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Stability to vector Liénard equation with constant deviating argument

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Abstract In this paper, we consider the vector Liénard equation with the constant deviating argument, $\tau > 0$,

$$X''(t) + F\big(X(t), X'(t)\big)X'(t) + H\big(X(t-\tau)\big) = P(t)$$

in two cases: (i) $P(.) \equiv 0$, (ii) $P(.) \neq 0$. Based on the Lyapunov–Krasovskii functional approach, the asymptotic stability of the zero solution and the boundedness of all solutions are discussed for these cases. We give an example to illustrate the theoretical analysis made in this work and to show the effectiveness of the method utilized here.

Keywords Vector Liénard equation · Stability · Constant deviating argument

1 Introduction

In applied sciences, some practical problems concerning mechanics, the engineering technique fields, economy, control theory, physics, chemistry, biology, medicine, atomic energy, information theory, etc. are associated with Liénard or modified Liénard equation. By this time, the qualitative properties of solutions of scalar Liénard or modified Liénard equation with and without a deviating argument have been intensively discussed and are still being investigated in the literature. We refer the reader to the papers or books of Ahmad and Rama Mohana Rao [1], Burton [3, 4], Burton and Zhang [5], Caldeira-Saraiva [6], Cantarelli [7], Èl'sgol'ts [8], Èl'sgol'ts and Norkin [9], Gao and Zhao [10], Hale [11], Hara and Yoneyama [12, 13], Heidel [14, 15], Huang and Yu [16], Jitsuro and Yusuke [17], Kato [18, 19], Kolmanovskii and Myshkis [20], Krasovskiì [21], Li [22], Liu and Huang [23, 24], Liu and Xu [25], Liu [26], Long and Zhang [27], Luk [28], Malyseva [29], Muresan [30], Nápoles Valdés [31], Sugie [32], Sugie and Amano [33], Sugie et al. [34], Tunç [38–44], C. Tunç and E. Tunç [46], Yang [47], Ye et al. [48], Yoshizawa [49], Zhang [50, 51], Zhang and Yan [52], Zhou and Jiang [53], Zhou and Liu [54], Zhou and Xiang [55], Wei and Huang [56], Wiandt [57], and the references therein.

However, to the best of our knowledge from the literature, the stability and boundedness of solutions for vector Liénard equation with a deviating argument has not been discussed in the literature, yet.

In this paper, we are interested in the vector Liénard equation with the constant deviating argument $\tau > 0$:

$$X''(t) + F\big(X(t), X'(t)\big)X'(t) + H\big(X(t-\tau)\big) = P(t), \tag{1}$$

in which $t \in \Re^+$, $\Re^+ = [0, \infty)$, $t - \tau \geq 0$, and $X \in \Re^n$; F is a continuous symmetric $n \times n$ matrix, $H : \Re^n \to \Re^n$ and $P : \Re^n \to \Re^n$ are continuous, and H is

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also differentiable with $H(0)=0$. The existence and uniqueness of the solutions of Eq. (1) is assumed.

Equation (1) is the vector version for systems of real second order nonlinear differential equations of the form:

$$x_i''+\sum_{k=1}^{n} f_{ik}\big(x_1,\dots,x_n;x_1',\dots,x_n'\big)x_k' + h_i\big(x_1(t-\tau),\dots,x_n(t-\tau)\big)=p_i(t) \quad (i=1,2,\dots,n).$$

Obviously, we can write Eq. (1) in the differential system form as

$$\begin{aligned} X' &= Y,\\ Y' &= -F(X,Y)Y - H(X) + \int_{t-\tau}^{t} J_H\big(X(s)\big)Y(s)ds + P(t), \end{aligned} \tag{2}$$

which was obtained by setting $X'=Y$, where $X(t)$ and $Y(t)$ are respectively abbreviated as X and Y throughout the paper.

The Jacobian matrix of $H(X)$ is given by

$$J_H(X)=\left(\frac{\partial h_i}{\partial x_j}\right) \quad (i,j=1,2,\dots,n),$$

where $(x_1,x_2,\dots,x_n)$ and $(h_1,h_2,\dots,h_n)$ are the components of X and H, respectively. It is also assumed that the Jacobian matrix $J_H(X)$ exists and is continuous.

