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A geometrically exact approach to the overall dynamics of elastic rotating blades—part 2: flapping nonlinear normal modes

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Abstract The geometrically exact equations of motion about the prestressed state discussed in part 1 (i.e., the nonlinear equilibrium under centrifugal forces) are expanded in the Taylor series of the incremental displacements and rotations to obtain the third-order perturbed form. The expanded form is amenable to a perturbation treatment to unfold the nonlinear features of free undamped flapping dynamics. The method of multiple scales is thus applied directly to the partial-differential equations of motion to construct the backbone curves of the flapping modes and their nonlinear approximations when they are away from internal resonances with other modes. The effective nonlinearity coefficients of the lowest three flapping modes of elastic isotropic blades are investigated when the angular speed is changed from low- to high-speed regimes. The novelty of the current findings is in the fact that the nonlinearity of the flapping modes is shown to depend critically on the angular speed since it can switch from hardening to softening and vice versa at certain speeds. The asymptotic results are compared with previous literature results. Moreover, 2:1 internal resonances between flapping and axial modes are exhibited as singularities of the effective nonlinearity coefficients. These nonlinear interactions can entail fundamental changes in the blade local and global dynamics.

Keywords Rotating isotropic blade · Nonlinear flapping vibrations · Nonlinear normal modes · Method of multiple scales

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1 Introduction

The study of dynamics of rotating blades such as helicopter blades or wind turbines is important for design, optimization, and control. The majority of previous studies on nonlinear vibrations of rotating blades are based on Euler–Bernoulli beam models, while a few authors have considered more refined mechanical blade models.

Linear vibrations of rotating blades are studied extensively in the literature (see, e.g., Stafford and Giurgiutiu [1, 2] and Wright et al. [3]). On the other hand, in the context of studies on nonlinear vibrations of rotating beams, some dealt with flexural-flexural-torsional vibrations in 3D space while others addressed in-plane vibrations. Crespo da Silva and Hodges [4] investigated the effects of higher-order terms as well as aerodynamic forces on the stability of the coupled elastic flapping, lead-lagging, and torsional motions of uniform straight rotating blades. They recognized the cubic-order structural geometric nonlinearities in the torsional equation as the most important nonlinear terms while the bending terms were shown to be less important.

Chen and Peng [5] studied the dynamic stability of a rotating blade subjected to axial periodic forces by Lagrange's equation and a Galerkin-based finite element method. The model included the effects of geometric nonlinearity, shear deformation, and rotary inertia. They applied the iterative method to obtain the mode shapes and nonlinear frequencies. They showed that with increasing the flapping amplitude of the beam tip, the instability regions shift toward high-frequency ratios and the widths of the instability regions decrease.

Crespo da Silva [6] derived the fully nonlinear partial-differential equations of motion of rotating blades including the geometric nonlinearities through a variational process. He found the equilibrium solution and the eigenpairs of the perturbed system under the aerodynamic forces by numerical integration of a nonlinear two-point boundary-value problem. He then analyzed the stability of the perturbed blade about its equilibrium.

Pesheck et al. [7] presented a method based on invariant manifolds for reduced-order modeling of rotating beams including the case of internal resonances. They employed the Rayleigh–Ritz procedure to obtain a discrete system of equations with quadratic and cubic geometric nonlinearities.

Apiwattanalunggarn et al. [8] compared the conventional asymptotic approach and the Galerkin-based approach for a rotating beam with internal resonances. They showed that the latter provides a more accurate reduction, especially at large amplitudes.

Hodges [9] derived the exact nonlinear equations of motion of precurved and pretwisted anisotropic beams. Using the Kirchhoff analogy, he demonstrated that a time-discretization scheme for the nonlinear dynamics of rigid bodies can be applied, in the same fashion, in space for the nonlinear static treatment of beams.

Pierre et al. [10] developed an efficient Galerkin projection method for constructing the nonlinear modes up to large amplitudes for systems featuring internal resonance and for systems subjected to external excitation. They illustrated the accuracy of the proposed Galerkin-based method in the construction of the nonlinear normal modes (NNMs) for an isotropic rotating Euler–Bernoulli beam.

Das et al. [11] studied free out-of-plane vibrations of a rotating beam with a nonlinear spring-mass system. They obtained the nonlinear frequencies of the flapping modes by applying the direct method of multiple time scales to the equations of motion and boundary conditions.

Avramov et al. [12] studied nonlinear flexural-flexural-torsional vibrations of rotating beams with asymmetric cross sections. They investigated nonlinear free vibrations of a rotating beam by discussing the backbone curves and the surfaces describing the nonlinear modes. A softening behavior was predicted for the first and fourth modes while a hardening behavior was found for the second and third modes, respectively.

Ghorashi and Nitzsche [14] investigated the elastodynamic response of an accelerating rotating hingeless composite beam previously studied in [13] by solving the nonlinear intrinsic differential equations of a beam. They analyzed the stability of the steady-state response of rotating beams. They also inspected the effect of embedded anisotropic piezocomposite actuators in the beam response.

Turhan and Bulut [15] applied the Lindstedt–Poincaré method and the method of multiple scales to the discretized integropartial-differential equations of motion of a rotating Euler–Bernoulli beam with third-order curvature nonlinearity for analyzing in-plane vibrations. They observed that the changes in angular speed have some unusual effects on nonlinear vibrations of a rotating beam, such as the switching from hardening to softening and harmonic or superharmonic jump phenomena when the beam is under an external periodic excitation.

Palacios [16] obtained the NNMs of beams from intrinsic equations, in the framework of the fully intrinsic equations of Hodges [9], in which the velocities and strains were considered as primary unknowns. Using the Cosserat exact kinematic description, the NNMs were obtained via an asymptotic approximation to the invariant manifolds in the space of the intrinsic modal coordinates. He considered homogeneous isotropic and composite cantilevered beams as illustrative examples.

Arvin and Bakhtiari-Nejad [17] applied the method of multiple scales to the discretized equations of motion which were obtained using Hamilton's principle, in order to construct the NNMs with or without internal resonances. They investigated the stability and bifurcations of the NNMs under three-to-one and two-to-one internal resonances between two flapping modes and between one flapping mode and one axial mode, respectively.

In this paper, the equations of motion for arbitrary linearly elastic, isotropic beams obtained in part 1 [18] via a geometrically exact approach are expanded in Taylor series about the prestressed equilibrium to obtain the third-order perturbed form. The method of multiple scales is applied directly to the partial-differential equations and boundary conditions to construct the backbone curves of the flapping modes along the lines of previous works on the analytical asymptotic construction of NNMs of general one-dimensional distributed-parameter systems [19–24]. The results are compared with the results of [17].

2 Equations of motion

The equations of motion were derived in part 1 [18] by defining the geometry of deformation of the rotating blade, thus enforcing the local balance of linear and angular momentum. The equations governing the prestressed solution read

$$\partial_s \boldsymbol{n}^{\mathrm{o}}(s,t) + \boldsymbol{f}^{\mathrm{o}} = \partial_t \breve{\boldsymbol{l}}^{\mathrm{o}}, \tag{1}$$

$$\partial_s \boldsymbol{m}^{\mathrm{o}}(s,t) + \boldsymbol{\nu}^{\mathrm{o}}(s,t) \times \boldsymbol{n}^{\mathrm{o}}(s,t) + \boldsymbol{c}^{\mathrm{o}} = \partial_t \breve{\boldsymbol{h}}^{\mathrm{o}}. \tag{2}$$

The component representation of the generalized stress resultant $\boldsymbol{n}^{\mathrm{o}}$ and moment $\boldsymbol{m}^{\mathrm{o}}$ in the local frame $(\boldsymbol{b}_1^{\mathrm{o}}, \boldsymbol{b}_2^{\mathrm{o}}, \boldsymbol{b}_3^{\mathrm{o}})$ gives $\boldsymbol{n}^{\mathrm{o}} = Q_1^{\mathrm{o}}\boldsymbol{b}_1^{\mathrm{o}} + Q_2^{\mathrm{o}}\boldsymbol{b}_2^{\mathrm{o}} + N^{\mathrm{o}}\boldsymbol{b}_3^{\mathrm{o}}$ and $\boldsymbol{m}^{\mathrm{o}} = M_1^{\mathrm{o}}\boldsymbol{b}_1^{\mathrm{o}} + M_2^{\mathrm{o}}\boldsymbol{b}_2^{\mathrm{o}} + T^{\mathrm{o}}\boldsymbol{b}_3^{\mathrm{o}}$ where $(Q_1^{\mathrm{o}}, Q_2^{\mathrm{o}})$ are the shear forces, N^{o} is the tension, $(M_1^{\mathrm{o}}, M_2^{\mathrm{o}})$ are the bending moments, and T^{o} is the torque. ∂_s denotes differentiation with respect to the arclength s along the base line of the blade in its natural state and ∂_t indicates differentiation with respect to time t. The vectors $\breve{\boldsymbol{l}}^{\mathrm{o}}$ and $\breve{\boldsymbol{h}}^{\mathrm{o}}$ denote the linear and angular momentum per unit reference length of the blade, respectively. The vector $\boldsymbol{\nu}^{\mathrm{o}} = \partial_s \boldsymbol{r}^{\mathrm{o}}$ is the *stretch vector* [25] where $\boldsymbol{r}^{\mathrm{o}}$ is the position vector of the blade base line in the prestressed state.

On the other hand, the equations of motion for the current configuration, obtained via a total Lagrangian formulation, are

$$\partial_s \breve{\boldsymbol{n}}(s,t) + \breve{\boldsymbol{f}} = \partial_t \breve{\boldsymbol{l}}, \tag{3}$$

$$\partial_s \breve{\boldsymbol{m}}(s,t) + \breve{\boldsymbol{\nu}}(s,t) \times \breve{\boldsymbol{n}}(s,t) + \breve{\boldsymbol{c}} = \partial_t \breve{\boldsymbol{h}}, \tag{4}$$

where $\breve{\boldsymbol{n}} = \breve{Q}_1\boldsymbol{b}_1 + \breve{Q}_2\boldsymbol{b}_2 + \breve{N}\boldsymbol{b}_3$ and $\breve{\boldsymbol{m}} = \breve{M}_1\boldsymbol{b}_1 + \breve{M}_2\boldsymbol{b}_2 + \breve{T}\boldsymbol{b}_3$ are the total generalized stress and moment resultants in the current configuration $\mathcal{B}$.

The natural decompositions for the generalized stress resultants in two parts, one related to the prestressed solution and to the incremental part, read: $\breve{\boldsymbol{n}}(s,t) = \boldsymbol{n}^{\mathrm{o}}(s,t) + \boldsymbol{n}(s,t)$ and $\breve{\boldsymbol{m}}(s,t) = \boldsymbol{m}^{\mathrm{o}}(s,t) + \boldsymbol{m}(s,t)$ where $\boldsymbol{n}(s,t)$ and $\boldsymbol{m}(s,t)$ are the incremental contact force and contact couple. The same decomposition holds for the mechanical data: $\breve{\boldsymbol{f}}(s,t) = \boldsymbol{f}^{\mathrm{o}}(s,t) + \boldsymbol{f}(s,t)$ and $\breve{\boldsymbol{c}}(s,t) = \boldsymbol{c}^{\mathrm{o}}(s,t) + \boldsymbol{c}(s,t)$. By introducing these decompositions together with the balance equations (1) and (2) for the prestressed state, Eqs. (3) and (4) are simplified into the following incremental equations of motion:

$$\partial_s \boldsymbol{n}(s,t) + \boldsymbol{f} = \partial_t \breve{\boldsymbol{l}} - \partial_t \breve{\boldsymbol{l}}^{\mathrm{o}}, \tag{5}$$

$$\begin{aligned}&\partial_s \boldsymbol{m}(s,t) + \breve{\boldsymbol{\nu}}(s,t) \times \boldsymbol{n}(s,t)\\ &\quad + \partial_s \boldsymbol{u}(s,t) \times \boldsymbol{n}^{\mathrm{o}}(s,t) + \boldsymbol{c} = \partial_t \breve{\boldsymbol{h}} - \partial_t \breve{\boldsymbol{h}}^{\mathrm{o}},\end{aligned} \tag{6}$$

where $\boldsymbol{f}(s,t)$ and $\boldsymbol{c}(s,t)$ are the incremental external force and couple per unit reference length, respectively. The vector $\breve{\boldsymbol{\nu}} = \partial_s \breve{\boldsymbol{r}}$ is the *total stretch vector* [25] where $\breve{\boldsymbol{r}}$ is the position vector of the blade base line in the current configuration while $\boldsymbol{u}$ is the displacement vector from the prestressed state to the current configuration.

As shown in part 1 [18], by accounting for the low shear compliance of slender blades, the unshearability kinematic constraints can be enforced so as to reduce the equations of motion to four governing equations via elimination of the shear forces. The boundary conditions respectively for the root and tip sections of the rotating blade are $\breve{\boldsymbol{r}}(0,t) = \boldsymbol{r}_{\mathrm{o}}^{\mathrm{E}}(t)$ and $\boldsymbol{R}(0,t) = \boldsymbol{I}$ and $\breve{\boldsymbol{n}}(L,t) = \mathbf{o}$ and $\breve{\boldsymbol{m}}(L,t) = \mathbf{o}$ where L is the span of the blade.

The constitutive laws for the incremental stress resultants of linearly elastic isotropic blades are assumed in the form $\hat{N} = EA(\nu - 1)$, $\hat{M}_1 = EJ_{11}^{\mathrm{S}}\mu_1$, $\hat{M}_2 = EJ_{22}^{\mathrm{S}}\mu_2$, and $\hat{T} = GJ_{33}^{\mathrm{S}}\mu_3$ where ν is the incremental stretch and (μ_1, μ_2, μ_3) are the incremental bending and torsional curvatures, while $(EA, EJ_{11}^{\mathrm{S}}, EJ_{22}^{\mathrm{S}}, GJ_{33}^{\mathrm{S}})$ denote the blade axial stiffness, the bending stiffnesses about $\boldsymbol{b}_1$ and $\boldsymbol{b}_2$ and the torsional stiffness, respectively.

A suitable nondimensionalization is introduced by dividing all space variables by the span L and time by $1/\omega_f$. The following nondimensional parameters are thus defined:

$$r := d_3/L, \quad \lambda_t := \sqrt{GJ_{33}^S/(\rho J_{33}^S \omega_f^2 L^2)},$$
$$\lambda_b^{(2)} := \sqrt{EJ_{22}^S/(\rho AL^4 \omega_f^2)},$$
$$\lambda_b^{(1)} := \sqrt{EJ_{11}^S/(\rho AL^4 \omega_f^2)},$$
$$\lambda_a := \sqrt{EA/(\rho AL^2 \omega_f^2)}, \quad \bar{\omega}_R := \omega_R/\omega_f,$$
$$\bar{\omega}_R^a := \bar{\omega}_R/\lambda_a, \quad \alpha_2 := \sqrt{(\rho AL^2)/(\rho J_{22}^S)},$$
$$\alpha_1 := \sqrt{(\rho AL^2)/(\rho J_{11}^S)}, \quad \alpha_3 := \sqrt{\rho J_{11}^S \alpha_1^2/\rho J_{33}^S}.$$

A suitable choice for the characteristic frequency is $\omega_f := \sqrt{EJ_{22}^S/(\rho AL^4)}$ by which $\lambda_b^{(1)} = 1$, $\lambda_b^{(2)} = \sqrt{EJ_{11}^S/EJ_{22}^S}$, $\lambda_a = \sqrt{EAL^2/EJ_{22}^S}$.

3 Perturbation treatment via the method of multiple scales

The only nontrivial strain generated by nonlinear flapping vibrations of unshearable rotating blades is the longitudinal stretch. Hence, to capture the geometric (stiffening) effects on flapping vibrations, a third-order Taylor expansion of the equations governing flapping-axial motions is carried out to yield

$$\begin{aligned}\mathbf{I}\cdot\ddot{\mathbf{u}}+\mathbf{L}\cdot\mathbf{u} &= \mathbf{i}_{21}(\dot{\mathbf{u}},\dot{\mathbf{u}})+\mathbf{i}_{22}(\mathbf{u},\ddot{\mathbf{u}})+\mathbf{i}_{31}(\mathbf{u},\dot{\mathbf{u}},\dot{\mathbf{u}})\\ &\quad+\mathbf{i}_{32}(\mathbf{u},\mathbf{u},\ddot{\mathbf{u}})+\mathbf{n}_2(\mathbf{u},\mathbf{u})+\mathbf{n}_3(\mathbf{u},\mathbf{u},\mathbf{u}) \qquad (7)\end{aligned}$$

where $\mathbf{u} = [u_1, u_3]^\mathsf{T}$ collects the flapping and longitudinal displacement components; $\mathbf{I}$ and $\mathbf{L}$ are the linear mass and stiffness operators, respectively; $\mathbf{i}_{21}$, $\mathbf{i}_{22}$, and $\mathbf{n}_2$ are the quadratic inertia and stiffness operators, respectively; $\mathbf{i}_{31}$, $\mathbf{i}_{32}$, and $\mathbf{n}_3$ are the cubic inertia and stiffness operators, respectively. The linear mass and stiffness together with the quadratic and cubic inertia and stiffness operators are given in Appendix 1.

