b + x^*)(b + x)} + \frac{k_2 d}{2kk_1}\right] > 0$$

i.e. if

$$\frac{d}{c} < 2k \quad \text{and} \quad x > \frac{2kmaby^*}{(b + x^*)[2rmb + d(b + x^*)]} - b$$
$$\text{and} \quad (x, y, E) \neq (x^*, y^*, E^*),$$

then

$$\frac{dv}{dt} < 0.$$

Hence, the equilibrium point P_5 is globally asymptotically stable.

Therefore, we have the following theorem.

Theorem 4 *If $\frac{d}{c} < 2k$, the equilibrium point $P_5(x^*, y^*, E^*)$ is globally asymptotically stable in the region $x > \frac{2kmaby^*}{(b+x^*)[2rmb+d(b+x^*)]} - b$.*

Prey–predator models with constant parameters are often found to approach a steady state in which the species co-exist in equilibrium. But if parameters used in the model are changed, other types of dynamical behavior may occur and the critical parameter values at which such transitions happen are called bifurcation points. From an ecological point of view, bifurcations endanger the existence of a particular species in a prey–predator system. When a stable steady state goes through a bifurcation, in general, it either loses its stability or disappears entirely. However, in order to understand the general mechanisms leading to bifurcations, we take the growth rate 'd' of the predator due to alternative prey as the bifurcation parameter.

8 Bifurcation analysis

8.1 Bifurcation for the parameter 'd'

The characteristic equation (4) has two purely imaginary roots if and only if $m_1 m_2 = m_3$ for some value of d, say d^*. We find that

$$d^* = \frac{k(b + 2x^*)}{ax^* y^*} \left[-\frac{crx^* y^*}{b + x^*} \left(1 - \frac{b}{2k} - \frac{2x^*}{k} \right) \right. $$
$$\left. + \frac{mabrx^*(1 - \frac{x^*}{k})}{(b + x^*)^2} + \frac{c^2 q E^* y^*}{m_1} \right].$$

Thus, there exist a unique d^* such that $m_1 m_2 = m_3$. Therefore, there is only one value of d, at which we have a Hopf bifurcation. Thus, in the neighborhood of d^* the characteristic equation (4) cannot have real roots.

For $d = d^*$, we have

$$(\lambda^2 + m_2)(\lambda + m_1) = 0.$$

This equation has two purely imaginary roots and a real root as

$$\lambda_1 = i\sqrt{m_2}, \qquad \lambda_2 = -i\sqrt{m_2}, \qquad \lambda_3 = -m_1.$$

The roots are in general of the form

$$\lambda_1(d^*) = p(d^*) + iq(d^*),$$
$$\lambda_2(d^*) = p(d^*) - iq(d^*),$$
$$\lambda_3(d^*) = -m_1(d^*).$$

To apply the Hopf bifurcation theorem as stated in Marsden and McCracken [17], we need to verify the transversality condition

$$\left[\frac{dp}{dd} \right]_{d=d^*} \neq 0.$$

Substituting $\lambda_1(d^*) = p(d^*) + iq(d^*)$ in Eq. (4) and differentiating the resulting equation w.r.t. d and setting $p(d^*) = 0$ and $q(d^*) = \sqrt{m_2} = q_1$, we get

$$\frac{dp}{dd}(-3q_1^2 + m_2) + \frac{dq}{dd}(-2m_1 q_1) = m_1' q_1^2 - m_3',$$
$$\frac{dp}{dd}(2m_1 q_1) + \frac{dq}{dd}(-3q_1^2 + m_2) = -m_2' q_1,$$

where m_1, m_2, and m_3 are a function of the bifurcation parameter d and

$$m_1' = \frac{dm_1}{dd}, \qquad m_2' = \frac{dm_2}{dd}, \qquad m_3' = \frac{dm_3}{dd}.$$

Solving for $\frac{dp}{dd}$ and $\frac{dq}{dd}$, we have

$$\left[\frac{dp}{dd} \right]_{d=d^*} = -\frac{m_1' m_2 + m_1 m_2' - m_3'}{2(m_2^2 + m_1^2 m_2)}.$$

To establish the Hopf bifurcation at $d = d^*$, we need to show that

$$\left[\frac{dp}{dd} \right]_{d=d^*} \neq 0,$$

i.e. $m_1' m_2 + m_1 m_2' - m_3' \neq 0$.