The symbol $\langle X,Y\rangle$ corresponding to any pair X,Y in $\Re^n$ stands for the usual scalar product $\sum_{i=1}^{n} x_i y_i$, that is, $\langle X,Y\rangle=\sum_{i=1}^{n} x_i y_i$; thus $\langle X,X\rangle=\|X\|^2$, and $\lambda_i(\Omega)(i=1,2,\dots,n)$ are the eigenvalues of the real symmetric $n\times n$ matrix Ω. The matrix Ω is said to be negative-definite, when $\langle \Omega X,X\rangle\le 0$ for all nonzero X in $\Re^n$.

The motivation of this paper has been inspired by the results established in the above mentioned papers and the recent papers of Tunç [35–37] and Tunç and Ateş [45]. This paper is also the first attempt to investigate the stability and boundedness of solutions to vector Liénard equation with a deviating argument, and it is a new improvement and has a contribution to the subject in the literature, and it may be useful for researchers working on the qualitative behaviors of solutions.

2 Preliminaries

We need the following preliminary results.

Lemma 1 (Bellman [2]) *Let A be a real symmetric $n\times n$ matrix and*

$$\bar a \ge \lambda_i(A) \ge a > 0 \quad (i=1,2,\dots,n),$$

where $\bar a$ and a are constants.

Then

$$\bar a\langle X,X\rangle \ge \langle AX,X\rangle \ge a\langle X,X\rangle$$

and

$$\bar a^2\langle X,X\rangle \ge \langle AX,AX\rangle \ge a^2\langle X,X\rangle.$$

For a given number $r\ge 0$, let C^n denote the space of continuous functions mapping the interval $[-r,0]$ into $\Re^n$ and for $\phi\in C^n$, $\|\phi\|=\sup_{-r\le\phi\le 0}\|\phi(\theta)\|$. C_H^n will denote the set of ϕ in C^n for which $\|\phi\|<H$. For any continuous function $x(u)$ defined on $-r\le u\le B$, $B>0$, any fixed t, $0\le t\le B$, the symbol x_t will denote the function $x(t+\theta)$, $-r\le\theta\le 0$.

If $g(\phi)$ is a functional defined for every ϕ in C_H^n and $x'(t)$ is the right-hand side derivative of $x(t)$, we consider the autonomous functional differential equation:

$$x'(t)=g(x_t), \quad t\ge 0. \tag{3}$$

We say $x(\phi)$ is a solution of Eq. (3) with the initial condition ϕ in C_H^n at $t=0$ if there is a $B>0$ such that $x(\phi)$ is a function from $[-r,B)$ into $\Re^n$ such that $x_t(\phi)$ is in C_H^n for $0\le t<B$, $x_0(\phi)=\phi$ and $x(\phi)(t)$ satisfies Eq. (3) for $0\le t<B$.

Definition 1 (Burton [4]) A continuous function $W:\Re^n\to\Re^+$ with $W(0)=0$, $W(s)>0$ if $s>0$, and W strictly increasing is a wedge. (We denote wedges by W or W_i, where i is an integer.)

Definition 2 (Hale [11]) Let V be a continuous scalar functional in C_H^n. The derivative of V along the solutions of Eq. (3) will be defined by

$$V'(\phi)=\limsup_{h\to 0^+}\frac{V(x_h(\phi))-V(\phi)}{h}.$$

Lemma 2 (Hale [11]) *Suppose $g(0)=0$. Let V be a continuous functional defined on C_H^n with $V(0)=0$ and let $u(s)$ be a function, non-negative and continuous for $0\le s<\infty$, $u(s)\to\infty$ as $s\to\infty$ with $u(0)=0$. If for all ϕ in C_H^n, $u(\|\phi(0)\|)\le V(\phi)$, $V'(\phi)\le 0$, then the solution $x=0$ of Eq. (3) is stable.*

Let $R\subset C_H^n$ be a set of all functions $\phi\in C_H^n$ where $V'(\phi)=0$. If $\{0\}$ is the largest invariant set in R, then the solution $x=0$ of Eq. (3) is asymptotically stable.

Let us consider non-autonomous delay differential system

$$x' = f(t, x_t), \quad x_t = x(t+\theta), \\ -r \le \theta \le 0, \quad t \ge 0, \tag{4}$$

where $f : \Re^+ \times C_H \to \Re^n$ is a continuous mapping, $f(t,0) = 0$, and we suppose that F takes closed bounded sets into bounded sets of $\Re^n$. Here $(C, \|.\|)$ is the Banach space of continuous functions $\phi : [-r, 0] \to \Re^n$ with the supremum norm, $r > 0$; C_H is the open H-ball in C; $C_H := \{\phi \in (C[-r,0], \Re^n) : \|\phi\| < H\}$. Let S be the set of $\varphi \in C$ such that $\|\varphi\| \ge H$. We shall denote by $S^\bullet$ the set of all functions $\varphi \in C$ such that $|\varphi(0)| \ge H$, where H is large enough.