While carrying out the perturbation expansion, care must be exercised (see., e.g., [26]) on the fact that the quadratic operators such as $\mathbf{i}_{21}$, etc. are noncommutative operators, that is, $\mathbf{i}_{21}(\dot{\mathbf{u}},\dot{\mathbf{v}}) \neq \mathbf{i}_{21}(\dot{\mathbf{v}},\dot{\mathbf{u}})$. A uniform perturbation expansion of the NNMs is sought by the method of multiple scales [27]. In particular, the main interest is directed toward the analytical construction of the backbone curves of the flapping modes together with their approximations.

To capture the nonlinear influence of longitudinal effects on flapping vibrations, in consonance with [28] the following expansions of $u_1(s,t)$ and $u_3(s,t)$ are considered in terms of a formal perturbation parameter denoted by ε:

$$\begin{aligned}u_1(s,T_o,T_1,T_2) &= \varepsilon u_{1,0}(s,T_o,T_1,T_2)+\varepsilon^2 u_{1,1}(s,T_o,T_1,T_2)\\ &\quad+\varepsilon^3 u_{1,2}(s,T_o,T_1,T_2)+O(\varepsilon^4), \qquad (8)\end{aligned}$$

$$\begin{aligned}u_3(s,T_o,T_1,T_2) &= \varepsilon^2 u_{3,1}(s,T_o,T_1,T_2)\\ &\quad+\varepsilon^3 u_{3,2}(s,T_o,T_1,T_2)+O(\varepsilon^4), \qquad (9)\end{aligned}$$

where $T_o := t$ is the fast time scale, $T_1 := \varepsilon t$ and $T_2 := \varepsilon^2 t$ are the slow time scales introduced to describe the nonlinear modulation of the amplitude and phase of the flapping NNMs. Hence, the first and second time derivatives respectively take the following forms:

$$\frac{d}{dt} = D_o + \varepsilon D_1 + \varepsilon^2 D_2, \qquad (10)$$
$$\frac{d^2}{dt^2} = D_o^2 + 2\varepsilon D_o D_1 + \varepsilon^2\left(2D_o D_2 + D_1^2\right), \qquad (11)$$

where $D_n(\cdot) := \partial/\partial T_n(\cdot)$. Substituting the assumed expansions of the displacements, velocities, and accelerations in the equations of motion (7) and retaining up to third-order terms leads to the following first, second, and third perturbations of the equations governing flapping motions:

$O(\varepsilon)$:

$$I_f[u_{1,0}] + L_f[u_{1,0}] = 0, \qquad (12)$$

$O(\varepsilon^2)$:

$$\begin{aligned}I_f[u_{1,1}] + L_f[u_{1,1}] &= 2D_1 D_o u_{1,0} - 2D_1 D_o \partial_s^2 u_{1,0}/\left(\alpha_2^2 \nu_o^2\right)\\ &\quad+ 4\nu_o' D_1 D_o \partial_s u_{1,0}/\left(\alpha_2^2 \nu_o^3\right), \qquad (13)\end{aligned}$$

$$\begin{aligned}I_a[u_{3,1}] + L_a[u_{3,1}] &= D_o^2 \partial_s u_{1,0} \partial_s^2 u_{1,0}/\left(\alpha_2^2 \nu_o^3\right) - \lambda_a^2 \nu_o' \partial_s u_{1,0}^2/\nu_o^2\\ &\quad+ {\lambda_b^{(2)}}^2 \nu_o'' \partial_s u_{1,0} \partial_s^2 u_{1,0}/\nu_o^4\end{aligned}$$

$$+ D_\mathrm{o}^2 u_{1,0}\partial_s u_{1,0}/\nu_\mathrm{o} + 2{\lambda_\mathrm{b}^{(2)}}^2 \nu_\mathrm{o}'\partial_s^2 u_{1,0}^2/\nu_\mathrm{o}^4$$
$$+ 2{\lambda_\mathrm{b}^{(2)}}^2 \nu_\mathrm{o}'^3\partial_s u_{1,0}^2/\nu_\mathrm{o}^6 - 2\lambda_\mathrm{a}^2\partial_s u_{1,0}\partial_s^2 u_{1,0}/\nu_\mathrm{o}^2$$
$$- {\lambda_\mathrm{b}^{(2)}}^2\partial_s^3 u_{1,0}\partial_s^2 u_{1,0}/\nu_\mathrm{o}^3 - {\lambda_\mathrm{b}^{(2)}}^2 \nu_\mathrm{o}''\nu_\mathrm{o}'\partial_s u_{1,0}^2/\nu_\mathrm{o}^5$$
$$+ \lambda_\mathrm{a}^2\partial_s u_{1,0}\partial_s^2 u_{1,0}/\nu_\mathrm{o}$$
$$- \bar{\omega}_\mathrm{R}^2\partial_s u_{1,0}\partial_s^2 u_{1,0}/\left(\alpha_2^2\nu_\mathrm{o}^3\right)$$
$$- 4{\lambda_\mathrm{b}^{(2)}}^2 \nu_\mathrm{o}'^2\partial_s u_{1,0}\partial_s^2 u_{1,0}/\nu_\mathrm{o}^5$$
$$+ 2\lambda_\mathrm{a}^2\nu_\mathrm{o}'\partial_s u_{1,0}^2/\nu_\mathrm{o}^3$$
$$+ {\lambda_\mathrm{b}^{(2)}}^2 \nu_\mathrm{o}'\partial_s^3 u_{1,0}\partial_s u_{1,0}/\nu_\mathrm{o}^4$$
$$+ \bar{\omega}_\mathrm{R}^2\nu_\mathrm{o}'\partial_s u_{1,0}^2/\left(\alpha_2^2\nu_\mathrm{o}^4\right)$$
$$- \nu_\mathrm{o}' D_\mathrm{o}^2\partial_s u_{1,0}\partial_s u_{1,0}/\left(\alpha_2^2\nu_\mathrm{o}^4\right), \tag{14}$$

$O(\varepsilon^3)$:

$$I_f[u_{1,2}] + L_f[u_{1,2}]$$
$$= \partial_s u_{1,0} D_\mathrm{o}^2\partial_s^2 u_{3,1}/\left(\alpha_2^2\nu_\mathrm{o}^3\right)$$
$$- 2D_2 D_\mathrm{o}\partial_s^2 u_{1,0}/\left(\alpha_2^2\nu_\mathrm{o}^2\right) - 2{\lambda_\mathrm{b}^{(2)}}^2\partial_s^2 u_{1,0}^3/\nu_\mathrm{o}^4$$
$$- D_1^2\partial_s^2 u_{1,0}/\left(\alpha_2^2\nu_\mathrm{o}^2\right) - {\lambda_\mathrm{b}^{(2)}}^2\partial_s u_{1,0}\partial_s^4 u_{3,1}/\nu_\mathrm{o}^3$$
$$- 2\bar{\omega}_\mathrm{R}^2\partial_s^2 u_{1,0}\partial_s u_{3,1}/\left(\alpha_2^2\nu_\mathrm{o}^3\right)$$
$$- 2{\lambda_\mathrm{b}^{(2)}}^2\partial_s^4 u_{1,0}\partial_s u_{3,1}/\nu_\mathrm{o}^3 + 2D_1 D_\mathrm{o} u_{1,1}$$
$$+ 2D_\mathrm{o}^2\partial_s u_{1,0}\partial_s^2 u_{3,1}/\left(\alpha_2^2\nu_\mathrm{o}^3\right)$$
$$+ 2D_\mathrm{o}^2\partial_s^2 u_{1,0}\partial_s u_{3,1}/\left(\alpha_2^2\nu_\mathrm{o}^3\right)$$
$$- 8\nu_\mathrm{o}' D_\mathrm{o}\partial_s u_{1,0}^2\partial_s u_{1,0}/\left(\alpha_2^2\nu_\mathrm{o}^5\right)$$
$$+ 13{\lambda_\mathrm{b}^{(2)}}^2 \nu_\mathrm{o}'\partial_s u_{1,0}^2\partial_s^3 u_{1,0}/\nu_\mathrm{o}^5$$
$$- 3\nu_\mathrm{o}'\partial_s u_{1,0} D_\mathrm{o}^2\partial_s u_{3,1}/\left(\alpha_2^2\nu_\mathrm{o}^4\right)$$
$$- 24{\lambda_\mathrm{b}^{(2)}}^2 \nu_\mathrm{o}'^2\partial_s u_{1,0}\partial_s^2 u_{3,1}/\nu_\mathrm{o}^5$$
$$+ 7{\lambda_\mathrm{b}^{(2)}}^2 \nu_\mathrm{o}''\partial_s u_{1,0}\partial_s^2 u_{3,1}/\nu_\mathrm{o}^4$$
$$- (9/2)\bar{\omega}_\mathrm{R}^2\partial_s u_{1,0}^2\partial_s^2 u_{1,0}/\left(\alpha_2^2\nu_\mathrm{o}^4\right)$$
$$+ (3/2)\lambda_\mathrm{a}^2\partial_s u_{1,0}^2\partial_s^2 u_{1,0}/\nu_\mathrm{o}^2$$
$$- (3/2){\lambda_\mathrm{b}^{(2)}}^2\partial_s u_{1,0}^2\partial_s^4 u_{1,0}/\nu_\mathrm{o}^4$$
$$+ (3/2)D_\mathrm{o}^2\partial_s^2 u_{1,0}\partial_s u_{1,0}^2/\left(\alpha_2^2\nu_\mathrm{o}^4\right)$$
$$- 4{\lambda_\mathrm{b}^{(2)}}^2\partial_s^3 u_{1,0}\partial_s^2 u_{3,1}/\nu_\mathrm{o}^3 - 2\lambda_\mathrm{a}^2\partial_s u_{1,0}^2\partial_s^2 u_{1,0}/\nu_\mathrm{o}^3$$
$$+ (3/2){\lambda_\mathrm{b}^{(2)}}^2 \nu_\mathrm{o}'''\partial_s u_{1,0}^3/\nu_\mathrm{o}^5$$
$$+ 4\nu_\mathrm{o}' D_\mathrm{o} D_2\partial_s u_{1,0}/\left(\alpha_2^2\nu_\mathrm{o}^3\right)$$
$$+ 4\nu_\mathrm{o}' D_1 D_\mathrm{o}\partial_s u_{1,1}/\left(\alpha_2^2\nu_\mathrm{o}^3\right)$$
$$+ 2\nu_\mathrm{o}' D_1^2\partial_s u_{1,0}/\left(\alpha_2^2\nu_\mathrm{o}^3\right)$$
$$+ 7{\lambda_\mathrm{b}^{(2)}}^2 \nu_\mathrm{o}'\partial_s u_{1,0}\partial_s^3 u_{3,1}/\nu_\mathrm{o}^4$$
$$- 32{\lambda_\mathrm{b}^{(2)}}^2 \nu_\mathrm{o}'^2\partial_s^2 u_{1,0}\partial_s u_{3,1}/\nu_\mathrm{o}^5$$
$$- \lambda_\mathrm{a}^2\nu_\mathrm{o}'\partial_s u_{1,0}\partial_s u_{3,1}/\nu_\mathrm{o}^2$$
$$+ 40{\lambda_\mathrm{b}^{(2)}}^2 \nu_\mathrm{o}'^3\partial_s u_{1,0}\partial_s u_{3,1}/\nu_\mathrm{o}^6$$
$$- 6\nu_\mathrm{o}' D_\mathrm{o}^2\partial_s u_{1,0}\partial_s u_{1,0}^2/\left(\alpha_2^2\nu_\mathrm{o}^5\right)$$
$$- 28{\lambda_\mathrm{b}^{(2)}}^2 \nu_\mathrm{o}'\nu_\mathrm{o}''\partial_s u_{1,0}\partial_s u_{3,1}/\nu_\mathrm{o}^5$$
$$+ 12{\lambda_\mathrm{b}^{(2)}}^2 \nu_\mathrm{o}'\partial_s u_{3,1}\partial_s^3 u_{1,0}/\nu_\mathrm{o}^4$$
$$+ 16{\lambda_\mathrm{b}^{(2)}}^2 \nu_\mathrm{o}'\partial_s^2 u_{1,0}\partial_s^2 u_{3,1}/\nu_\mathrm{o}^4$$
$$- 2\bar{\omega}_\mathrm{R}^2\partial_s u_{1,0}\partial_s^2 u_{3,1}/\left(\alpha_2^2\nu_\mathrm{o}^3\right)$$
$$- 3{\lambda_\mathrm{b}^{(2)}}^2\partial_s^2 u_{1,0}\partial_s^3 u_{3,1}/\nu_\mathrm{o}^3$$
$$+ 3\lambda_\mathrm{a}^2\nu_\mathrm{o}'\partial_s u_{1,0}\partial_s u_{3,1}/\nu_\mathrm{o}^3$$
$$+ 2D_\mathrm{o}\partial_s u_{1,0} D_\mathrm{o}\partial_s^2 u_{3,1}/\left(\alpha_2^2\nu_\mathrm{o}^3\right)$$
$$- 2\lambda_\mathrm{a}^2\partial_s^2 u_{1,0}\partial_s u_{3,1}/\nu_\mathrm{o}^2$$
$$- 2D_1 D_\mathrm{o}\partial_s^2 u_{1,1}/\left(\alpha_2^2\nu_\mathrm{o}^2\right)$$
$$- 50{\lambda_\mathrm{b}^{(2)}}^2 \nu_\mathrm{o}'^2\partial_s u_{1,0}^2\partial_s^2 u_{1,0}/\nu_\mathrm{o}^6$$
$$- 7{\lambda_\mathrm{b}^{(2)}}^2\partial_s u_{1,0}\partial_s^3 u_{1,0}\partial_s^2 u_{1,0}/\nu_\mathrm{o}^4$$
$$+ 2D_\mathrm{o}\partial_s u_{1,0}^2\partial_s^2 u_{1,0}/\left(\alpha_2^2\nu_\mathrm{o}^4\right)$$
$$+ 4D_\mathrm{o}\partial_s u_{1,0}\partial_s u_{1,0} D_\mathrm{o}\partial_s^2 u_{1,0}/\left(\alpha_2^2\nu_\mathrm{o}^4\right)$$
$$+ \bar{\omega}_\mathrm{R}^2 u_{3,1}\partial_s u_{1,0}/\nu_\mathrm{o} + 2\lambda_\mathrm{a}^2\nu_\mathrm{o}'\partial_s u_{1,0}^3/\nu_\mathrm{o}^4$$
$$- \lambda_\mathrm{a}^2\nu_\mathrm{o}'\partial_s u_{1,0}^3/\nu_\mathrm{o}^3 + 6\bar{\omega}_\mathrm{R}^2\nu_\mathrm{o}'\partial_s u_{1,0}^3/\left(\alpha_2^2\nu_\mathrm{o}^5\right)$$
$$+ 2D_\mathrm{o}\partial_s^2 u_{1,0} D_\mathrm{o}\partial_s u_{3,1}/\left(\alpha_2^2\nu_\mathrm{o}^3\right)$$
$$+ 3{\lambda_\mathrm{b}^{(2)}}^2 \nu_\mathrm{o}'''\partial_s u_{1,0}\partial_s u_{3,1}/\nu_\mathrm{o}^4$$
$$+ 22{\lambda_\mathrm{b}^{(2)}}^2 \nu_\mathrm{o}'\partial_s u_{1,0}\partial_s^2 u_{1,0}^2/\nu_\mathrm{o}^5$$
$$- D_\mathrm{o}^2 u_{3,1}\partial_s u_{1,0}/\nu_\mathrm{o}$$
$$- 1/2 D_\mathrm{o}^2 u_{1,0}\partial_s u_{1,0}^2/\nu_\mathrm{o}^2 + 30{\lambda_\mathrm{b}^{(2)}}^2 \nu_\mathrm{o}'^3\partial_s u_{1,0}^3/\nu_\mathrm{o}^7$$

$$- 6\nu_{\rm o}' D_{\rm o} \partial_s u_{1,0} D_{\rm o} \partial_s u_{3,1} / \big(\alpha_2^2 \nu_{\rm o}^4\big)$$
$$+ (23/2) \lambda_{\rm b}^{(2)^2} \nu_{\rm o}'' \partial_s u_{1,0}^2 \partial_s^2 u_{1,0} / \nu_{\rm o}^5$$
$$- \lambda_{\rm a}^2 \partial_s u_{1,0} \partial_s^2 u_{3,1} / \nu_{\rm o}^2$$
$$+ \partial_s^2 u_{1,0} D_{\rm o}^2 \partial_s u_{3,1} / \big(\alpha_2^2 \nu_{\rm o}^3\big)$$
$$+ \lambda_{\rm a}^2 \partial_s^2 u_{1,0} \partial_s u_{3,1} / \nu_{\rm o}$$
$$+ 3 D_{\rm o}^2 \partial_s u_{1,0} \partial_s u_{1,0} \partial_s^2 u_{1,0} / \big(\alpha_2^2 \nu_{\rm o}^4\big)$$
$$+ D_1^2 u_{1,0} + 9 \lambda_{\rm b}^{(2)^2} \nu_{\rm o}'' \partial_s^2 u_{1,0} \partial_s u_{3,1} / \nu_{\rm o}^4$$
$$+ \lambda_{\rm a}^2 \partial_s u_{1,0} \partial_s^2 u_{3,1} / \nu_{\rm o}$$
$$- 6\nu_{\rm o}' D_{\rm o}^2 \partial_s u_{1,0} \partial_s u_{3,1} / \big(\alpha_2^2 \nu_{\rm o}^4\big)$$
$$+ 6 \bar{\omega}_{\rm R}^2 \nu_{\rm o}' \partial_s u_{1,0} \partial_s u_{3,1} / \big(\alpha_2^2 \nu_{\rm o}^4\big)$$
$$- (35/2) \lambda_{\rm b}^{(2)^2} \nu_{\rm o}'' \nu_{\rm o}' \partial_s u_{1,0}^3 / \nu_{\rm o}^6 + 2 D_2 D_{\rm o} u_{1,0}, \qquad (15)$$