Here,

$$m_1' = \frac{dm_1}{dd} = 0,$$
$$m_2' = \frac{dm_2}{dd} = cq \frac{dE^*}{dd} + \frac{ax^*}{b + x^*} \left(-\frac{y^*}{k} \right)$$
$$= \left(c - \frac{rx^*}{k} \right) \left(1 - \frac{x^*}{k} \right),$$
$$m_3' = \frac{dm_3}{dd} = cq(m_1 - cy^*) \frac{dE^*}{dd}$$
$$+ cqE^* \left(\frac{dm_1}{dd} - c\frac{dy^*}{dd} \right)$$
$$= -\frac{crx^*}{b + x^*} \left(1 - \frac{x^*}{k} \right) \left(1 - \frac{b}{k} - \frac{2x^*}{k} \right).$$

Therefore,

$$m_1' m_2 + m_1 m_2' - m_3'$$
$$= \left(1 - \frac{x^*}{k} \right) \left[cy^* \left(c - \frac{rx^*}{k} \right) \right.$$
$$\left. + \frac{r^2 x^{*2}}{k(b + x^*)} \left(1 - \frac{b}{k} - \frac{2x^*}{k} \right) \right]$$
$$< 0 \quad \text{if } c - \frac{rx^*}{k} < 0 \text{ and } 1 - \frac{b}{k} - \frac{2x^*}{k} < 0.$$

Thus, if

$$k < \min \left\{ \frac{rx^*}{c}, b + 2x^* \right\},$$

then

$$m_1' m_2 + m_1 m_2' - m_3' < 0$$

and hence

$$\left[\frac{dp}{dd} \right]_{d=d^*} \neq 0.$$

Thus, we get a sufficient condition that whenever

$$k < \min \left\{ \frac{rx^*}{c}, b + 2x^* \right\},$$

the Hopf bifurcation occurs at $d = d^*$, that is the system is stable when $d < d^*$ and unstable when $d > d^*$.

9 Optimal taxation policy

The objective of the regulatory agency is to maximize the total discounted net revenues that the society derives from the fishery. Symbolically, this objective amounts to maximizing the present value J of a continuous time-stream of revenues given by

$$J = \int_0^\infty e^{-\delta t}(pqy - c)\,dt,$$

where δ denotes the instantaneous annul rate of discount, c is the fishing cost per unit effort and p is the price per unit biomass of y. To solve this optimization problem, we utilize the Pontryagin's maximal principle. We treat τ as the control variable and wish to determine a tax policy $\tau = \tau(t)$ which maximizes J subject to the system (2).

The Hamiltonian of this control problem is

$$
\begin{aligned}
H = {} & e^{-\delta t}(pqy - c)E \\
& + \lambda_1\left\{rx\left(1 - \frac{x}{k}\right) - \frac{axy}{b+x}\right\} \\
& + \lambda_2\left\{\frac{maxy}{b+x} + dy\left(1 - \frac{x}{k}\right) - cy^2 - qEy\right\} \\
& + \lambda_3\left\{(p - \tau)qy - c\right\}E,
\end{aligned}
$$

(6)

where λ_1, λ_2, and λ_3 are additional unknown functions called the adjoint variables. The Hamiltonian (6) must be maximized for τ. Assuming that the control constraints are not binding (i.e. the optimal solution does not occur at $\tau = \tau_{\min}$ or $\tau = \tau_{\max}$), we have singular control given by $\frac{\partial H}{\partial \tau} = 0$.

Now, $\frac{\partial H}{\partial \tau} = 0$ gives $\lambda_3 \lambda E q y = 0$.

We use a singular control and find the singular path. For this, we take $\lambda_3 = 0$.