Definition 3 (Burton [4]) Let D be an open set in $\Re^n$ with $0 \in D$. A function $V : [0,\infty) \times D \to [0,\infty)$ is called positive definite if $V(t,0) = 0$ and if there is a wedge W_1 with $V(t,x) \ge W_1(|x|)$, and is called a decrescent function if there is a wedge W_2 with $V(t,x) \le W_2(|x|)$.

Theorem 1 (*Burton* [4]) *If there is a Lyapunov functional for* (4) *and wedges satisfying*:

(i) $W_1(|\varphi(0)|) \le V(t,\varphi) \le W_2(\|\varphi\|)$, (*where* $W_1(r)$ *and* $W_2(r)$ *are wedges*),
(ii) $V'(t,\varphi) \le 0$,

then the zero solution of (4) *is uniformly stable.*

Theorem 2 (*Yoshizawa* [49]) *Suppose that there exists a continuous Lyapunov functional* $V(t,\varphi)$ *defined for all* $t \in \Re^+$ *and* $\varphi \in S^\bullet$, *which satisfies the following conditions*:

(i) $a(|\varphi(0)|) \le V(t,\varphi) \le b_1(|\varphi(0)|) + b_2(\|\varphi\|)$,

where $a(r), b_1(r), b_2(r) \in CI$ (CI *denotes the set of continuous increasing functions*), *and are positive for* $r > H$ *and* $a(r) - b_2(r) \to \infty$ *as* $r \to \infty$,

(ii) $V'(t,\varphi) \le 0$,

then the solutions of (4) *are uniformly bounded.*

3 Main results

Let $P(.) \equiv 0$.

Our first result is the following theorem.

Theorem 3 *In addition to the basic assumptions imposed on* F *and* G *that appear in Eq.* (1), *we assume that there exist some positive constants* a_0, a_1 *and* a_2 *such that the following conditions hold*:

(i) *The matrix* F *is symmetric, and* $\lambda_i(F(.)) \ge a_1$ *for all* $X, Y \in \Re^n$.
(ii) $H(0) = 0$, $H(X) \ne 0$, $(X \ne 0)$, $J_H(X)$ *is symmetric and* $a_2 \le \lambda_i(J_H(X)) \le a_0$ *for all* $X \in \Re^n$.

If

$$\tau < \frac{a_1}{\sqrt{n a_0}},$$

then the solution $X = 0$ *of Eq.* (1) *is asymptotically stable.*

Proof We define a continuous differentiable Lyapunov–Krasovskii functional $V(.) = V(X_t, Y_t)$ by

$$V(.) = \int_0^1 \langle H(\sigma X), X\rangle d\sigma + \frac{1}{2}\langle Y, Y\rangle + \lambda \int_{-\tau}^0 \int_{t+s}^t \|Y(\theta)\|^2 d\theta\, ds.$$

It is clear that $V(0,0) = 0$. On the other hand, since $H(0) = 0$, $\frac{\partial}{\partial\sigma} H(\sigma X) = J_H(\sigma X)X$ and $\lambda_i(J_H(X)) \ge a_2$, then we can write

$$H(X) = \int_0^1 J_H(\sigma X) X\, d\sigma$$

so that

$$\begin{aligned}\int_0^1 \langle H(\sigma X), X\rangle d\sigma &= \int_0^1 \int_0^1 \langle \sigma_1 J_H(\sigma_1\sigma_2 X)X, X\rangle d\sigma_2\, d\sigma_1 \\ &\ge \int_0^1 \int_0^1 \langle \sigma_1 a_2 X, X\rangle\, d\sigma_2\, d\sigma_1 \ge \frac{a_2}{2}\|X\|^2.\end{aligned}$$

Hence, it follows that

$$\begin{aligned} V &\ge \frac{1}{2}\big(a_2\|X\|^2 + \|Y\|^2\big) + \lambda \int_{-\tau}^0 \int_{t+s}^t \|Y(\theta)\|^2 d\theta\, ds \\ &\ge D_1\big(\|X\|^2 + \|Y\|^2\big) + \lambda \int_{-\tau}^0 \int_{t+s}^t \|Y(\theta)\|^2 d\theta\, ds, \\ &= D_1\big(\|X\|^2 + \|Y\|^2\big) + \lambda \int_{-\tau}^0 \|Y(t+\theta)\|^2 d\theta \\ &= D_1\big(\|\phi(0)\|^2\big) + \lambda \int_{-\tau}^0 \|Y(t+\theta)\|^2 d\theta, \end{aligned}$$

where $D_1 = \min\{a_2, 1\}$.