where the subscripts f and a denote the flapping and axial components, respectively, $(\cdot)'$ denotes differentiation with respect to s, and the linear inertia and stiffness operators governing linear flapping and axial vibrations are given by

$$I_f[\cdot] := - D_{\rm o}^2(\cdot) + D_{\rm o}^2 \partial_s^2(\cdot) / \big(\alpha_2^2 \nu_{\rm o}^2\big) - 2\nu_{\rm o}' D_{\rm o}^2 \partial_s(\cdot) / \big(\alpha_2^2 \nu_{\rm o}^3\big),$$
$$L_f[\cdot] := - \lambda_{\rm a}^2 \partial_s^2(\cdot)/\nu_{\rm o} + 2\bar{\omega}_{\rm R}^2 \nu_{\rm o}' \partial_s(\cdot) / \big(\alpha_2^2 \nu_{\rm o}^3\big) - \bar{\omega}_{\rm R}^2 \partial_s^2(\cdot) / \big(\alpha_2^2 \nu_{\rm o}^2\big) + 3 \lambda_{\rm b}^{(2)^2} \nu_{\rm o}'' \partial_s^2(\cdot)/\nu_{\rm o}^3 + 4 \lambda_{\rm b}^{(2)^2} \nu_{\rm o}' \partial_s^3(\cdot)/\nu_{\rm o}^3 - 7 \lambda_{\rm b}^{(2)^2} \nu_{\rm o}'' \nu_{\rm o}' \partial_s(\cdot)/\nu_{\rm o}^4 + 8 \lambda_{\rm b}^{(2)^2} \nu_{\rm o}'^3 \partial_s(\cdot)/\nu_{\rm o}^5 - 8 \lambda_{\rm b}^{(2)^2} \nu_{\rm o}'^2 \partial_s^2(\cdot)/\nu_{\rm o}^4 - \lambda_{\rm b}^{(2)^2} \partial_s^4(\cdot)/\nu_{\rm o}^2 + \lambda_{\rm a}^2 \partial_s^2(\cdot) + \lambda_{\rm a}^2 \nu_{\rm o}' \partial_s(\cdot)/\nu_{\rm o}^2 + \lambda_{\rm b}^{(2)^2} \nu_{\rm o}''' \partial_s(\cdot)/\nu_{\rm o}^3,$$
$$I_a[\cdot] := - D_{\rm o}^2(\cdot),$$
$$L_a[\cdot] := \bar{\omega}_{\rm R}^2(\cdot) + \lambda_{\rm a}^2 \partial_s^2(\cdot)/\nu_{\rm o} - \lambda_{\rm a}^2 \nu_{\rm o}' \partial_s(\cdot)/\nu_{\rm o}^2. \qquad (16)$$

For simplicity, the explicit appearance of the arguments $(s, T_{\rm o}, T_1, T_2)$ is, here and henceforth, omitted. The first-order flapping equation, given by Eq. (12), admits the generating solution $u_{1,0}$ as

$$u_{1,0} = \psi_{1,{\rm k}}(s) A_{\rm k}(T_1, T_2) e^{i\omega_{\rm k} T_{\rm o}} + cc, \qquad (17)$$

where $\psi_{1,{\rm k}}(s)$ is the kth linear flapping mode of the rotating beam, $\omega_{\rm k}$ is the associated frequency, $A_{\rm k}(T_1, T_2)$ is the complex-valued amplitude, and cc stands for the complex conjugate of the preceding terms. The basis of linear normal modes of the rotating beam, denoted by $\psi_{1,{\rm k}}(s)$ and used to discretize $u_{1,0}$, has been obtained using the Galerkin method in [18] as a combination of linear normal modes of the nonrotating beam. Substituting $u_{1,0}$ in the right-hand side of the second-order flapping equation, given by Eq. (13), yields the following right-hand side:

$$2i\omega_{\rm k} D_1 A_{\rm k}(T_1, T_2) e^{i\omega_{\rm k} T_{\rm o}} \big[-\psi_{1,{\rm k}}''(s) \nu_{\rm o}(s) + \psi_{1,{\rm k}}(s) \alpha_2^2 \nu_{\rm o}^3(s) + 2\nu_{\rm o}'(s) \psi_{1,{\rm k}}'(s)\big] / \big[\alpha_2^2 \nu_{\rm o}(s)^3\big] + cc. \qquad (18)$$

The solvability condition via Fredholm's alternative theorem applied to this equation leads to $D_1 A_{\rm k}(T_1, T_2) = 0$ which implies that A_k is function of the time scale T_2 only and that the solution at this order is trivial, $u_{1,1} = 0$.

On the other hand, substituting $u_{1,0}$ in the second-order equation governing axial (slaved) motions, given by Eq. (14), yields

$O(\varepsilon^2)$:

$$I_a[u_{3,1}] + L_a[u_{3,1}] = -\big[-\bar{\omega}_{\rm R}^2 \nu_{\rm o}' \psi_{1,{\rm k}}'(s)^2 \nu_{\rm o}^2/\alpha_2^2 - \nu_{\rm o}' \psi_{1,{\rm k}}'(s)^2 \omega_{\rm k}^2 \nu_{\rm o}^2/\alpha_2^2 + \psi_{1,{\rm k}}(s) \omega_{\rm k}^2 \psi_{1,{\rm k}}'(s) \nu_{\rm o}^5 + \lambda_{\rm a}^2 \nu_{\rm o}' \psi_{1,{\rm k}}'(s)^2 \nu_{\rm o}^4 - 2\lambda_{\rm b}^{(2)^2} \nu_{\rm o}'^3 \psi_{1,{\rm k}}'(s)^2 - \lambda_{\rm a}^2 \psi_{1,{\rm k}}'(s) \psi_{1,{\rm k}}''(s) \nu_{\rm o}^5 + 2\lambda_{\rm a}^2 \psi_{1,{\rm k}}'(s) \psi_{1,{\rm k}}''(s) \nu_{\rm o}^4 - 2\lambda_{\rm b}^{(2)^2} \nu_{\rm o}' \psi_{1,{\rm k}}''(s)^2 \nu_{\rm o}^2 - 2\lambda_{\rm a}^2 \nu_{\rm o}' \psi_{1,{\rm k}}'(s)^2 \nu_{\rm o}^3 + \lambda_{\rm b}^{(2)^2} \psi_{1,{\rm k}}'''(s) \psi_{1,{\rm k}}''(s) \nu_{\rm o}^3 + \psi_{1,{\rm k}}'(s) \omega_{\rm k}^2 \psi_{1,{\rm k}}''(s) \nu_{\rm o}^3/\alpha_2^2 + \bar{\omega}_{\rm R}^2 \psi_{1,{\rm k}}'(s) \psi_{1,{\rm k}}''(s) \nu_{\rm o}^3/\alpha_2^2 + \lambda_{\rm b}^{(2)^2} \nu_{\rm o}'' \nu_{\rm o}' \psi_{1,{\rm k}}'(s)^2 \nu_{\rm o} - \lambda_{\rm b}^{(2)^2} \nu_{\rm o}'' \psi_{1,{\rm k}}'(s) \psi_{1,{\rm k}}''(s) \nu_{\rm o}^2 - \lambda_{\rm b}^{(2)^2} \nu_{\rm o}' \psi_{1,{\rm k}}'''(s) \psi_{1,{\rm k}}'(s) \nu_{\rm o}^2$$

$$+ 4\lambda_b^{(2)^2} \nu_o'^2 \psi_{1,k}'(s)\psi_{1,k}''(s)\nu_o)\big]/\nu_o^6 e^{2i\omega_k T_o} A_k^2$$
$$- \big[2\lambda_a^2 \nu_o' \psi_{1,k}'(s)^2 \nu_o^4 + 2\lambda_b^{(2)^2} \nu_o'' \nu_o' \psi_{1,k}'(s)^2 \nu_o$$
$$+ 8\lambda_b^{(2)^2} \nu_o'^2 \psi_{1,k}'(s)\psi_{1,k}''(s)\nu_o$$
$$- 2\bar{\omega}_R^2 \nu_o' \psi_{1,k}'(s)^2 \nu_o^2/\alpha_2^2 - 4\lambda_b^{(2)^2} \nu_o'^3 \psi_{1,k}'(s)^2$$
$$+ 4\lambda_a^2 \psi_{1,k}'(s)\psi_{1,k}''(s)\nu_o^4$$
$$+ 2\bar{\omega}_R^2 \psi_{1,k}'(s)\psi_{1,k}''(s)\nu_o^3/\alpha_2^2$$
$$+ 2\psi_{1,k}(s)\omega_k^2 \psi_{1,k}'(s)\nu_o^5 - 2\lambda_a^2 \psi_{1,k}'(s)\psi_{1,k}''(s)\nu_o^5$$
$$+ 2\psi_{1,k}'(s)\omega_k^2 \psi_{1,k}''(s)\nu_o^3/\alpha_2^2$$
$$- 2\nu_o' \psi_{1,k}'(s)^2 \omega_k^2 \nu_o^2/\alpha_2^2 - 4\lambda_a^2 \nu_o' \psi_{1,k}'(s)^2 \nu_o^3$$
$$- 2\lambda_b^{(2)^2} \nu_o' \psi_{1,k}'''(s)\psi_{1,k}'(s)\nu_o^2$$
$$- 2\lambda_b^{(2)^2} \nu_o'' \psi_{1,k}'(s)\psi_{1,k}''(s)\nu_o^2$$
$$+ 2\lambda_b^{(2)^2} \psi_{1,k}'''(s)\psi_{1,k}''(s)\nu_o^3$$
$$- 4\lambda_b^{(2)^2} \nu_o' \psi_{1,k}''(s)^2 \nu_o^2\big]/\big(2\nu_o^6\big) A_k \bar{A}_k + cc, \quad (19)$$

where the overbar indicates the complex conjugate. The right-hand side of Eq. (19) suggests the following form for the second-order solution expressing the flapping-induced longitudinal motion:

$$u_{3,1} = h_{31}(s) A_k^2(T_2) e^{2i\omega_k T_o} + h_{32}(s) A_k(T_2)\bar{A}_k(T_2) + cc. \quad (20)$$

Substituting Eq. (20) into Eq. (19) leads to the following two two-point boundary-value problems for h_{31} and h_{32}:

$$h_{31}'' - \nu_o'(s)h_{31}'/\nu_o(s) + \big(4\nu_o(s)\omega_k^2/\lambda_a^2 + \nu_o(s)\bar{\omega}_R^2/\lambda_a^2\big)h_{31}(s) + f(s) = 0, \quad (21)$$

$$h_{32}'' - \nu_o'(s)h_{32}'/\nu_o(s) + \nu_o(s)\bar{\omega}_R^2 h_{32}(s)/\lambda_a^2 + f(s) = 0 \quad (22)$$

with the pertinent boundary conditions and the expression of $f(s)$ given in Appendix 2. The functions $h_{31}(s)$ and $h_{32}(s)$ have been found as solutions of two-point boundary-value problems using the Galerkin method based on the linear normal modes of the rotating beam as trial functions.

Substituting $u_{1,0}$ and $u_{3,1}$ in the right-hand side of the third-order flapping equation, given by Eq. (15), delivers a lengthy expression which is reported in Appendix 2. The solvability condition for this equation (see Eq. (15)) reads

$$\int_0^1 \big[\psi_{1,j}(s) \cdot C_1(s, T_1, T_2)\big] ds = 0, \quad (23)$$

where C_1 is the coefficient of $e^{i\omega_k T_o}$ in the right-hand side of the equation at $O(\varepsilon^3)$, which corresponds to Eq. (77). This condition turns out to be

$$A_k'(T_2) = 1/4i\, \Gamma_{k,k} A_k^2(T_2)\bar{A}_k(T_2), \quad (24)$$

where $\Gamma_{k,k}$ is the effective nonlinearity coefficient given in Appendix 3. The integral equations of Appendix 3 are numerically computed using Simpson's rule.

By substituting

$$A_k(T_2) = 1/2 a_k(T_2) e^{i\beta_k(T_2)} \quad (25)$$

into Eq. (24), the time evolution of a_k and β_k on the times scale T_2 is found in the form:

$$a_k(T_2) = a_k^o, \quad \beta_k(T_2) = \beta_k^{NL} a_k^2 T_2 + \beta_k^o, \quad (26)$$

where a_k^o and β_k^o are defined from the initial conditions and $\beta_k^{NL} = (1/16)\Gamma_{k,k}$. Hence, the second-order approximation of the nonlinear frequency of the kth flapping mode is expressed as

$$\omega_k^{NL} = \omega_k + \beta_k^{NL} a_k^2. \quad (27)$$

By substituting Eq. (24) into the right-hand side of the equation at third order, given by Eq. (77), it turns out that $u_{1,2}$ has to be in the form:

$$u_{1,2} = h_{23}(s) A_k^3 e^{3i\omega_k T_o} + h_{21}(s) A_k^2 \bar{A}_k e^{i\omega_k T_o} + cc. \quad (28)$$

Putting $u_{1,2}$ in the third-order flapping equation, represented by Eq. (15), yields two two-point boundary-value problems governing $h_{21}(s)$ and $h_{23}(s)$ (see Appendix 4). The functions $h_{21}(s)$ and $h_{23}(s)$ have been found as solutions of the two-point boundary-value problems using the Galerkin method and the linear normal modes of the rotating beam as trial functions. The approximation of the flapping deflection associated to the kth NNM can be obtained by putting together $u_{1,0}$, $u_{1,1}$, and $u_{1,2}$ in Eq. (8), putting $T_o = t$, $T_2 = \varepsilon^2 t$, and setting $\varepsilon = 1$. The ensuing flapping deflection is

$$
\begin{aligned}
&u_1(s,t)\\
&\quad = a_k \cos\left[\left(\omega_k + \beta_k^{NL} a_k^2\right)t + \beta_k^o\right]\psi_{1,k}(s)\\
&\qquad + 1/4 a_k^3 \cos\left[3\left(\omega_k + \beta_k^{NL} a_k^2\right)t + 3\beta_k^o\right] h_{23}(s)\\
&\qquad + 1/4 a_k^3 \cos\left[\left(\omega_k + \beta_k^{NL} a_k^2\right)t + \beta_k^o\right] h_{21}(s). \qquad (29)
\end{aligned}
$$

4 Softening vs. hardening nonlinearity of the flapping modes

In this section, the results obtained by the present perturbation treatment for the backbone curves of the lowest three flapping modes are compared with those obtained in [17]. Therein the blade mechanical model was based on the von Karman strain-displacement relationships (i.e., second-order truncation for the stretch) as the only source of geometric nonlinearity. In [17], the method of multiple scales was applied to the Galerkin-reduced discretized version of the equations of motion to construct the NNMs.

The beam geometric, material properties and the angular speed reported in [17] are shown in Table 1. The backbone curves of the lowest three flapping modes are shown in Figs. 1, 2, 3. A good qualitative and quantitative agreement is found. The obtained linear frequencies of the lowest three flapping modes are lower than those obtained in [17] because the current mechanical model is more relaxed, and thus exhibits higher flexibility. Both methods predict the same type of nonlinearity for the considered angular speeds, i.e., a softening behavior for the lowest flapping mode and a hardening behavior for the second and third flapping modes. Moreover, another systematic feature of the backbones obtained by the present treatment of the *geometrically exact* equations of motion for unshearable blades is the lower bending with respect to the truncated *ad hoc* model in [17] which incorporates linear flexural curvature and higher nonlinear inertia (softening) contribution.