The adjoint equations are

$$
\begin{aligned}
\frac{d\lambda_1}{dt} = {} & -\frac{\partial H}{\partial x} \\
= {} & -\left[\lambda_1\left\{r\left(1 - \frac{2x}{k}\right) - \frac{aby}{(b+x)^2}\right\}\right. \\
& \left. + \lambda_2\left\{\frac{maby}{(b+x)^2} - \frac{dy}{k}\right\}\right],
\end{aligned}
$$

(7)

$$
\begin{aligned}
\frac{d\lambda_2}{dt} = {} & -\frac{\partial H}{\partial y} \\
= {} & -\left[e^{-\delta t}pqE - \lambda_1\frac{ax}{b+x}\right.
\end{aligned}
$$

$$
\begin{aligned}
& + \lambda_2\left\{\frac{max}{b+x} + d\left(1 - \frac{x}{k}\right)\right. \\
& \left.\left. - 2cy - qE\right\} + \lambda_3(p - \tau)qE\right],
\end{aligned}
$$

(8)

$$
\begin{aligned}
\frac{d\lambda_3}{dt} = {} & -\frac{\partial H}{\partial E} \\
= {} & -\left[e^{-\delta t}(pqy - c) - \lambda_2 qy\right. \\
& \left. + \lambda_3\left\{(p - \tau)qy - c\right\}\right].
\end{aligned}
$$

(9)

Since $\lambda_3 = 0$, we have from (9)

$$\lambda_2 = e^{-\delta t}\left(p - \frac{c}{qy}\right).$$

(10)

We seek to find optimal equilibrium solution of the problem so that x, y, and E can be treated as constants.

Substituting λ_2 in (7), we get

$$\frac{d\lambda_1}{dt} = A\lambda_1 + Be^{-\delta t},$$

where

$$
\begin{aligned}
A &= \frac{rx}{k} - r\left(1 - \frac{x}{k}\right) + \frac{aby}{(b+x)^2}, \\
B &= \left\{\frac{dy}{k} - \frac{mab}{(b+x)^2}\right\}\left(p - \frac{c}{qy}\right).
\end{aligned}
$$

(11)

The solution of this linear equation is

$$\lambda_1 = -\frac{B}{A + \delta}e^{-\delta t} + K_0 e^{At},$$

where k_0 is a constant.

The shadow price $\lambda_1 e^{-\delta t}$ is bounded as $t \to \infty$ iff $k_0 = 0$.

Therefore,

$$\lambda_1 = -\frac{B}{A + \delta}e^{-\delta t}.$$

(12)

Using (8), we get

$$
\begin{aligned}
\delta\left(p - \frac{c}{qy}\right) = {} & pqE + \left(\frac{B}{A + \delta}\right)\left(\frac{ax}{b+x}\right) \\
& + \left(p - \frac{c}{qy}\right)\left\{\frac{max}{b+x}d\left(1 - \frac{x}{k}\right)\right. \\
& \left. - 2cy - qE\right\}.
\end{aligned}
$$

(13)

Now for the optimal equilibrium solution, we have from (2)

$$
\begin{aligned}
r\left(1 - \frac{x^*}{k}\right) - \frac{ay^*}{b + x^*} &= 0, \\
\frac{max^*}{b + x^*} + d\left(1 - \frac{x^*}{k}\right) - cy^* - q^*E &= 0, \\
(p - \tau)qy^* - c &= 0.
\end{aligned}
$$

(14)

Using these equations in (13), we get

$$\delta\left(p - \frac{c}{qy^*}\right)(cy^* + \delta)$$
$$- \left(\frac{B}{A+\delta}\right)\left(\frac{ax^*}{b+x^*}\right) - pqE^* = 0, \qquad (15)$$

where

$$A = \frac{aby^*}{(b+x^*)^2} - r\left(1 - \frac{x^*}{k}\right),$$
$$B = \left\{\frac{dy^*}{k} - \frac{mab}{(b+x^*)^2}\right\}\left(p - \frac{c}{qy^*}\right). \qquad (16)$$

Equation (15) together with Eqs. (14) gives the optimal tax $\tau = \tau^*$ and optimal equilibrium solutions x^*, y^*, E^*.