If ϕ is in C_H^n, then $V(x_t(\phi))$ non-increasing implies $u(\|x(\phi)(t)\|) \le V(x_t(\phi)) \le V(\phi)$ for all $t \ge 0$.

Therefore, we can find a continuous function $u(s)$ such that

$$u\big(\|\phi(0)\|\big) \le V(\phi), \qquad u\big(\|\phi(0)\|\big) \ge 0.$$

Calculating the time derivative of the Lyapunov–Krasovskii functional $V(.)$, along any solution $(X(t), Y(t))$ of (2), we get

$$\begin{aligned} V'(.) &= \frac{d}{dt} V(X_t, Y_t) \\ &= -\langle H(X), Y\rangle - \langle F(X,Y)Y, Y\rangle \\ &\quad + \left\langle \int_{t-\tau}^{t} J_H\big(X(s)\big)Y(s)\,ds, Y\right\rangle \\ &\quad + \frac{d}{dt}\int_0^1 \langle H(\sigma X), X\rangle\, d\sigma \\ &\quad + \frac{d}{dt}\lambda \int_{-\tau}^{0}\int_{t+s}^{t} \big\|Y(\theta)\big\|^2\, d\theta\, ds. \end{aligned}$$

Recall that

$$\begin{aligned} &\frac{d}{dt}\int_0^1 \langle H(\sigma X), X\rangle\, d\sigma \\ &\quad = \int_0^1 \sigma\langle J_H(\sigma X)Y, X\rangle\, d\sigma + \int_0^1 \langle H(\sigma X), Y\rangle\, d\sigma \\ &\quad = \int_0^1 \sigma \frac{\partial}{\partial\sigma}\langle H(\sigma X), Y\rangle\, d\sigma + \int_0^1 \langle H(\sigma X), Y\rangle\, d\sigma \\ &\quad = \sigma\langle H(\sigma X), Y\rangle\big|_0^1 = \langle H(X), Y\rangle \end{aligned}$$

and

$$\begin{aligned} &\frac{d}{dt}\lambda \int_{-\tau}^{0}\int_{t+s}^{t} \big\|Y(\theta)\big\|^2\, d\theta\, ds \\ &\quad = \lambda\|Y\|^2\tau - \lambda\int_{t-\tau}^{t} \big\|Y(\theta)\big\|^2\, d\theta. \end{aligned}$$

By the above estimates, we obtain

$$\begin{aligned} V'(.) &= -\langle F(X,Y)Y, Y\rangle + \lambda\tau\langle Y, Y\rangle \\ &\quad + \left\langle \int_{t-\tau}^{t} J_H\big(X(s)\big)Y(s)\,ds, Y\right\rangle \\ &\quad - \lambda\int_{t-\tau}^{t} \big\|Y(\theta)\big\|^2\, d\theta. \end{aligned}$$

Using the assumptions $\lambda_i(F(X,Y)) \ge a_1$, $\lambda_i(J_H(X)) \le a_0$ and the inequality $2|a||b| \le a^2 + b^2$ (with a and b are real numbers) combined with the classical Cauchy–Schwartz inequality, it follows that

$$-\langle F(X,Y)Y, Y\rangle \le -a_1\|Y\|^2,$$

$$\begin{aligned} &\left\langle \int_{t-\tau}^{t} J_H\big(X(s)\big)Y(s)\,ds, Y\right\rangle \\ &\quad \le \|Y\| \left\| \int_{t-\tau}^{t} J_H\big(X(s)\big)Y(s)\,ds \right\| \\ &\quad \le \sqrt{n}a_0\|Y\| \int_{t-\tau}^{t} \big\|Y(s)\big\|\,ds \\ &\quad \le \frac{1}{2}\sqrt{n}a_0 \int_{t-\tau}^{t} \big(\big\|Y(t)\big\|^2 + \big\|Y(s)\big\|^2\big)\,ds \\ &\quad \le \frac{1}{2}\sqrt{n}a_0\tau\|Y\|^2 + \frac{1}{2}\sqrt{n}a_0 \int_{t-\tau}^{t} \big\|Y(s)\big\|^2\,ds \end{aligned}$$