The dependence of the effective nonlinearity coefficients associated with the lowest three flapping modes on the angular speed is shown in Figs. 4, 5, 6. In all of these plots, there are singularities to infinity of the coefficients at certain angular speeds whenever the considered flapping mode is involved in a 1:2 internal resonance with one of the axial modes (see, e.g., [17, 29, 30]). In the neighborhood of the singularity, the obtained approximation of the NNMs breaks down. The expansion should be constructed differently consider-

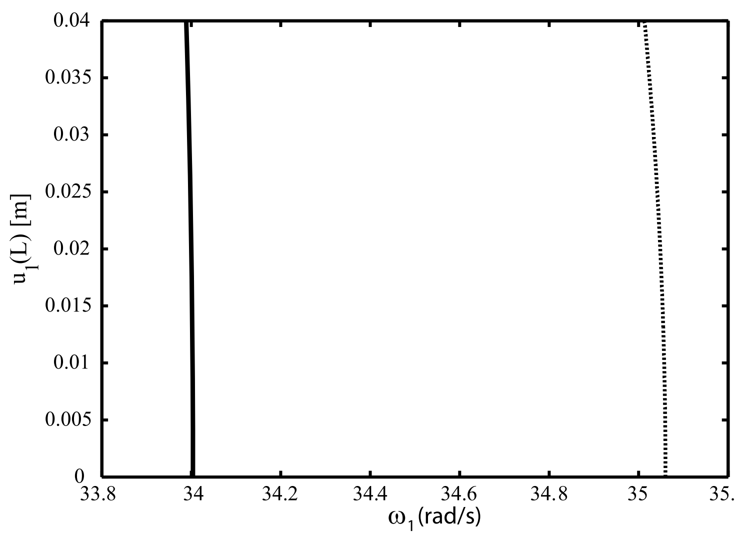

Fig. 1 Backbone curves of the lowest flapping mode calculated via the current geometrically exact approach (*solid lines*) and according to [17] (*dashed lines*)

Table 1 Geometric and Material Properties from [17]

Mass per unit length (kg/m)	Axial Stiffness (N)	Flexural Stiffness (N m^2)	Length (m)	Rotor Radius (m)	Angular Speed (rad/s)
10	2.23×10^8	3.99×10^5	9	0.5	30

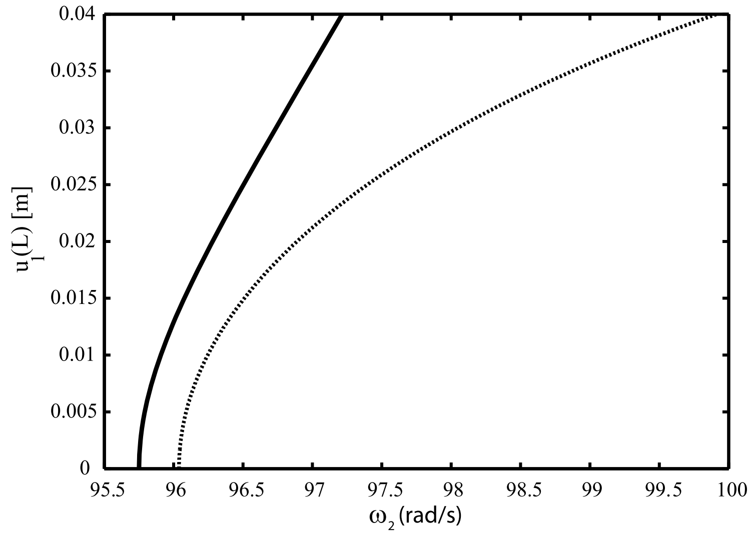

Fig. 2 Backbone curves of the second flapping mode calculated via the current geometrically exact approach (*solid lines*) and according to [17] (*dashed lines*)

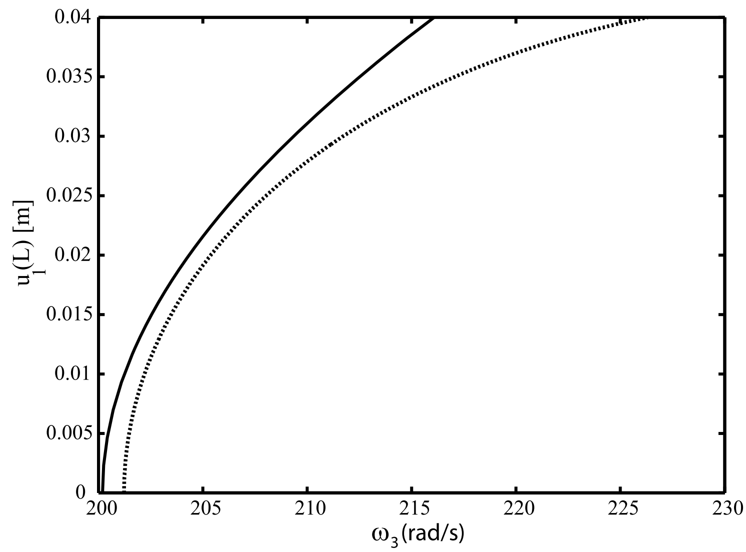

Fig. 3 Backbone curves of the third flapping mode calculated via the current geometrically exact approach (*solid lines*) and according to [17] (*dashed lines*)

ing the linear modes involved in the internal resonance in the generating solution as shown in [29] in the study of NNMs of nonshallow suspended cables. To highlight the fact that there is a region close to the singularity where the current solution is no longer valid, a shaded grey area centered about the singularity has been superimposed. The first effective nonlinearity in Fig. 4 exhibits a singularity at about 3,387 rpm where a 1:2 internal resonance between the lowest flapping mode and the lowest axial mode occurs. While the softening character is more pronounced at low speeds, it decreases significantly with increasing the speed up to 1,349 rpm. Above this speed, the mode becomes hardening and after the singularity it switches to softening. The width of the region where the effective nonlinearity calculated with one mode in the generating solution breaks down cannot be ascertained a priori. Therefore, the shaded areas highlight the fact that there are uncertain ranges of angular speeds in the neighborhood of the singularities where the effective nonlinearity coefficients cannot be predicted by the present analysis.

The second flapping mode in Fig. 5 exhibits two singularities at 1,481 rpm and 4,336 rpm, respectively. At the first singularity, a 1:2 internal resonance between the second flapping mode and the lowest axial

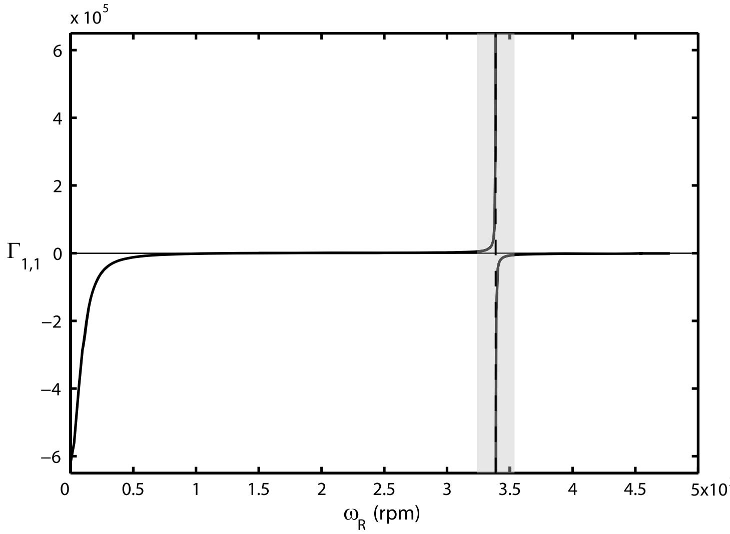

Fig. 4 The effective nonlinearity coefficient of the first flapping mode $\Gamma_{1,1}$ vs. the angular speed ω_{R}

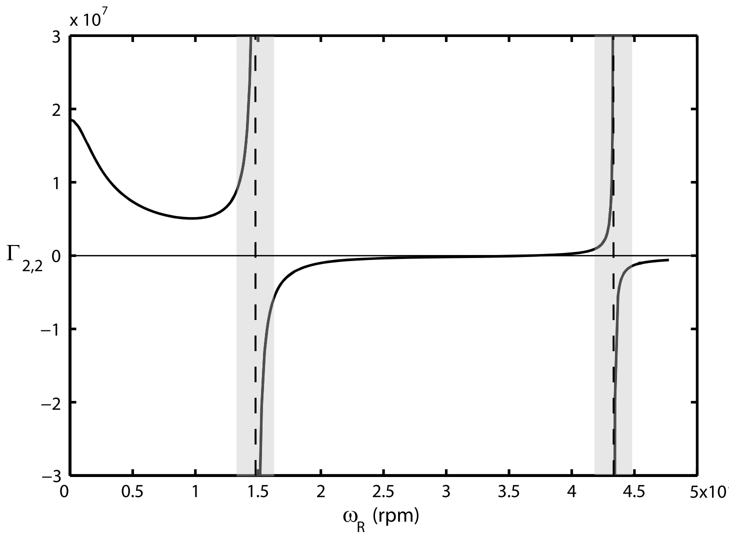

Fig. 5 The effective nonlinearity coefficient of the second flapping mode $\Gamma_{2,2}$ vs. the angular speed ω_{R}

mode occurs. On the other hand, at the second singularity (at 4,336 rpm), the second flapping mode and the second axial mode are in 1:2 internal resonance. The second flapping mode has a hardening nonlinearity until the first singularity, past which it becomes softening until 3,677 rpm. Past this angular speed, it becomes hard until the second singularity, past which it switches to softening again.

The third effective nonlinearity coefficient shown in Fig. 6 exhibits three singularities at 861, 2,798, and 4,553 rpm. Differently from the first and second effective nonlinearity coefficients, it is invariant under the variation of the angular speed, i.e., it remains always hardening except near the singularities. At the three singularities, 1:2 internal resonances occur between the third flapping mode and the lowest axial mode at 861 rpm, the third flapping mode and the second axial mode at 2,798 rpm, the third flapping mode and the third axial mode at 4,553 rpm.

To appreciate the effects of the nonlinearities on the shape of the NNMs in terms of spatial nonlinear corrections (spatial over-tones), the deflection profiles of the lowest three flapping NNMs are calculated at time $t = 0$ and $\omega_{\mathrm{R}} = 30$ rad/s for increasing amplitudes a_k. The results are shown in Figs. 7, 8, 9 in comparison with the corresponding linear normal modes. It is shown that with increasing the amplitude a_{k} the shape of the NNMs deviates from the corresponding linear

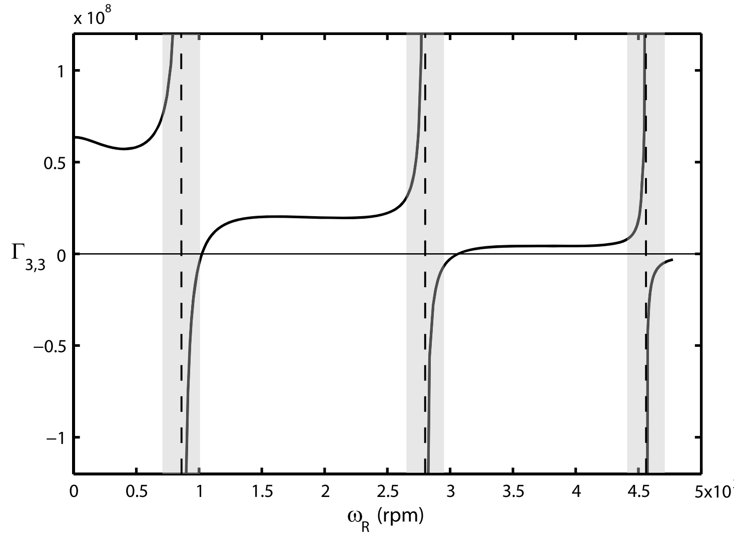

Fig. 6 The effective nonlinearity coefficient of the third flapping mode $\Gamma_{3,3}$ vs. the angular speed ω_R

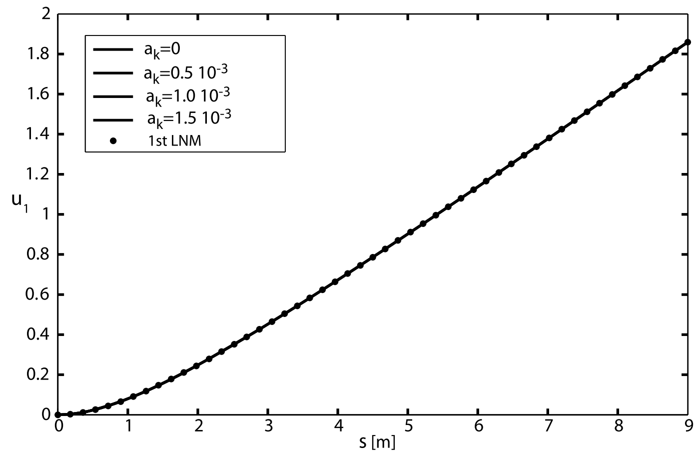

Fig. 7 Deflection profiles of the lowest flapping NNM for different amplitudes a_k, compared with the linear normal mode (*filled circles*)

normal modes due to the nonlinear space-wise distortions at third order which are captured by the functions $h_{23}(s)$ and $h_{21}(s)$, solutions of the stated two-point boundary-value problems.

A series of snapshots of the deflections of the mentioned flapping modes (for $a_k = 1.5 \times 10^{-3}$) during one-half of their nonlinear periods T_{NL} are presented in Figs. 10, 11, 12.

5 Conclusions

The third-order Taylor expansion of the geometrically exact equations of motion about the prestressed state of the blades under centrifugal forces was treated by a perturbation method to unfold the nonlinear features of the individual flapping modes away from internal resonances.

The geometrically exact equations, obtained within the context of an updated Lagrangian formulation in the companion paper [18], can describe overall motions of elastic isotropic blades of arbitrary cross sections. The equations include all geometric terms without any restriction on the geometry of deformation. This is a substantial aspect of the present contribution since the obtained equations of motion lend themselves to global studies about the overall blade dynamics arising from instabilities and resonances. The perturbation approach applied directly to the geometrically exact model overcomes the limitations of truncated models based on *ad hoc* kinematic assumptions.

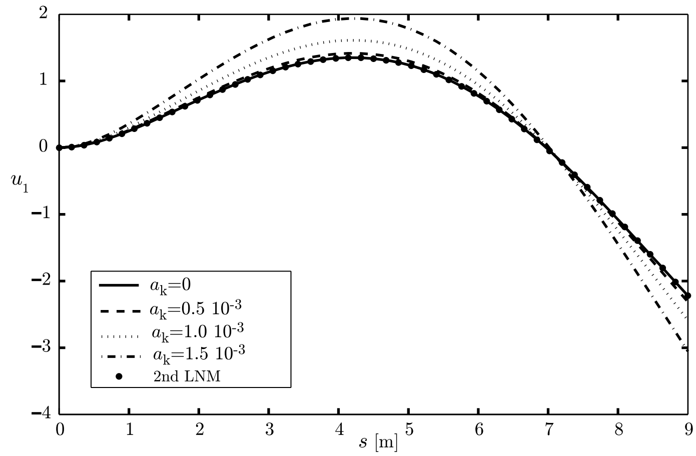

Fig. 8 Deflection profiles of the second flapping NNM for different amplitudes a_k, compared with the linear normal mode (*filled circles*)

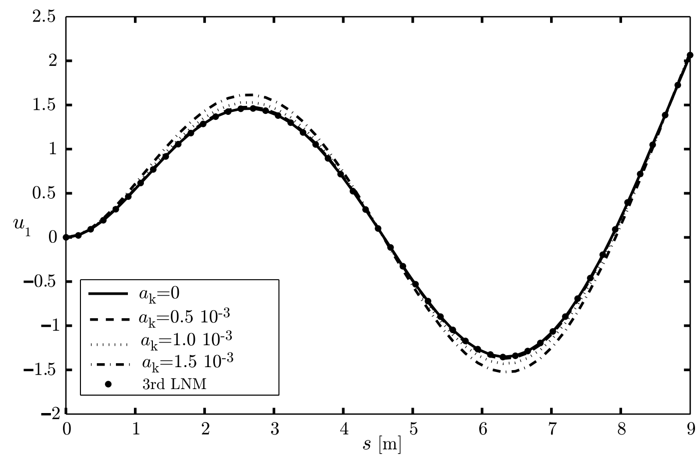

Fig. 9 Deflection profiles of the third flapping NNM for different amplitudes a_k, compared with the linear normal mode (*filled circles*)

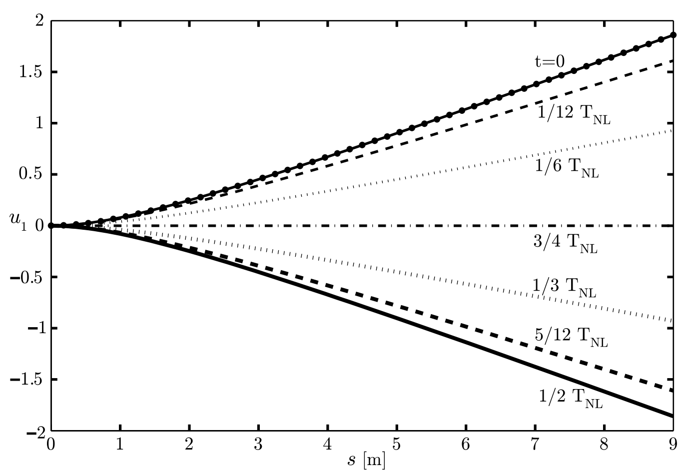

Fig. 10 Snapshots of the deflection profiles of the lowest flapping NNM during one-half of the corresponding nonlinear period T_{NL}

The method of multiple scales was thus applied directly to the system of partial-differential equations of motion to construct the so-called individual flapping nonlinear normal modes away from internal resonances. The virtue of the direct perturbation approach is in the fact that space discretization errors are over-