10 Numerical simulation

In this section, we present some numerical simulations of the system (2) to verify the analytical predictions obtained in the previous sections. Using numerical simulation instead of real world data, which of course would be of great interest, has some advantages. It may be noted that the simulations presented in this paper should be considered from a qualitative, rather than a quantitative point of view. However, numerous scenarios covering the breadth of the biological feasible parameter space were conducted and the results display the gamut of dynamical results collected from all the scenarios tested.
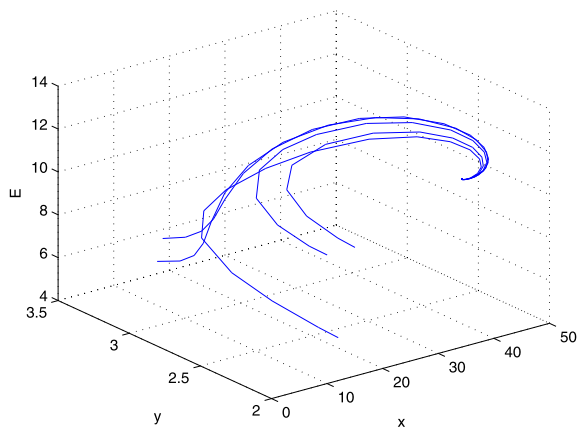

Fig. 2 Phase space trajectories corresponding to the optimal tax $\tau^* = 6.56075$ beginning with different initial levels for the model system (2). Trajectories converge to the positive equilibrium (42.6345, 2.32609, 10.2578)

(i) Let us take $r = 6, a = 20, b = 10, c = 4, d = 10, k = 50, p = 10, q = 0.5, \delta = 0.01, m = 0.8$ in appropriate units. Then from Eqs. (14) and (15), we find that for the optimal tax $\tau^* = 6.56075$, the system (2) has a positive equilibrium (42.6345, 2.32609, 10.2578) and is globally asymptotically stable as seen from Figs. 2 and 3.

(ii) From Figs. 4, 5, and 6, we observe that the prey population decreases and the predator population increases with the increase of tax whereas the harvesting effort always decreases with the increase of tax when the other parameters remain the same. This is realistic because whenever tax increases, the people are less interested to harvest
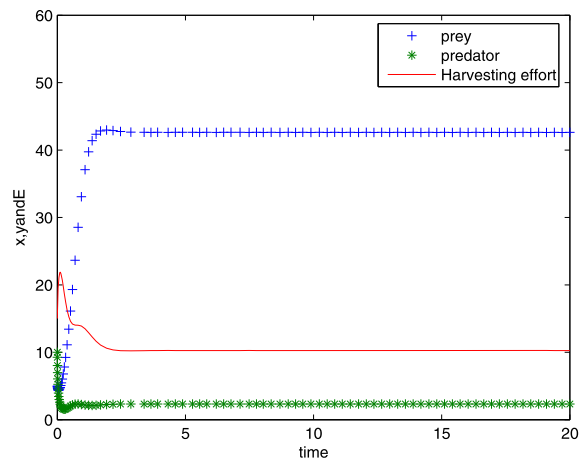

Fig. 3 Time evolution of populations for the model system (2) corresponding to the optimal tax $\tau^* = 6.56075$
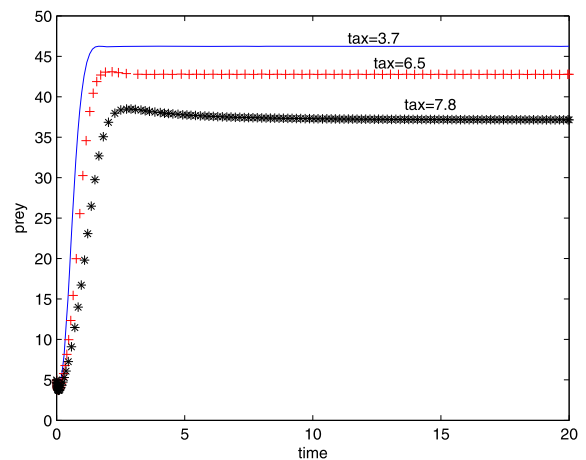