so that

$$\begin{aligned} V'(.) &\le -\left\{a_1 - \left(\frac{1}{2}\sqrt{n}a_0 + \lambda\right)\tau\right\}\|Y\|^2 \\ &\quad - \lambda\int_{t-\tau}^{t} \big\|Y(\theta)\big\|^2\, d\theta \\ &\quad + \frac{1}{2}\sqrt{n}a_0\int_{t-\tau}^{t} \big\|Y(s)\big\|^2\, ds. \end{aligned}$$

Let

$$\lambda = \frac{1}{2}\sqrt{n}a_0.$$

Hence

$$V'(.) \le -(a_1 - \sqrt{n}a_0\tau)\|Y\|^2.$$

If $\tau < \frac{a_1}{\sqrt{n}a_0}$, then we have for some positive constant α that

$$V'(.) \le -\alpha\|Y\|^2 \le 0.$$

We also observe from previous estimate of $V'(.)$ that $V'(.) = 0$ $(t \ge 0)$ necessarily implies that $Y = 0$ for all $t \ge 0$, and therefore also that $X = \xi$ (a constant vector). The substitution of the estimates

$$X = \xi, \quad Y = 0 \quad (t \ge 0)$$

in (2) leads to the result $H(\xi) = 0 \Leftrightarrow \xi = 0$. Thus $V'(.) = 0$ $(t \ge 0)$ implies that

$$X = Y = 0 \quad \text{for all } t \ge 0.$$

The last estimate shows that the largest invariant set in Z is $Q = \{0\}$, where $Z = \{\phi \in C_H : V'(\phi) = 0\}$. Further, it can be seen that the only solution of Eq. (1) for which $V'(X_t, Y_t) \equiv 0$ is the solution $X \equiv 0$. Thus, subject to the above discussion, one can conclude that the zero solution of Eq. (1) is asymptotically stable.

This completes the proof of Theorem 3. □

Corollary 1 *Under the assumptions of Theorem 3, the zero solution of Eq. (1) is uniformly stable.*

Let $P(.) \neq 0$.

Our second result is the following theorem.

Theorem 4 *Let us assume that all the assumptions of Theorem 3 and*

$$\|P(t)\| \leq Q(t)$$

hold, where $Q \in L^1(0,\infty)$, $L^1(0,\infty)$ *is the space of Lebesgue-integrable functions.*

If

$$\tau < \frac{a_1}{\sqrt{n}a_0},$$

then there exists a positive constant K *such that the solution* $X(t)$ *of Eq. (1) defined by the initial functions*

$$X(t) = \Phi(t), \quad X'(t) = \Phi'(t), \quad t_0 - \tau \leq t \leq t_0,$$

satisfies the estimates

$$\|X(t)\| \leq K, \quad \|X'(t)\| \leq K$$

for all $t \geq t_0$, *where* $\Phi \in C^1([t_0-\tau, t_0], \Re)$.

Proof We reconsider the Lyapunov–Krasovskii functional which is defined above. Then, under the assumptions of Theorem 4, it can be easily seen that

$$V(.) \geq D_1\left(\|X\|^2 + \|Y\|^2\right).$$

Since $P \neq 0$, the time derivative of V can be bounded as follows:

$$\begin{aligned} V'(.) &\leq -\alpha\|Y\|^2 + \langle Y, P(t)\rangle \\ &\leq \|Y\|\|P(t)\| \\ &\leq \|Y\|Q(t). \end{aligned}$$

In view of the estimate

$$\|Y\| \leq 1 + \|Y\|^2,$$

it follows that

$$\begin{aligned} V'(.) &\leq Q(t) + Q(t)\|Y\|^2 \\ &\leq Q(t) + D_2 Q(t) V(.) \end{aligned}$$

where $D_2 = D_1^{-1}$.