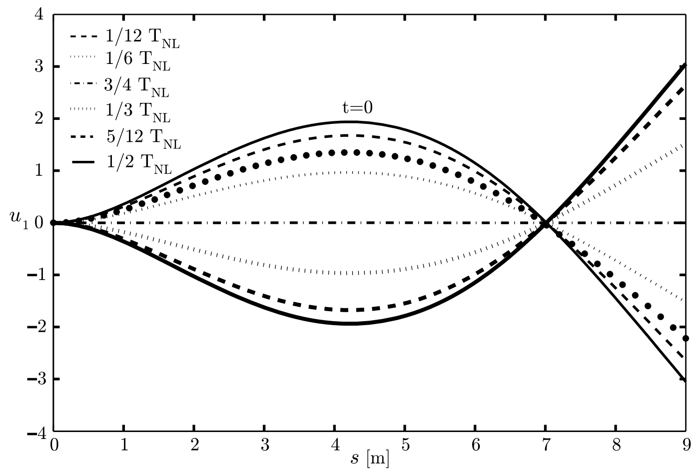

Fig. 11 Snapshots of the deflection profiles of the second flapping NNM during one-half of the corresponding nonlinear period T_{NL}

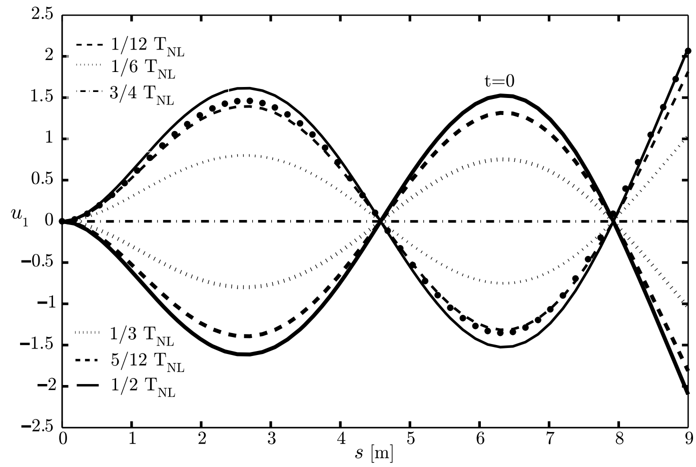

Fig. 12 Snapshots of the deflection profiles of the third flapping NNM during one-half of the corresponding nonlinear period T_{NL}

come and all spatial over-tones (i.e., nonlinear corrections to the linear mode shapes) can be described accurately through functions which are solutions of two-point boundary-value problems. The obtained results confirm a good agreement with previous results in the prediction of the backbones of the lowest flapping modes at given angular speeds. The most significant results can be summarized as follows: (i) the effective nonlinearity coefficients based on the direct perturbation expansion blow up to infinity at angular speeds where 2:1 internal resonances between one of the axial modes and the considered flapping mode occur. The centrifugal forces are the geometric tuning mechanism by which the frequencies of the flapping modes increase drastically thus reducing the typical frequency gap between the flapping modes and the axial modes to a level such that 2:1 frequency ratios can be attained; (ii) the lowest flapping mode is softening at low angular speeds with a decreasing trend of its softening nonlinear character with increasing the speed up to a threshold value at which it becomes hardening until undergoing a singularity point. Past this angular speed, the nonlinearity switches to softening; (iii) the third flapping mode is always hardening; (iv) the second flapping mode exhibits a hardening nonlinearity until its first singularity point past which it becomes softening until a threshold angular speed. Then it becomes hardening and undergoes a second singularity, past which it becomes softening again. The implication is that the dynamics arising from modal interactions between flapping and axial modes can exhibit significantly different nonlinear phenomena depending on the operating range of angular speeds. There are various 2:1 internal resonances by which the coupling between flapping and axial modes can give rise to coupled nonlinear modes with a continuous transfer of energy together with the associated nonlinear phenomenology typical of 2:1 autoparametric resonances such as saturation, Hopf bifurcations, chaos, etc.

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Appendix 1: The linear mass, stiffness, second- and third-order inertia and stiffness operators

$$I_{11} := \frac{\partial}{\partial s}\left(1/\left(\nu_o^2\alpha_2^2\right)\frac{\partial}{\partial s}(\quad)\right) - 1, \tag{30}$$

$$I_{12} := 0, \tag{31}$$

$$I_{21} := 0, \tag{32}$$

$$I_{22} := -1, \tag{33}$$

$$\begin{aligned} L_{11} := \frac{\partial}{\partial s}\Bigg(&-\lambda_a^2/\nu_o\frac{\partial}{\partial s}(\quad) \\ &-\left(\lambda_b^{(2)^2}\alpha_2^2\frac{\partial^3}{\partial s^3}(\quad) + \bar{\omega}_R^2\frac{\partial}{\partial s}(\quad)\right)/\left(\nu_o^2\alpha_2^2\right) \\ &+ 2\nu_o'\lambda_b^{(2)^2}/\nu_o^3\frac{\partial^2}{\partial s^2}(\quad) \\ &- 2\nu_o'^2\lambda_b^{(2)^2}/\nu_o^4\frac{\partial}{\partial s}(\quad) \\ &+ \nu_o''\lambda_b^{(2)^2}/\nu_o^3\frac{\partial}{\partial s}(\quad) + \lambda_a^2\frac{\partial}{\partial s}(\quad)\Bigg), \end{aligned} \tag{34}$$

$$L_{12} := 0, \tag{35}$$

$$L_{21} := 0, \tag{36}$$

$$L_{22} := \frac{\partial}{\partial s}\left(\lambda_a^2/\nu_o\frac{\partial}{\partial s}(\quad)\right) + \bar{\omega}_R^2, \tag{37}$$

$$\begin{aligned} n_{21}^{(1)} := &-3u_1'u_3'\nu_o'/\nu_o^3 + 2u_1''u_3'/\nu_o^2 \\ &+ u_1'u_3''/\nu_o^2 - u_1'u_3''/\nu_o \\ &- u_1''u_3'/\nu_o + u_1'u_3'\nu_o'/\nu_o^2, \end{aligned} \tag{38}$$

$$n_{221i}^{(1)} := 2u_1'u_3'/\left(\nu_o^3\alpha_2^2\right), \tag{39}$$

$$n_{222}^{(1)} := -u_3u_1'/\nu_o, \tag{40}$$

$$\begin{aligned} n_{23i}^{(1)} := &\left(u_1'u_3''' + 2u_1''u_3'' + 2u_1'''u_3'\right)/\nu_o^3 \\ &- 6u_1''u_3'\nu_o'/\nu_o^4 - 4u_1'u_3''\nu_o'/\nu_o^4 \\ &+ 8u_1'u_3'\nu_o'^2/\nu_o^5 - 3u_1'u_3'\nu_o''/\nu_o^4, \end{aligned} \tag{41}$$

$$\begin{aligned} n_2^{(1)} = &\lambda_a^2 n_{21}^{(1)} \\ &+ \bar{\omega}_R^2\left(\frac{\partial}{\partial s}\left(n_{221i}^{(1)}\right) + n_{222}^{(1)}\right) + \lambda_b^{(2)^2}\frac{\partial}{\partial s}\left(n_{23i}^{(1)}\right), \end{aligned} \tag{42}$$

$$i_{21i}^{(1)} := -2\dot{u}_1'\dot{u}_3'/\left(\nu_o^3\alpha_2^2\right), \tag{43}$$

$$i_{221i}^{(1)} := -\left(u_1'\ddot{u}_3' + 2\ddot{u}_1'u_3'\right)/\left(\nu_o^3\alpha_2^2\right), \tag{44}$$

$$i_{222}^{(1)} := \ddot{u}_3u_1'/\nu_o, \tag{45}$$

$$i_{21}^{(1)} := \frac{\partial}{\partial s}\left(i_{21i}^{(1)}\right), \tag{46}$$

$$i_{22}^{(1)} := \frac{\partial}{\partial s}\left(i_{221i}^{(1)}\right) + i_{222}^{(1)}, \tag{47}$$

$$\begin{aligned} n_{31}^{(1)} := &-2u_1''u_3'^2/\nu_o^3 \\ &+ u_1''u_3'^2/\nu_o^2 - (3/2)u_1'^2u_1''/\nu_o^2 \\ &- 3u_1'u_3'u_3''/\nu_o^3 - 2u_1'^3\nu_o'/\nu_o^4 \\ &+ 5u_1'u_3'^2\nu_o'/\nu_o^4 + 2u_1'^2u_1''/\nu_o^3 \\ &- 2u_1'u_3'^2\nu_o'/\nu_o^3 + 2u_1'u_3'u_3''/\nu_o^2 \end{aligned} \tag{48}$$

$$+ u_1'^3\nu_o'/\nu_o^3, \tag{49}$$

$$n_{321i}^{(1)} := (3/2)\left(u_1'^3 - 2u_1'u_3'^2\right)/\left(\nu_o^4\alpha_2^2\right), \tag{50}$$

$$n_{322}^{(1)} := u_3u_1'u_3'/\nu_o^2, \tag{51}$$

$$\begin{aligned} n_{33i}^{(1)} := &(3/8)\big(-(16/3)u_1'u_3''^2 + 4u_1'^2u_1''' - 16u_1''u_3'u_3'' \\ &- 8u_1'u_3'u_3''' + (16/3)u_1'u_1''^2 - 8u_1'''u_3'^2\big)/\nu_o^4 \\ &- 20u_1'u_3'^2\nu_o'^2/\nu_o^6 + 16u_1'u_3'\nu_o'u_3''/\nu_o^5 \\ &+ 12u_1''u_3'^2\nu_o'/\nu_o^5 + 5u_1'^3\nu_o'^2/\nu_o^6 \\ &- 7u_1'^2\nu_o'u_1''/\nu_o^5 \\ &- \left(3/2\left(u_1'^3/\nu_o^5 - 4u_1'u_3'^2/\nu_o^5\right)\right)\nu_o'', \end{aligned} \tag{52}$$

$$\begin{aligned} n_3^{(1)} := &n_{31}^{(1)}\lambda_a^2 + \left(\frac{\partial}{\partial s}\left(n_{321i}^{(1)}\right) + n_{322}^{(1)}\right)\bar{\omega}_R^2 \\ &+ \frac{\partial}{\partial s}\left(n_{33i}^{(1)}\right)\lambda_b^{(2)^2}, \end{aligned} \tag{53}$$

$$i_{31i}^{(1)} := \left(2u_1'\dot{u}_3'^2 + 6u_3'\dot{u}_1'\dot{u}_3' - 2\dot{u}_1'^2u_1'\right)/\left(\nu_o^4\alpha_2^2\right), \quad (54)$$

$$i_{321i}^{(1)} := \left(3\ddot{u}_1'u_3'^2 - 3/2\ddot{u}_1'u_1'^2 + 3u_3'u_1'\ddot{u}_3'\right)/\left(\nu_o^4\alpha_2^2\right), \quad (55)$$

$$i_{322}^{(1)} = \frac{1}{2}\ddot{u}_1u_1'^2/\nu_o^2 - \ddot{u}_3u_1'u_3'/\nu_o^2, \quad (56)$$

$$i_{31}^{(1)} := \frac{\partial}{\partial s}\left(i_{31i}^{(1)}\right), \quad (57)$$

$$i_{32}^{(1)} := \frac{\partial}{\partial s}\left(i_{321i}^{(1)}\right) + i_{322}^{(1)}, \quad (58)$$

$$n_{21}^{(2)} := -u_1'u_1''/\nu_o - 2u_1'^2\nu_o'/\nu_o^3 + u_1'^2\nu_o'/\nu_o^2 + 2u_1'u_1''/\nu_o^2, \quad (59)$$

$$n_{22}^{(2)} := u_1'u_1''/\left(\alpha_2^2\nu_o^3\right) - u_1'^2\nu_o'/\left(\alpha_2^2\nu_o^4\right), \quad (60)$$

$$n_{23}^{(2)} := -u_1'''u_1'\nu_o'/\nu_o^4 + u_1'''u_1''/\nu_o^3 + u_1'^2\nu_o''\nu_o'/\nu_o^5 - 2u_1'^2\nu_o'^3/\nu_o^6 + 4u_1'\nu_o'^2u_1''/\nu_o^5 - 2u_1''^2\nu_o'/\nu_o^4 - u_1'\nu_o''u_1''/\nu_o^4, \quad (61)$$

$$n_2^{(2)} := \lambda_a^2 n_{21}^{(2)} + \bar{\omega}_R^2 n_{22}^{(2)} + \lambda_b^{(2)^2} n_{23}^{(2)}, \quad (62)$$

$$i_{221}^{(2)} := \ddot{u}_1'u_1'\nu_o'/\left(\alpha_2^2\nu_o^4\right) - \ddot{u}_1'u_1''/\left(\alpha_2^2\nu_o^3\right), \quad (63)$$

$$i_{222}^{(2)} := -\ddot{u}_1u_1'/\nu_o, \quad (64)$$

$$i_{21}^{(2)} := 0, \quad (65)$$

$$i_{22}^{(2)} := i_{221}^{(2)} + i_{222}^{(2)}, \quad (66)$$

$$n_{31}^{(2)} := -(3/2)u_1'^2u_3''/\nu_o^3 + (9/2)u_1'^2u_3'\nu_o'/\nu_o^4 - 3u_1'u_1''u_3'/\nu_o^3 - 3u_1'^2u_3'\nu_o'/\nu_o^3 + 2u_1'u_1''u_3'/\nu_o^2 + u_1'^2u_3''/\nu_o^2, \quad (67)$$

$$n_{32}^{(2)} := -\frac{1}{2}u_3u_1'^2/\nu_o^2 + 4u_1'^2u_3'\nu_o'/\left(\alpha_2^2\nu_o^5\right) - u_1'^2u_3''/\left(\alpha_2^2\nu_o^4\right) - 3u_1'u_1''u_3'/\left(\alpha_2^2\nu_o^4\right), \quad (68)$$

$$n_{33}^{(2)} := 8u_1'u_3''\nu_o'u_1''/\nu_o^5 + u_1'^2\nu_o''u_3''/\nu_o^5 - u_1'u_3'''u_1''/\nu_o^4 + u_1'^2u_3'''\nu_o'/\nu_o^5 + 8u_3'u_1''^2\nu_o'/\nu_o^5 - u_1'''u_1'u_3''/\nu_o^4 - 5u_3'u_1'^2\nu_o''\nu_o'/\nu_o^6 - 6u_1'^2u_3''\nu_o'^2/\nu_o^6 + 4u_1'''u_1'u_3'\nu_o'/\nu_o^5 - 3u_1'''u_1''u_3'/\nu_o^4 + 4u_3'u_1'\nu_o''u_1''/\nu_o^5 + 12u_3'u_1'^2\nu_o'^3/\nu_o^7 - 2u_1''^2u_3''/\nu_o^4 - 20u_3'u_1''\nu_o'^2u_1'/\nu_o^6, \quad (69)$$

$$n_3^{(2)} := \lambda_a^2 n_{31}^{(2)} + \bar{\omega}_R^2 n_{32}^{(2)} + \lambda_b^{(2)^2} n_{33}^{(2)}, \quad (70)$$

$$i_{31}^{(2)} := 2\dot{u}_1'\dot{u}_3'u_1''/\left(\alpha_2^2\nu_o^4\right) - 2\dot{u}_1'\dot{u}_3'u_1'\nu_o'/\left(\alpha_2^2\nu_o^5\right), \quad (71)$$

$$i_{321}^{(2)} := 3u_3'\ddot{u}_1'u_1''/\left(\alpha_2^2\nu_o^4\right) - u_1'^2\ddot{u}_3'\nu_o'/\left(\alpha_2^2\nu_o^5\right) + u_1'\ddot{u}_3'u_1''/\left(\alpha_2^2\nu_o^4\right) + \ddot{u}_1'u_1'u_3''/\left(\alpha_2^2\nu_o^4\right) - 4u_3'\ddot{u}_1'u_1'\nu_o'/\left(\alpha_2^2\nu_o^5\right), \quad (72)$$

$$i_{322}^{(2)} := \frac{1}{2}\ddot{u}_3u_1'^2/\nu_o^2 + \ddot{u}_1u_1'u_3'/\nu_o^2, \quad (73)$$

$$i_{31}^{(2)} := 0, \quad (74)$$

$$i_{32}^{(2)} := i_{321}^{(2)} + i_{322}^{(2)}. \quad (75)$$

Appendix 2: Inhomogeneous term of the two-point boundary-value problems

$$\begin{aligned} f(s) :=& -2\lambda_b^{(2)^2}\nu_o'^3\psi_{1,k}'(s)^2/\left(\nu_o^5\lambda_a^2\right) - \psi_{1,k}'(s)\psi_{1,k}''(s) \\ &+ \nu_o'\psi_{1,k}'(s)^2/\nu_o + \lambda_b^{(2)^2}\nu_o''\nu_o'\psi_{1,k}'(s)^2/\left(\nu_o^4\lambda_a^2\right) \\ &+ \psi_{1,k}(s)\omega_k^2\psi_{1,k}'(s)/\lambda_a^2 - 2\nu_o'\psi_{1,k}'(s)^2/\nu_o^2 \\ &- 2\lambda_b^{(2)^2}\nu_o'\psi_{1,k}''(s)^2/\left(\nu_o^3\lambda_a^2\right) \\ &+ \lambda_b^{(2)^2}\psi_{1,k}'''(s)\psi_{1,k}''(s)/\left(\nu_o^2\lambda_a^2\right) \\ &+ \psi_{1,k}'(s)\omega_k^2\psi_{1,k}''(s)/\left(\nu_o^2\alpha_2^2\lambda_a^2\right) \\ &+ \bar{\omega}_R^2\psi_{1,k}'(s)\psi_{1,k}''(s)/\left(\nu_o^2\alpha_2^2\lambda_a^2\right) \\ &- \bar{\omega}_R^2\nu_o'\psi_{1,k}'(s)^2/\left(\nu_o^3\alpha_2^2\lambda_a^2\right) \\ &+ 4\lambda_b^{(2)^2}\nu_o'^2\psi_{1,k}'(s)\psi_{1,k}''(s)/\left(\nu_o^4\lambda_a^2\right) \\ &- \lambda_b^{(2)^2}\nu_o''\psi_{1,k}'(s)\psi_{1,k}''(s)/\left(\nu_o^3\lambda_a^2\right) \\ &+ 2\psi_{1,k}'(s)\psi_{1,k}''(s)/\nu_o \\ &- \lambda_b^{(2)^2}\nu_o'\psi_{1,k}'''(s)\psi_{1,k}'(s)/\left(\nu_o^3\lambda_a^2\right) \\ &- \nu_o'\psi_{1,k}'(s)^2\omega_k^2/\left(\nu_o^3\alpha_2^2\lambda_a^2\right), \end{aligned} \quad (76)$$