Fig. 4 Variation of prey population against time for different tax levels; the other parameters remaining the same
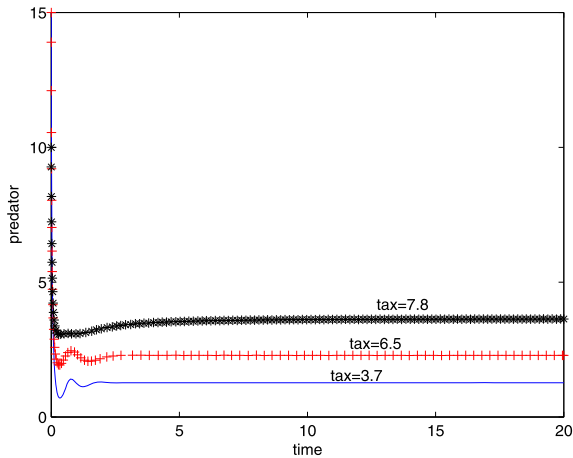
Fig. 5 Variation of predator population against time for different tax levels; the other parameters remaining the same
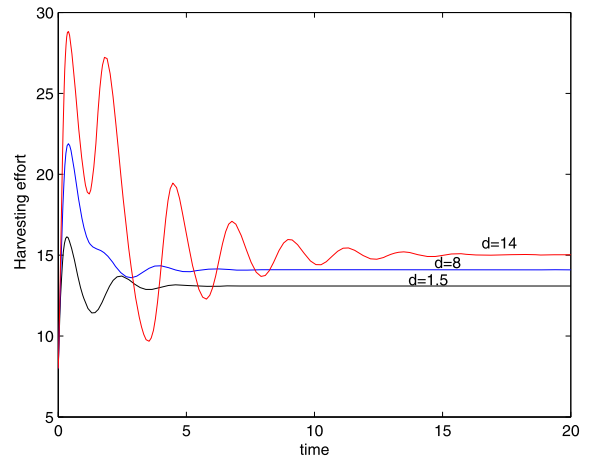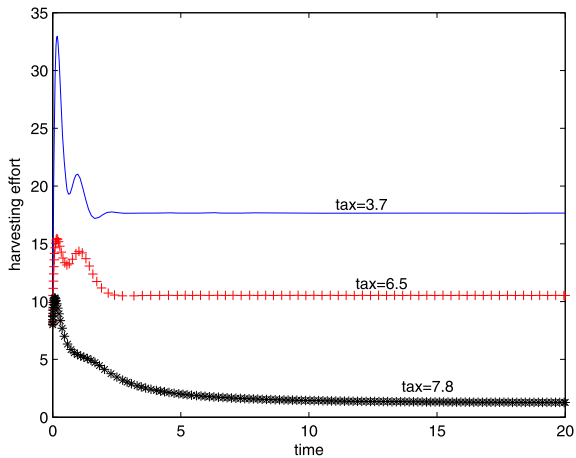

Fig. 6 Variation of harvesting effort against time for different tax levels; the other parameters remaining the same

Fig. 7 Variation of harvesting effort against time for different values of d; the values of the other parameters are $r = 5$, $a = 15$, $b = 5$, $c = 1$, $d = 1.5$, $k = 50$, $p = 2.8$, $q = 0.7$, $\delta = 0.01$, $m = 0.8$, $\tau^* = 2$
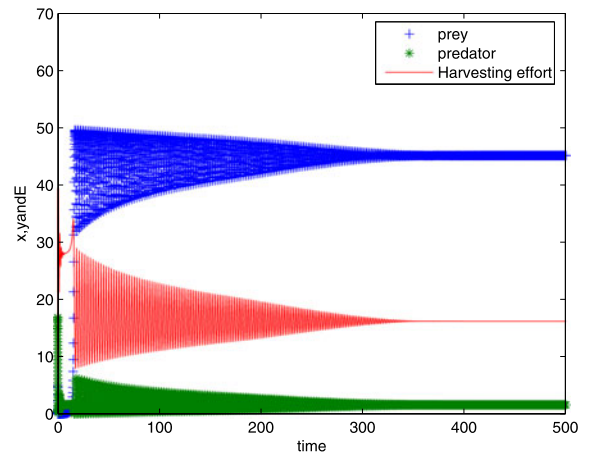

Fig. 8 This figure shows that when $d = 20 < d^* = 20.9212$, the equilibrium point (45.1975, 1.60714, 16.1133) is stable

predator and as a result, the predator population increases and consumption of the prey increases and the prey population decreases. Thus, numerically, this fact is seen from Figs. 4, 5, and 6.

(iii) From Fig. 7, we observe that as d, the growth rate of the predator due to alternative food increases, the harvesting effort increases as expected.