Integrating the last estimate from 0 to t $(t \geq 0)$, we have

$$\begin{aligned} &V(X_t, Y_t) - V(X_0, Y_0) \\ &\quad\leq \int_0^t Q(s)\,ds + D_2\int_0^t V(X_s, Y_s)Q(s)\,ds. \end{aligned}$$

Let $D_3 = V(X_0, Y_0) + \int_0^t Q(s)\,ds$. Using Gronwall–Bellman inequality, we obtain

$$V(X_t, Y_t) \leq D_3 \exp\left(D_2\int_0^t Q(s)\,ds\right).$$

This completes the proof of Theorem 4. □

Corollary 2 *Subject to the assumptions of Theorem 4, all the solutions of Eq. (1) are uniformly bounded.*

Corollary 3 *Consider the vector Liénard equation with the variable deviating argument* $\tau(t) > 0$:

$$\begin{aligned} &X''(t) + F\big(X(t), X'(t)\big)X'(t) + H\big(X\big(t-\tau(t)\big)\big) \\ &\quad = P(t), \end{aligned} \tag{5}$$

in which $t \in \Re^+$, $\Re^+ = [0,\infty)$, $t - \tau(t) \geq 0$, *and* $X \in \Re^n$; F *is a continuous symmetric* $n \times n$ *matrix,* $H: \Re^n \to \Re^n$ *and* $P: \Re^n \to \Re^n$ *are continuous, and* H *and* $\tau(t)$ *are also differentiable with* $H(0) = 0$.

Let us define a continuous differentiable Lyapunov–Krasovskii functional $V_1(.) = V_1(X_t, Y_t)$ *by*

$$\begin{aligned} V_1(.) &= \int_0^1 \big\langle H(\sigma X), X\big\rangle d\sigma + \frac{1}{2}\langle Y, Y\rangle \\ &\quad + \lambda\int_{-\tau(t)}^0 \int_{t+s}^t \|Y(\theta)\|^2\,d\theta\,ds. \end{aligned}$$

Then, it can be proved that the zero solution of Eq. (5) is asymptotically stable and uniformly stable when $P(.) \equiv 0$, *and all the solutions of Eq. (5) are bounded and uniformly bounded, when* $P(.) \neq 0$.

Example As a special case of Eq. (1) for $n = 2$, we consider the vector Liénard equation with the constant deviating argument, $\tau > 0$,

$$\begin{aligned} &x_i'' + \sum_{k=1}^2 f_{ik}(x_1,\ldots,x_n; x_1',\ldots,x_n')x_k' \\ &\quad + h_i\big(x_1(t-\tau),\ldots,x_n(t-\tau)\big) = p_i(t) \quad (i = 1, 2). \end{aligned}$$

Let

$$F(X, X') = \begin{bmatrix} 3 + x_1^4 + x_1'^4 & 0 \\ 0 & 3 + x_2^4 + x_2'^4 \end{bmatrix},$$

$$H\big(X(t-\tau)\big) = \begin{bmatrix} x_1(t-\tau) \\ x_2(t-\tau) \end{bmatrix},$$

and

$$P(t) = \begin{bmatrix} \frac{\sin t}{1+t^2} \\ \frac{\cos t}{1+t^2} \end{bmatrix}.$$

Then, clearly, the eigenvalues of the matrix $F(.)$ are given by

$$\lambda_1(F) = 3 + x_1^4 + x_1'^4 \quad \text{and} \quad \lambda_2(F) = 3 + x_2^4 + x_2'^4$$

so that

$$\lambda_i(F) \geq 3 = a_1 > 0 \quad (i = 1, 2).$$

The Jacobian matrix of $H(x(t-\tau))$ is given by

$$J_H(X) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

so that $\lambda_i(J_H(X)) = 1 > 0$ and

$$1 \leq \lambda_i\big(J_H(X)\big) < 2 \quad (i = 1, 2).$$

Further, it follows that

$$\|P(t)\| = \left\| \begin{bmatrix} \frac{\sin t}{1+t^2} \\ \frac{\cos t}{1+t^2} \end{bmatrix} \right\| \leq \frac{2}{1+t^2} = Q(t),$$

$$\int_0^\infty Q(s)\, ds = 2\int_0^\infty \frac{1}{1+s^2}\, ds = \pi,$$

that is, $Q \in L^1(0, \infty)$.

Thus, all the conditions of Theorem 3 and Theorem 4 hold.

Remark It is necessary to remark that all the conditions of Theorem 3 and Theorem 4 are fulfilled whenever $\tau < \frac{3}{2\sqrt{2}}$.

4 Conclusion

A vector Liénard equation with a constant deviating argument is considered. The stability and boundedness of solutions of this equation is discussed. In proving our results, we employ the Lyapunov–Krasovskii functional approach by defining a new Lyapunov–Krasovskii functional. An example is also constructed to illustrate our theoretical findings.

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