$$\begin{aligned} \mathrm{RHS}\left(O\left(\varepsilon^3\right)\right) :=& -1/2(4\lambda_b^{(2)^2}\psi_{1,k}''(s)^3\alpha_2^2\nu_o^3 \\ &+ 21\psi_{1,k}'(s)^2\omega_k^2\psi_{1,k}''(s)\nu_o^3 \\ &+ 4\bar{\omega}_R^2\psi_{1,k}''(s)\nu_o^4h_{31}'(s) \end{aligned}$$

$$
\begin{aligned}
&-26\lambda_{\mathrm{b}}^{(2)^2}\nu_{\mathrm{o}}'\psi_{1,\mathrm{k}}'(s)^2\psi_{1,\mathrm{k}}'''(s)\alpha_2^2\nu_{\mathrm{o}}^2\\
&+4\bar{\omega}_{\mathrm{R}}^2\psi_{1,\mathrm{k}}'(s)\nu_{\mathrm{o}}^4h_{31}''(s)\\
&+9\bar{\omega}_{\mathrm{R}}^2\psi_{1,\mathrm{k}}'(s)^2\psi_{1,\mathrm{k}}''(s)\nu_{\mathrm{o}}^3\\
&+20\psi_{1,\mathrm{k}}'(s)\omega_{\mathrm{k}}^2\nu_{\mathrm{o}}^4h_{31}''(s)\\
&+8\lambda_{\mathrm{b}}^{(2)^2}\psi_{1,\mathrm{k}}'''(s)\alpha_2^2\nu_{\mathrm{o}}^4h_{31}''(s)\\
&+20\psi_{1,\mathrm{k}}''(s)h_{31}'(s)\omega_{\mathrm{k}}^2\nu_{\mathrm{o}}^4\\
&-3\lambda_{\mathrm{b}}^{(2)^2}\nu_{\mathrm{o}}'''\psi_{1,\mathrm{k}}'(s)^3\alpha_2^2\nu_{\mathrm{o}}^2\\
&+4\lambda_{\mathrm{a}}^2\psi_{1,\mathrm{k}}'(s)^2\psi_{1,\mathrm{k}}''(s)\alpha_2^2\nu_{\mathrm{o}}^4\\
&-3\lambda_{\mathrm{a}}^2\psi_{1,\mathrm{k}}'(s)^2\psi_{1,\mathrm{k}}''(s)\alpha_2^2\nu_{\mathrm{o}}^5\\
&+3\lambda_{\mathrm{b}}^{(2)^2}\psi_{1,\mathrm{k}}'(s)^2\psi_{1,\mathrm{k}}''''(s)\alpha_2^2\nu_{\mathrm{o}}^3\\
&+2\lambda_{\mathrm{b}}^{(2)^2}\psi_{1,\mathrm{k}}'(s)\alpha_2^2\nu_{\mathrm{o}}^4h_{31}''''(s)\\
&-2\lambda_{\mathrm{a}}^2\psi_{1,\mathrm{k}}'(s)\alpha_2^2\nu_{\mathrm{o}}^6h_{31}''(s)\\
&+48\lambda_{\mathrm{b}}^{(2)^2}\nu_{\mathrm{o}}'^2\psi_{1,\mathrm{k}}'(s)\alpha_2^2\nu_{\mathrm{o}}^2h_{31}''(s)\\
&-2\lambda_{\mathrm{a}}^2\psi_{1,\mathrm{k}}''(s)\alpha_2^2\nu_{\mathrm{o}}^6h_{31}'(s)\\
&+4\lambda_{\mathrm{a}}^2\psi_{1,\mathrm{k}}''(s)\alpha_2^2\nu_{\mathrm{o}}^5h_{31}'(s)\\
&-32\lambda_{\mathrm{b}}^{(2)^2}\nu_{\mathrm{o}}'\psi_{1,\mathrm{k}}''(s)\alpha_2^2\nu_{\mathrm{o}}^3h_{31}''(s)\\
&-28\nu_{\mathrm{o}}'\psi_{1,\mathrm{k}}'(s)^3\omega_{\mathrm{k}}^2\nu_{\mathrm{o}}^2\\
&-12\bar{\omega}_{\mathrm{R}}^2\nu_{\mathrm{o}}'\psi_{1,\mathrm{k}}'(s)^3\nu_{\mathrm{o}}^2\\
&-60\lambda_{\mathrm{b}}^{(2)^2}\nu_{\mathrm{o}}'^3\psi_{1,\mathrm{k}}'(s)^3\alpha_2^2\\
&-14\lambda_{\mathrm{b}}^{(2)^2}\nu_{\mathrm{o}}''\psi_{1,\mathrm{k}}'(s)\alpha_2^2\nu_{\mathrm{o}}^3h_{31}''(s)\\
&+14\lambda_{\mathrm{b}}^{(2)^2}\psi_{1,\mathrm{k}}'(s)\psi_{1,\mathrm{k}}'''(s)\psi_{1,\mathrm{k}}''(s)\alpha_2^2\nu_{\mathrm{o}}^3\\
&+56\lambda_{\mathrm{b}}^{(2)^2}\nu_{\mathrm{o}}'\nu_{\mathrm{o}}''\psi_{1,\mathrm{k}}'(s)\alpha_2^2\nu_{\mathrm{o}}^2h_{31}'(s)\\
&+35\lambda_{\mathrm{b}}^{(2)^2}\nu_{\mathrm{o}}''\nu_{\mathrm{o}}'\psi_{1,\mathrm{k}}'(s)^3\alpha_2^2\nu_{\mathrm{o}}\\
&+4\lambda_{\mathrm{b}}^{(2)^2}\psi_{1,\mathrm{k}}''''(s)\alpha_2^2\nu_{\mathrm{o}}^4h_{31}'(s)\\
&-6\lambda_{\mathrm{a}}^2\nu_{\mathrm{o}}'\psi_{1,\mathrm{k}}'(s)\alpha_2^2\nu_{\mathrm{o}}^4h_{31}'(s)\\
&+2\lambda_{\mathrm{a}}^2\nu_{\mathrm{o}}'\psi_{1,\mathrm{k}}'(s)\alpha_2^2\nu_{\mathrm{o}}^5h_{31}'(s)\\
&-80\lambda_{\mathrm{b}}^{(2)^2}\nu_{\mathrm{o}}'^3\psi_{1,\mathrm{k}}'(s)\alpha_2^2\nu_{\mathrm{o}}h_{31}'(s)\\
&-24\lambda_{\mathrm{b}}^{(2)^2}\nu_{\mathrm{o}}'\psi_{1,\mathrm{k}}'''(s)\alpha_2^2\nu_{\mathrm{o}}^3h_{31}'(s)
\end{aligned}
$$

$$
\begin{aligned}
&+100\lambda_{\mathrm{b}}^{(2)^2}\nu_{\mathrm{o}}'^2\psi_{1,\mathrm{k}}'(s)^2\psi_{1,\mathrm{k}}''(s)\alpha_2^2\nu_{\mathrm{o}}\\
&+6\lambda_{\mathrm{b}}^{(2)^2}\psi_{1,\mathrm{k}}''(s)\alpha_2^2\nu_{\mathrm{o}}^4h_{31}'''(s)\\
&+2\lambda_{\mathrm{a}}^2\psi_{1,\mathrm{k}}'(s)\alpha_2^2\nu_{\mathrm{o}}^5h_{31}''(s)\\
&-14\lambda_{\mathrm{b}}^{(2)^2}\nu_{\mathrm{o}}'\psi_{1,\mathrm{k}}'(s)\alpha_2^2\nu_{\mathrm{o}}^3h_{31}'''(s)\\
&-44\lambda_{\mathrm{b}}^{(2)^2}\nu_{\mathrm{o}}'\psi_{1,\mathrm{k}}'(s)\psi_{1,\mathrm{k}}''(s)^2\alpha_2^2\nu_{\mathrm{o}}^2\\
&-6\lambda_{\mathrm{b}}^{(2)^2}\nu_{\mathrm{o}}'''\psi_{1,\mathrm{k}}'(s)\alpha_2^2\nu_{\mathrm{o}}^3h_{31}'(s)\\
&-23\lambda_{\mathrm{b}}^{(2)^2}\nu_{\mathrm{o}}''\psi_{1,\mathrm{k}}'(s)^2\psi_{1,\mathrm{k}}''(s)\alpha_2^2\nu_{\mathrm{o}}^2\\
&-4\lambda_{\mathrm{a}}^2\nu_{\mathrm{o}}'\psi_{1,\mathrm{k}}'(s)^3\alpha_2^2\nu_{\mathrm{o}}^3\\
&+2\lambda_{\mathrm{a}}^2\nu_{\mathrm{o}}'\psi_{1,\mathrm{k}}'(s)^3\alpha_2^2\nu_{\mathrm{o}}^4\\
&-\psi_{1,\mathrm{k}}(s)\omega_{\mathrm{k}}^2\psi_{1,\mathrm{k}}'(s)^2\alpha_2^2\nu_{\mathrm{o}}^5\\
&-2\bar{\omega}_{\mathrm{R}}^2\psi_{1,\mathrm{k}}'(s)\alpha_2^2\nu_{\mathrm{o}}^6h_{31}(s)\\
&-12\bar{\omega}_{\mathrm{R}}^2\nu_{\mathrm{o}}'\psi_{1,\mathrm{k}}'(s)\nu_{\mathrm{o}}^3h_{31}'(s)\\
&-8h_{31}(s)\omega_{\mathrm{k}}^2\psi_{1,\mathrm{k}}'(s)\alpha_2^2\nu_{\mathrm{o}}^6\\
&-60\nu_{\mathrm{o}}'\psi_{1,\mathrm{k}}'(s)\omega_{\mathrm{k}}^2\nu_{\mathrm{o}}^3h_{31}'(s)\\
&+64\lambda_{\mathrm{b}}^{(2)^2}\nu_{\mathrm{o}}'^2\psi_{1,\mathrm{k}}''(s)\alpha_2^2\nu_{\mathrm{o}}^2h_{31}'(s)\\
&-18\lambda_{\mathrm{b}}^{(2)^2}\nu_{\mathrm{o}}''\psi_{1,\mathrm{k}}''(s)\alpha_2^2\nu_{\mathrm{o}}^3h_{31}'(s)\\
&\times e^{3I\omega_{\mathrm{k}}T_{\mathrm{o}}}A_{\mathrm{k}}(T_2)^3/\left(\alpha_2^2\nu_{\mathrm{o}}^7\right)\\
&-1/2(12\lambda_{\mathrm{b}}^{(2)^2}\psi_{1,\mathrm{k}}''(s)^3\alpha_2^2\nu_{\mathrm{o}}^3\\
&+8\psi_{1,\mathrm{k}}'(s)\omega_{\mathrm{k}}^2\nu_{\mathrm{o}}^4h_{32}''(s)\\
&+15\psi_{1,\mathrm{k}}'(s)^2\omega_{\mathrm{k}}^2\psi_{1,\mathrm{k}}''(s)\nu_{\mathrm{o}}^3\\
&-64\lambda_{\mathrm{b}}^{(2)^2}\nu_{\mathrm{o}}'\psi_{1,\mathrm{k}}''(s)\alpha_2^2\nu_{\mathrm{o}}^3h_{32}''(s)\\
&+4\bar{\omega}_{\mathrm{R}}^2\psi_{1,\mathrm{k}}''(s)\nu_{\mathrm{o}}^4h_{31}'(s)\\
&-78\lambda_{\mathrm{b}}^{(2)^2}\nu_{\mathrm{o}}'\psi_{1,\mathrm{k}}'(s)^2\psi_{1,\mathrm{k}}'''(s)\alpha_2^2\nu_{\mathrm{o}}^2\\
&+4\bar{\omega}_{\mathrm{R}}^2\psi_{1,\mathrm{k}}'(s)\nu_{\mathrm{o}}^4h_{31}''(s)\\
&+27\bar{\omega}_{\mathrm{R}}^2\psi_{1,\mathrm{k}}'(s)^2\psi_{1,\mathrm{k}}''(s)\nu_{\mathrm{o}}^3\\
&+4\psi_{1,\mathrm{k}}'(s)\omega_{\mathrm{k}}^2\nu_{\mathrm{o}}^4h_{31}''(s)\\
&+8\lambda_{\mathrm{b}}^{(2)^2}\psi_{1,\mathrm{k}}''''(s)\alpha_2^2\nu_{\mathrm{o}}^4h_{32}'(s)\\
&+8\lambda_{\mathrm{b}}^{(2)^2}\psi_{1,\mathrm{k}}'''(s)\alpha_2^2\nu_{\mathrm{o}}^4h_{31}''(s)
\end{aligned}
$$