(iv) For the values $r = 5$, $a = 15$, $b = 5$, $c = 0.9$, $k = 50$, $p = 2.8$, $q = 0.7$, $\delta = 0.01$, $m = 0.8$, $\tau = 2$, we obtain the critical value $d^* = 20.9212$ from Sect. 8. The values of the parameters also satisfy the sufficient condition for the Hopf bifurcation. From Figs. 8 and 9, we see that when $d < d^*$, the system is stable and as d crosses its criti-

cal value d^* the system becomes unstable, i.e. a Hopf-bifurcation occurs at the critical value d^*.

(v) Figure 10 shows that the system has a cyclic behavior for $d = d^* = 20.9212$.

The numerical study presented here shows that, using parameter d as the control parameter, it is possible to break the unstable behavior of the system (2) and drive it to a stable state. Also, it is possible to keep the population levels at a required state using the above control.
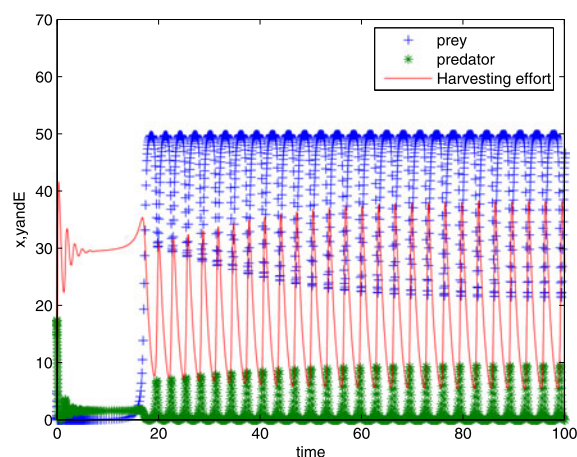
Fig. 9 This figure shows that when $d = 22 > d^* = 20.9212$, the system (2) becomes unstable
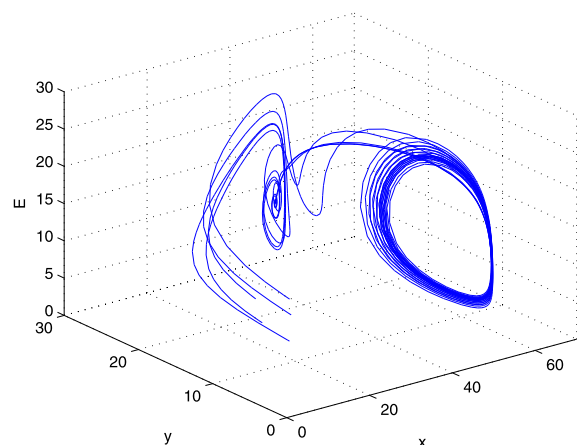

Fig. 10 Bifurcation for the critical value of the parameter $d = d^* = 20.9212$ when the other parameters remain same

11 Conclusion

Nowadays, the biological resources are mostly harvested with the aim of achieving economic interest. Thus, unregulated exploitation and extinction of many natural and biological resources is a major problem of present day. In this work, we consider a bio-economic prey–predator model with the provision of alternative food to the predator and only the predator species is harvested. We force the fishing effort to remain continuous over time and consider tax as a control instrument. The important feature of this model is that it assumes a fully dynamic interaction between the fishing effort and the perceived rent.

From the model, it is seen that the alternative food plays an important role in stability of the system. Bifurcation analysis shows that under certain conditions, the system changes its state from stable to unstable whenever the growth rate of the predator due to alternative prey crosses its critical value. Also, the optimal taxation policy is discussed. Numerical simulations show the consistency of the theoretical results.

In our model, we have considered the catch-rate function based on catch-per-unit-effort hypothesis. But this type of catch-rate function embodies some defects of such as (i) it assumes random search for fish, (ii) it assumes equal likelihood of being captured for every fish, (iii) there is unbounded linear increase in the catch with respect to effort, and (iv) there is unbounded linear increase in the catch with respect to population for a fixed effort. These unrealistic features can largely be removed by adopting the alternative functional form

$$h = \frac{qEy}{aE + by}$$

where a and b are two positive parameters, but we leave it for our future research work.

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