$$
\begin{aligned}
& -36\lambda_{\mathrm{b}}^{(2)^2}\nu_{\mathrm{o}}''\psi_{1,\mathrm{k}}''(s)\alpha_2^2\nu_{\mathrm{o}}^3h_{32}'(s)\\
& +4\psi_{1,\mathrm{k}}''(s)h_{31}'(s)\omega_{\mathrm{k}}^2\nu_{\mathrm{o}}^4\\
& -9\lambda_{\mathrm{b}}^{(2)^2}\nu_{\mathrm{o}}'''\psi_{1,\mathrm{k}}'(s)^3\alpha_2^2\nu_{\mathrm{o}}^2\\
& +12\lambda_{\mathrm{a}}^2\psi_{1,\mathrm{k}}'(s)^2\psi_{1,\mathrm{k}}''(s)\alpha_2^2\nu_{\mathrm{o}}^4\\
& -9\lambda_{\mathrm{a}}^2\psi_{1,\mathrm{k}}'(s)^2\psi_{1,\mathrm{k}}''(s)\alpha_2^2\nu_{\mathrm{o}}^5\\
& +9\lambda_{\mathrm{b}}^{(2)^2}\psi_{1,\mathrm{k}}'(s)^2\psi_{1,\mathrm{k}}''''(s)\alpha_2^2\nu_{\mathrm{o}}^3\\
& +2\lambda_{\mathrm{b}}^{(2)^2}\psi_{1,\mathrm{k}}'(s)\alpha_2^2\nu_{\mathrm{o}}^4h_{31}''''(s)\\
& -2\lambda_{\mathrm{a}}^2\psi_{1,\mathrm{k}}'(s)\alpha_2^2\nu_{\mathrm{o}}^6h_{31}''(s)\\
& +48\lambda_{\mathrm{b}}^{(2)^2}\nu_{\mathrm{o}}'^2\psi_{1,\mathrm{k}}'(s)\alpha_2^2\nu_{\mathrm{o}}^2h_{31}''(s)\\
& -2\lambda_{\mathrm{a}}^2\psi_{1,\mathrm{k}}''(s)\alpha_2^2\nu_{\mathrm{o}}^6h_{31}'(s)\\
& +16\lambda_{\mathrm{b}}^{(2)^2}\psi_{1,\mathrm{k}}'''(s)\alpha_2^2\nu_{\mathrm{o}}^4h_{32}''(s)\\
& +4\lambda_{\mathrm{a}}^2\psi_{1,\mathrm{k}}''(s)\alpha_2^2\nu_{\mathrm{o}}^5h_{31}'(s)\\
& -160\lambda_{\mathrm{b}}^{(2)^2}\nu_{\mathrm{o}}'^3\psi_{1,\mathrm{k}}'(s)\alpha_2^2\nu_{\mathrm{o}}h_{32}'(s)\\
& +4\lambda_{\mathrm{a}}^2\nu_{\mathrm{o}}'\psi_{1,\mathrm{k}}'(s)\alpha_2^2\nu_{\mathrm{o}}^5h_{32}'(s)\\
& -12\lambda_{\mathrm{a}}^2\nu_{\mathrm{o}}'\psi_{1,\mathrm{k}}'(s)\alpha_2^2\nu_{\mathrm{o}}^4h_{32}'(s)\\
& -32\lambda_{\mathrm{b}}^{(2)^2}\nu_{\mathrm{o}}'\psi_{1,\mathrm{k}}''(s)\alpha_2^2\nu_{\mathrm{o}}^3h_{31}''(s)\\
& -20\nu_{\mathrm{o}}'\psi_{1,\mathrm{k}}'(s)^3\omega_{\mathrm{k}}^2\nu_{\mathrm{o}}^2\\
& -36\bar{\omega}_{\mathrm{R}}^2\nu_{\mathrm{o}}'\psi_{1,\mathrm{k}}'(s)^3\nu_{\mathrm{o}}^2\\
& -180\lambda_{\mathrm{b}}^{(2)^2}\nu_{\mathrm{o}}'^3\psi_{1,\mathrm{k}}'(s)^3\alpha_2^2\\
& -12\lambda_{\mathrm{b}}^{(2)^2}\nu_{\mathrm{o}}'''\psi_{1,\mathrm{k}}'(s)\alpha_2^2\nu_{\mathrm{o}}^3h_{32}'(s)\\
& -14\lambda_{\mathrm{b}}^{(2)^2}\nu_{\mathrm{o}}''\psi_{1,\mathrm{k}}'(s)\alpha_2^2\nu_{\mathrm{o}}^3h_{31}''(s)\\
& +8\lambda_{\mathrm{a}}^2\psi_{1,\mathrm{k}}''(s)\alpha_2^2\nu_{\mathrm{o}}^5h_{32}'(s)\\
& +42\lambda_{\mathrm{b}}^{(2)^2}\psi_{1,\mathrm{k}}'(s)\psi_{1,\mathrm{k}}'''(s)\psi_{1,\mathrm{k}}''(s)\alpha_2^2\nu_{\mathrm{o}}^3\\
& +56\lambda_{\mathrm{b}}^{(2)^2}\nu_{\mathrm{o}}'\nu_{\mathrm{o}}''\psi_{1,\mathrm{k}}'(s)\alpha_2^2\nu_{\mathrm{o}}^2h_{31}'(s)\\
& +105\lambda_{\mathrm{b}}^{(2)^2}\nu_{\mathrm{o}}''\nu_{\mathrm{o}}'\psi_{1,\mathrm{k}}'(s)^3\alpha_2^2\nu_{\mathrm{o}}\\
& +4\lambda_{\mathrm{b}}^{(2)^2}\psi_{1,\mathrm{k}}''''(s)\alpha_2^2\nu_{\mathrm{o}}^4h_{31}'(s)\\
& -6\lambda_{\mathrm{a}}^2\nu_{\mathrm{o}}'\psi_{1,\mathrm{k}}'(s)\alpha_2^2\nu_{\mathrm{o}}^4h_{31}'(s)\\
& +2\lambda_{\mathrm{a}}^2\nu_{\mathrm{o}}'\psi_{1,\mathrm{k}}'(s)\alpha_2^2\nu_{\mathrm{o}}^5h_{31}'(s)\\
& -80\lambda_{\mathrm{b}}^{(2)^2}\nu_{\mathrm{o}}'^3\psi_{1,\mathrm{k}}'(s)\alpha_2^2\nu_{\mathrm{o}}h_{31}'(s)\\
& -4\bar{\omega}_{\mathrm{R}}^2\psi_{1,\mathrm{k}}'(s)\alpha_2^2\nu_{\mathrm{o}}^6h_{32}(s)\\
& -24\bar{\omega}_{\mathrm{R}}^2\nu_{\mathrm{o}}'\psi_{1,\mathrm{k}}'(s)\nu_{\mathrm{o}}^3h_{32}'(s)\\
& -24\nu_{\mathrm{o}}'\psi_{1,\mathrm{k}}'(s)\omega_{\mathrm{k}}^2\nu_{\mathrm{o}}^3h_{32}'(s)\\
& -24\lambda_{\mathrm{b}}^{(2)^2}\nu_{\mathrm{o}}'\psi_{1,\mathrm{k}}'''(s)\alpha_2^2\nu_{\mathrm{o}}^3h_{31}'(s)\\
& +300\lambda_{\mathrm{b}}^{(2)^2}\nu_{\mathrm{o}}'^2\psi_{1,\mathrm{k}}'(s)^2\psi_{1,\mathrm{k}}''(s)\alpha_2^2\nu_{\mathrm{o}}\\
& +12\lambda_{\mathrm{b}}^{(2)^2}\psi_{1,\mathrm{k}}''(s)\alpha_2^2\nu_{\mathrm{o}}^4h_{32}'''(s)\\
& -4\lambda_{\mathrm{a}}^2\psi_{1,\mathrm{k}}''(s)\alpha_2^2\nu_{\mathrm{o}}^6h_{32}'(s)\\
& -28\lambda_{\mathrm{b}}^{(2)^2}\nu_{\mathrm{o}}'\psi_{1,\mathrm{k}}'(s)\alpha_2^2\nu_{\mathrm{o}}^3h_{32}'''(s)\\
& +4\lambda_{\mathrm{a}}^2\psi_{1,\mathrm{k}}'(s)\alpha_2^2\nu_{\mathrm{o}}^5h_{32}''(s)\\
& +6\lambda_{\mathrm{b}}^{(2)^2}\psi_{1,\mathrm{k}}''(s)\alpha_2^2\nu_{\mathrm{o}}^4h_{31}'''(s)\\
& +2\lambda_{\mathrm{a}}^2\psi_{1,\mathrm{k}}'(s)\alpha_2^2\nu_{\mathrm{o}}^5h_{31}''(s)\\
& +112\lambda_{\mathrm{b}}^{(2)^2}\nu_{\mathrm{o}}'\nu_{\mathrm{o}}''\psi_{1,\mathrm{k}}'(s)\alpha_2^2\nu_{\mathrm{o}}^2h_{32}'(s)\\
& -14\lambda_{\mathrm{b}}^{(2)^2}\nu_{\mathrm{o}}'\psi_{1,\mathrm{k}}'(s)\alpha_2^2\nu_{\mathrm{o}}^3h_{31}'''(s)\\
& -4\lambda_{\mathrm{a}}^2\psi_{1,\mathrm{k}}'(s)\alpha_2^2\nu_{\mathrm{o}}^6h_{32}''(s)\\
& -132\lambda_{\mathrm{b}}^{(2)^2}\nu_{\mathrm{o}}'\psi_{1,\mathrm{k}}'(s)\psi_{1,\mathrm{k}}''(s)^2\alpha_2^2\nu_{\mathrm{o}}^2\\
& -6\lambda_{\mathrm{b}}^{(2)^2}\nu_{\mathrm{o}}'''\psi_{1,\mathrm{k}}'(s)\alpha_2^2\nu_{\mathrm{o}}^3h_{31}'(s)\\
& -69\lambda_{\mathrm{b}}^{(2)^2}\nu_{\mathrm{o}}''\psi_{1,\mathrm{k}}'(s)^2\psi_{1,\mathrm{k}}''(s)\alpha_2^2\nu_{\mathrm{o}}^2\\
& -48\lambda_{\mathrm{b}}^{(2)^2}\nu_{\mathrm{o}}'\psi_{1,\mathrm{k}}'''(s)\alpha_2^2\nu_{\mathrm{o}}^3h_{32}'(s)\\
& +128\lambda_{\mathrm{b}}^{(2)^2}\nu_{\mathrm{o}}'^2\psi_{1,\mathrm{k}}''(s)\alpha_2^2\nu_{\mathrm{o}}^2h_{32}'(s)\\
& +4\lambda_{\mathrm{b}}^{(2)^2}\psi_{1,\mathrm{k}}'(s)\alpha_2^2\nu_{\mathrm{o}}^4h_{32}''''(s)\\
& -12\lambda_{\mathrm{a}}^2\nu_{\mathrm{o}}'\psi_{1,\mathrm{k}}'(s)^3\alpha_2^2\nu_{\mathrm{o}}^3\\
& +6\lambda_{\mathrm{a}}^2\nu_{\mathrm{o}}'\psi_{1,\mathrm{k}}'(s)^3\alpha_2^2\nu_{\mathrm{o}}^4\\
& -3\psi_{1,\mathrm{k}}(s)\omega_{\mathrm{k}}^2\psi_{1,\mathrm{k}}'(s)^2\alpha_2^2\nu_{\mathrm{o}}^5\\
& -2\bar{\omega}_{\mathrm{R}}^2\psi_{1,\mathrm{k}}'(s)\alpha_2^2\nu_{\mathrm{o}}^6h_{31}(s)\\
& -12\bar{\omega}_{\mathrm{R}}^2\nu_{\mathrm{o}}'\psi_{1,\mathrm{k}}'(s)\nu_{\mathrm{o}}^3h_{31}'(s)\\
& -8h_{31}(s)\omega_{\mathrm{k}}^2\psi_{1,\mathrm{k}}'(s)\alpha_2^2\nu_{\mathrm{o}}^6\\
& -12\nu_{\mathrm{o}}'\psi_{1,\mathrm{k}}'(s)\omega_{\mathrm{k}}^2\nu_{\mathrm{o}}^3h_{31}'(s)
\end{aligned}
$$

$$
\begin{aligned}
&- 28\lambda_{\mathrm{b}}^{(2)^2} \nu_{\mathrm{o}}'' \psi_{1,\mathrm{k}}'(s)\alpha_2^2 \nu_{\mathrm{o}}^3 h_{32}''(s) \\
&+ 64\lambda_{\mathrm{b}}^{(2)^2} \nu_{\mathrm{o}}'^2 \psi_{1,\mathrm{k}}''(s)\alpha_2^2 \nu_{\mathrm{o}}^2 h_{31}'(s) \\
&+ 96\lambda_{\mathrm{b}}^{(2)^2} \nu_{\mathrm{o}}'^2 \psi_{1,\mathrm{k}}'(s)\alpha_2^2 \nu_{\mathrm{o}}^2 h_{32}''(s) \\
&+ 8\bar{\omega}_{\mathrm{R}}^2 \psi_{1,\mathrm{k}}'(s)\nu_{\mathrm{o}}^4 h_{32}''(s) \\
&+ 8\bar{\omega}_{\mathrm{R}}^2 \psi_{1,\mathrm{k}}''(s)\nu_{\mathrm{o}}^4 h_{32}'(s) \\
&- 18\lambda_{\mathrm{b}}^{(2)^2} \nu_{\mathrm{o}}'' \psi_{1,\mathrm{k}}''(s)\alpha_2^2 \nu_{\mathrm{o}}^3 h_{31}'(s) \\
&+ 8\psi_{1,\mathrm{k}}''(s)\omega_{\mathrm{k}}^2 \nu_{\mathrm{o}}^4 h_{32}'(s)) e^{i\omega_{\mathrm{k}} T_{\mathrm{o}}} \\
&\times \bar{A}_{\mathrm{k}}(T_2) A_{\mathrm{k}}(T_2)^2 / \left(\alpha_2^2 \nu_{\mathrm{o}}^7\right) \\
&- 1/2\big(-8i\nu_{\mathrm{o}}' \psi_{1,\mathrm{k}}'(s)\omega_{\mathrm{k}} \nu_{\mathrm{o}}^4 \\
&+ 4i\psi_{1,\mathrm{k}}''(s)\omega_{\mathrm{k}} \nu_{\mathrm{o}}^5 \\
&- 4i\psi_{1,\mathrm{k}}(s)\omega_{\mathrm{k}} \alpha_2^2 \nu_{\mathrm{o}}^7\big) e^{i\omega_{\mathrm{k}} T_{\mathrm{o}}} \\
&\times A_{\mathrm{k}}'(T_2) / \left(\alpha_2^2 \nu_{\mathrm{o}}^7\right) + cc. \qquad (77)
\end{aligned}
$$

where RHS(ε^3) denotes the right-hand side of the third-order flapping problem.

Appendix 3: The effective nonlinearity coefficient of the kth flapping mode

$$
\begin{aligned}
\Gamma_{\mathrm{k,k}} = \big(&-96\lambda_{\mathrm{b}}^{(2)^2} h_{26,\mathrm{k,k}}^{(3)}\alpha_2^2 - 56\lambda_{\mathrm{b}}^{(2)^2} h_{44,\mathrm{k,k}}^{(3)}\alpha_2^2 \\
&+ 24\lambda_{\mathrm{b}}^{(2)^2} h_{16,\mathrm{k,k}}^{(3)}\alpha_2^2 + 2\lambda_{\mathrm{a}}^2 h_{31,\mathrm{k,k}}^{(3)}\alpha_2^2 \\
&+ 2\bar{\omega}_{\mathrm{R}}^2 h_{22,\mathrm{k,k}}^{(3)}\alpha_2^2 + 2\lambda_{\mathrm{a}}^2 h_{12,\mathrm{k,k}}^{(3)}\alpha_2^2 \\
&+ 80\lambda_{\mathrm{b}}^{(2)^2} h_{47,\mathrm{k,k}}^{(3)}\alpha_2^2 + 180\lambda_{\mathrm{b}}^{(2)^2} g_{14,\mathrm{k,k}}\alpha_2^2 \\
&+ 48\lambda_{\mathrm{b}}^{(2)^2} h_{5,\mathrm{k,k}}^{(3)}\alpha_2^2 - 16\lambda_{\mathrm{b}}^{(2)^2} h_{27,\mathrm{k,k}}^{(3)}\alpha_2^2 \\
&+ 14\lambda_{\mathrm{b}}^{(2)^2} h_{4,\mathrm{k,k}}^{(3)}\alpha_2^2 - 12\lambda_{\mathrm{b}}^{(2)^2} h_{19,\mathrm{k,k}}^{(3)}\alpha_2^2 \\
&- 4\lambda_{\mathrm{a}}^2 h_{30,\mathrm{k,k}}^{(3)}\alpha_2^2 - 12\lambda_{\mathrm{b}}^{(2)^2} g_{6,\mathrm{k,k}}\alpha_2^2 \\
&+ 28\lambda_{\mathrm{b}}^{(2)^2} h_{13,\mathrm{k,k}}^{(3)}\alpha_2^2 + 8\omega_{\mathrm{k}}^2 h_{22,\mathrm{k,k}}^{(3)}\alpha_2^2 \\
&- 4\lambda_{\mathrm{a}}^2 h_{11,\mathrm{k,k}}^{(3)}\alpha_2^2 - 6\lambda_{\mathrm{b}}^{(2)^2} h_{6,\mathrm{k,k}}^{(3)}\alpha_2^2 \\
&- 48\lambda_{\mathrm{b}}^{(2)^2} h_{42,\mathrm{k,k}}^{(3)}\alpha_2^2 - 4\lambda_{\mathrm{b}}^{(2)^2} h_{35,\mathrm{k,k}}^{(3)}\alpha_2^2 \\
&- 9\lambda_{\mathrm{b}}^{(2)^2} g_{12,\mathrm{k,k}}\alpha_2^2 - 6\lambda_{\mathrm{a}}^2 g_{15,\mathrm{k,k}}\alpha_2^2 - 8\omega_{\mathrm{k}}^2 h_{14,\mathrm{k,k}}^{(3)} \\
&- 300\lambda_{\mathrm{b}}^{(2)^2} g_{3,\mathrm{k,k}}\alpha_2^2 + 24\omega_{\mathrm{k}}^2 h_{39,\mathrm{k,k}}^{(3)} \\
&+ 12\omega_{\mathrm{k}}^2 h_{40,\mathrm{k,k}}^{(3)} + 12\bar{\omega}_{\mathrm{R}}^2 h_{40,\mathrm{k,k}}^{(3)} \\
&+ 160\lambda_{\mathrm{b}}^{(2)^2} h_{38,\mathrm{k,k}}^{(3)}\alpha_2^2 + 12\lambda_{\mathrm{a}}^2 g_{17,\mathrm{k,k}}\alpha_2^2 \\
&+ 3\omega_{\mathrm{k}}^2 g_{16,\mathrm{k,k}}\alpha_2^2 + 64\lambda_{\mathrm{b}}^{(2)^2} h_{43,\mathrm{k,k}}^{(3)}\alpha_2^2 \\
&- 12\lambda_{\mathrm{a}}^2 g_{7,\mathrm{k,k}}\alpha_2^2 + 9\lambda_{\mathrm{a}}^2 g_{1,\mathrm{k,k}}\alpha_2^2 \\
&- 2\lambda_{\mathrm{b}}^{(2)^2} h_{24,\mathrm{k,k}}^{(3)}\alpha_2^2 - 112\lambda_{\mathrm{b}}^{(2)^2} h_{25,\mathrm{k,k}}^{(3)}\alpha_2^2 \\
&+ 9\lambda_{\mathrm{b}}^{(2)^2} g_{11,\mathrm{k,k}}\alpha_2^2 + 69\lambda_{\mathrm{b}}^{(2)^2} g_{10,\mathrm{k,k}}\alpha_2^2 \\
&- 2\lambda_{\mathrm{a}}^2 h_{21,\mathrm{k,k}}^{(3)}\alpha_2^2 - 128\lambda_{\mathrm{b}}^{(2)^2} h_{36,\mathrm{k,k}}^{(3)}\alpha_2^2 \\
&+ 18\lambda_{\mathrm{b}}^{(2)^2} h_{34,\mathrm{k,k}}^{(3)}\alpha_2^2 + 78\lambda_{\mathrm{b}}^{(2)^2} g_{2,\mathrm{k,k}}\alpha_2^2 \\
&+ 12\lambda_{\mathrm{b}}^{(2)^2} h_{32,\mathrm{k,k}}^{(3)}\alpha_2^2 + 28\lambda_{\mathrm{b}}^{(2)^2} h_{8,\mathrm{k,k}}^{(3)}\alpha_2^2 \\
&- 8\lambda_{\mathrm{b}}^{(2)^2} h_{7,\mathrm{k,k}}^{(3)}\alpha_2^2 + 24\bar{\omega}_{\mathrm{R}}^2 h_{39,\mathrm{k,k}}^{(3)} \\
&+ 4\lambda_{\mathrm{a}}^2 h_{15,\mathrm{k,k}}^{(3)}\alpha_2^2 - 4\bar{\omega}_{\mathrm{R}}^2 h_{3,\mathrm{k,k}}^{(3)} \\
&- 15\omega_{\mathrm{k}}^2 g_{13,\mathrm{k,k}} - 15\omega_{\mathrm{k}}^2 g_{13,\mathrm{k,k}} \\
&- 27\bar{\omega}_{\mathrm{R}}^2 g_{13,\mathrm{k,k}} - 8\bar{\omega}_{\mathrm{R}}^2 h_{14,\mathrm{k,k}}^{(3)} \\
&- 4\bar{\omega}_{\mathrm{R}}^2 h_{1,\mathrm{k,k}}^{(3)} - 2\lambda_{\mathrm{a}}^2 h_{37,\mathrm{k,k}}^{(3)}\alpha_2^2 \\
&- 8\omega_{\mathrm{k}}^2 h_{2,\mathrm{k,k}}^{(3)} - 8\bar{\omega}_{\mathrm{R}}^2 h_{2,\mathrm{k,k}}^{(3)} - 4\omega_{\mathrm{k}}^2 h_{3,\mathrm{k,k}}^{(3)} \\
&- 4\lambda_{\mathrm{a}}^2 h_{41,\mathrm{k,k}}^{(3)}\alpha_2^2 - 8\lambda_{\mathrm{a}}^2 h_{33,\mathrm{k,k}}^{(3)}\alpha_2^2 \\
&+ 6\lambda_{\mathrm{a}}^2 h_{48,\mathrm{k,k}}^{(3)}\alpha_2^2 - 4\omega_{\mathrm{k}}^2 h_{1,\mathrm{k,k}}^{(3)} \\
&- 4\omega_{\mathrm{k}}^2 h_{3,\mathrm{k,k}}^{(3)} + 20\omega_{\mathrm{k}}^2 g_{9,\mathrm{k,k}} + 36\bar{\omega}_{\mathrm{R}}^2 g_{9,\mathrm{k,k}} \\
&+ 36\lambda_{\mathrm{b}}^{(2)^2} h_{45,\mathrm{k,k}}^{(3)}\alpha_2^2 + 4\lambda_{\mathrm{a}}^2 h_{29,\mathrm{k,k}}^{(3)}\alpha_2^2 \\
&+ 14\lambda_{\mathrm{b}}^{(2)^2} h_{20,\mathrm{k,k}}^{(3)}\alpha_2^2 - 42\lambda_{\mathrm{b}}^{(2)^2} g_{4,\mathrm{k,k}}\alpha_2^2 \\
&+ 6\lambda_{\mathrm{b}}{}^{(2)^2} h_{10,\mathrm{k,k}}^{(3)}\alpha_2^2 \\
&- 64\lambda_{\mathrm{b}}^{(2)^2} h_{17,\mathrm{k,k}}^{(3)}\alpha_2^2 + 32\lambda_{\mathrm{b}}^{(2)^2} h_{18,\mathrm{k,k}}^{(3)}\alpha_2^2 \\
&- 8\lambda_{\mathrm{b}}^{(2)^2} h_{9,\mathrm{k,k}}^{(3)}\alpha_2^2 \\
&+ 4\bar{\omega}_{\mathrm{R}}^2 h_{23,\mathrm{k,k}}^{(3)}\alpha_2^2\big) / \left(\omega_{\mathrm{k}}\left(2s_{1,\mathrm{k,k}} - s_{2,\mathrm{k,k}} + \alpha_2^2\right)\right), \qquad (78)
\end{aligned}
$$

where

$$
\int_{\mathrm{o}}^{1} \left(\psi_{1,\mathrm{j}}(s)\psi_{1,\mathrm{k}}(s)\right)\mathrm{ds} = \delta_{\mathrm{j,k}},
$$

$$\int_{\text{o}}^{1}\big(\psi_{1,\text{j}}(s)\nu_{\text{o}}'(s)\psi_{1,\text{k}}'(s)/\nu_{\text{o}}(s)^3\big)\text{ds}=s_{1,\text{j},\text{k}},$$

$$\int_{\text{o}}^{1}\big(\psi_{1,\text{j}}(s)\psi_{1,\text{k}}''(s)/\nu_{\text{o}}(s)^2\big)\text{ds}=s_{2,\text{j},\text{k}},$$

$$\int_{\text{o}}^{1}\big(\psi_{1,\text{j}}(s)\psi_{1,\text{k}}''(s)h_{31}'/\nu_{\text{o}}(s)^3\big)\text{ds}=h_{1,\text{j},\text{k}}^{(3)},$$

$$\int_{\text{o}}^{1}\big(\psi_{1,\text{j}}(s)\psi_{1,\text{k}}'(s)h_{32}''/\nu_{\text{o}}(s)^3\big)\text{ds}=h_{2,\text{j},\text{k}}^{(3)},$$

$$\int_{\text{o}}^{1}\big(\psi_{1,\text{j}}(s)\psi_{1,\text{k}}'(s)h_{31}''/\nu_{\text{o}}(s)^3\big)\text{ds}=h_{3,\text{j},\text{k}}^{(3)},$$

$$\int_{\text{o}}^{1}\big(\psi_{1,\text{j}}(s)\nu_{\text{o}}''(s)\psi_{1,\text{k}}'(s)h_{31}''/\nu_{\text{o}}(s)^4\big)\text{ds}=h_{4,\text{j},\text{k}}^{(3)},$$

$$\int_{\text{o}}^{1}\big(\psi_{1,\text{j}}(s)\nu_{\text{o}}'(s)\psi_{1,\text{k}}'''(s)h_{32}'/\nu_{\text{o}}(s)^4\big)\text{ds}=h_{5,\text{j},\text{k}}^{(3)},$$

$$\int_{\text{o}}^{1}\big(\psi_{1,\text{j}}(s)\psi_{1,\text{k}}''(s)h_{31}'''/\nu_{\text{o}}(s)^3\big)\text{ds}=h_{6,\text{j},\text{k}}^{(3)},$$

$$\int_{\text{o}}^{1}\big(\psi_{1,\text{j}}(s)\psi_{1,\text{k}}'''(s)h_{31}''/\nu_{\text{o}}(s)^3\big)\text{ds}=h_{7,\text{j},\text{k}}^{(3)},$$

$$\int_{\text{o}}^{1}\big(\psi_{1,\text{j}}(s)\nu_{\text{o}}''(s)\psi_{1,\text{k}}'(s)h_{32}''/\nu_{\text{o}}(s)^4\big)\text{ds}=h_{8,\text{j},\text{k}}^{(3)},$$

$$\int_{\text{o}}^{1}\big(\psi_{1,\text{j}}(s)\psi_{1,\text{k}}''''(s)h_{32}'/\nu_{\text{o}}(s)^3\big)\text{ds}=h_{9,\text{j},\text{k}}^{(3)},$$

$$\int_{\text{o}}^{1}\big(\psi_{1,\text{j}}(s)\big(\nu_{\text{o}}'''(s)\big)\psi_{1,\text{k}}'(s)h_{31}'/\nu_{\text{o}}(s)^4\big)\text{ds}=h_{10,\text{j},\text{k}}^{(3)},$$

$$\int_{\text{o}}^{1}\big(\psi_{1,\text{j}}(s)\psi_{1,\text{k}}''(s)h_{31}'/\nu_{\text{o}}(s)^2\big)\text{ds}=h_{11,\text{j},\text{k}}^{(3)},$$

$$\int_{\text{o}}^{1}\big(\psi_{1,\text{j}}(s)\psi_{1,\text{k}}''(s)h_{31}'/\nu_{\text{o}}(s)\big)\text{ds}=h_{12,\text{j},\text{k}}^{(3)},$$

$$\int_{\text{o}}^{1}\big(\psi_{1,\text{j}}(s)\nu_{\text{o}}'(s)\psi_{1,\text{k}}'(s)h_{32}'''/\nu_{\text{o}}(s)^4\big)\text{ds}=h_{13,\text{j},\text{k}}^{(3)},$$

$$\int_{\text{o}}^{1}\big(\psi_{1,\text{j}}(s)\psi_{1,\text{k}}'(s)^2\psi_{1,\text{k}}''(s)/\nu_{\text{o}}(s)^2\big)\text{ds}=g_{1,\text{j},\text{k}},$$

$$\int_{\text{o}}^{1}\big(\psi_{1,\text{j}}(s)\psi_{1,\text{k}}''(s)h_{32}'/\nu_{\text{o}}(s)^3\big)\text{ds}=h_{14,\text{j},\text{k}}^{(3)},$$

$$\int_{\text{o}}^{1}\big(\psi_{1,\text{j}}(s)\psi_{1,\text{k}}''(s)h_{32}'/\nu_{\text{o}}(s)\big)\text{ds}=h_{15,\text{j},\text{k}}^{(3)},$$

$$\int_{\text{o}}^{1}\big(\psi_{1,\text{j}}(s)\nu_{\text{o}}'(s)\psi_{1,\text{k}}'(s)^2\psi_{1,\text{k}}'''(s)/\nu_{\text{o}}(s)^5\big)\text{ds}=g_{2,\text{j},\text{k}},$$

$$\int_{\text{o}}^{1}\big(\psi_{1,\text{j}}(s)\nu_{\text{o}}'(s)\psi_{1,\text{k}}'''(s)h_{31}'/\nu_{\text{o}}(s)^4\big)\text{ds}=h_{16,\text{j},\text{k}}^{(3)},$$

$$\int_{\text{o}}^{1}\big(\psi_{1,\text{j}}(s)\nu_{\text{o}}'(s)^2\psi_{1,\text{k}}''(s)h_{31}'/\nu_{\text{o}}(s)^5\big)\text{ds}=h_{17,\text{j},\text{k}}^{(3)},$$

$$\int_{\text{o}}^{1}\big(\psi_{1,\text{j}}(s)\nu_{\text{o}}'(s)\psi_{1,\text{k}}''(s)h_{31}''/\nu_{\text{o}}(s)^4\big)\text{ds}=h_{18,\text{j},\text{k}}^{(3)},$$

$$\int_{\text{o}}^{1}\big(\psi_{1,\text{j}}(s)\psi_{1,\text{k}}''(s)h_{32}'''/\nu_{\text{o}}(s)^3\big)\text{ds}=h_{19,\text{j},\text{k}}^{(3)},$$

$$\int_{\text{o}}^{1}\big(\psi_{1,\text{j}}(s)\nu_{\text{o}}'(s)^2\psi_{1,\text{k}}'(s)^2\psi_{1,\text{k}}''(s)/\nu_{\text{o}}(s)^6\big)\text{ds}=g_{3,\text{j},\text{k}},$$

$$\int_{\text{o}}^{1}\big(\psi_{1,\text{j}}(s)\psi_{1,\text{k}}'(s)\psi_{1,\text{k}}''(s)\psi_{1,\text{k}}'''(s)/\nu_{\text{o}}(s)^4\big)\text{ds}=g_{4,\text{j},\text{k}},$$

$$\int_{\text{o}}^{1}\big(\psi_{1,\text{j}}(s)\nu_{\text{o}}'(s)\psi_{1,\text{k}}'(s)h_{31}'''/\nu_{\text{o}}(s)^4\big)\text{ds}=h_{20,\text{j},\text{k}}^{(3)},$$

$$\int_{\text{o}}^{1}\big(\psi_{1,\text{j}}(s)\nu_{\text{o}}'(s)\psi_{1,\text{k}}'(s)h_{31}'/\nu_{\text{o}}(s)^2\big)\text{ds}=h_{21,\text{j},\text{k}}^{(3)},$$

$$\int_{\text{o}}^{1}\big(\psi_{1,\text{j}}(s)h_{31}(s)\psi_{1,\text{k}}'(s)/\nu_{\text{o}}(s)\big)\text{ds}=h_{22,\text{j},\text{k}}^{(3)},$$

$$\int_{\text{o}}^{1}\big(\psi_{1,\text{j}}(s)\psi_{1,\text{k}}'(s)h_{32}(s)/\nu_{\text{o}}(s)\big)\text{ds}=h_{23,\text{j},\text{k}}^{(3)},$$

$$\int_{\text{o}}^{1}\big(\psi_{1,\text{j}}(s)\psi_{1,\text{k}}'(s)h_{31}''''/\nu_{\text{o}}(s)^3\big)\text{ds}=h_{24,\text{j},\text{k}}^{(3)},$$

$$\int_{\text{o}}^{1}\big(\psi_{1,\text{j}}(s)\nu_{\text{o}}'(s)\nu_{\text{o}}''(s)\psi_{1,\text{k}}'(s)h_{32}'/\nu_{\text{o}}(s)^5\big)\text{ds}=h_{25,\text{j},\text{k}}^{(3)},$$

$$\int_{\text{o}}^{1}\big(\psi_{1,\text{j}}(s)\nu_{\text{o}}'(s)^2\psi_{1,\text{k}}'(s)h_{32}''/\nu_{\text{o}}(s)^5\big)\text{ds}=h_{26,\text{j},\text{k}}^{(3)},$$

$$\int_{\text{o}}^{1}\big(\psi_{1,\text{j}}(s)\psi_{1,\text{k}}'''(s)h_{32}''/\nu_{\text{o}}(s)^3\big)\text{ds}=h_{27,\text{j},\text{k}}^{(3)},$$

$$\int_{\text{o}}^{1}\big(\psi_{1,\text{j}}(s)\nu_{\text{o}}'(s)\psi_{1,\text{k}}'(s)\psi_{1,\text{k}}''(s)^2/\nu_{\text{o}}(s)^5\big)\text{ds}=g_{5,\text{j},\text{k}},$$

$$\int_{\text{o}}^{1}\big(\psi_{1,\text{j}}(s)\psi_{1,\text{k}}'(s)h_{32}''''/\nu_{\text{o}}(s)^3\big)\text{ds}=h_{28,\text{j},\text{k}}^{(3)},$$

$$\int_{\text{o}}^{1}\big(\psi_{1,\text{j}}(s)\psi_{1,\text{k}}'(s)h_{32}''/\nu_{\text{o}}(s)\big)\text{ds}=h_{29,\text{j},\text{k}}^{(3)},$$

$$\int_{\text{o}}^{1}\big(\psi_{1,\text{j}}(s)\psi_{1,\text{k}}''(s)^3/\nu_{\text{o}}(s)^4\big)\text{ds}=g_{6,\text{j},\text{k}},$$

$$\int_{\text{o}}^{1}\big(\psi_{1,\text{j}}(s)\psi_{1,\text{k}}'(s)h_{32}''/\nu_{\text{o}}(s)^2\big)\text{ds}=h_{30,\text{j},\text{k}}^{(3)},$$

$$\int_o^1 \big(\psi_{1,j}(s)\psi'_{1,k}(s)h''_{31}/\nu_o(s)\big)ds = h^{(3)}_{31,j,k},$$

$$\int_o^1 \big(\psi_{1,j}(s)\big(\nu'''_o(s)\big)\psi'_{1,k}(s)h'_{32}/\nu_o(s)^4\big)ds = h^{(3)}_{32,j,k},$$

$$\int_o^1 \big(\psi_{1,j}(s)\psi''_{1,k}(s)h'_{32}/\nu_o(s)^2\big)ds = h^{(3)}_{33,j,k},$$

$$\int_o^1 \big(\psi_{1,j}(s)\psi'_{1,k}(s)^2\psi''_{1,k}(s)/\nu_o(s)^3\big)ds = g_{7,j,k},$$

$$\int_o^1 \big(\psi_{1,j}(s)\nu'_o(s)\nu''_o(s)\psi'_{1,k}(s)^3/\nu_o(s)^6\big)ds = g_{8,j,k},$$

$$\int_o^1 \big(\psi_{1,j}(s)\nu''_o(s)\psi''_{1,k}(s)h'_{31}/\nu_o(s)^4\big)ds = h^{(3)}_{34,j,k},$$

$$\int_o^1 \big(\psi_{1,j}(s)\psi''''_{1,k}(s)h'_{31}/\nu_o(s)^3\big)ds = h^{(3)}_{35,j,k},$$

$$\int_o^1 \big(\psi_{1,j}(s)\nu'_o(s)^2\psi''_{1,k}(s)h'_{32}/\nu_o(s)^5\big)ds = h^{(3)}_{36,j,k},$$

$$\int_o^1 \big(\psi_{1,j}(s)\psi'_{1,k}(s)h''_{31}/\nu_o(s)^2\big)ds = h^{(3)}_{37,j,k},$$

$$\int_o^1 \big(\psi_{1,j}(s)\nu'_o(s)\psi'_{1,k}(s)^3/\nu_o(s)^5\big)ds = g_{9,j,k},$$

$$\int_o^1 \big(\psi_{1,j}(s)\nu'_o(s)^3\psi'_{1,k}(s)h'_{32}/\nu_o(s)^6\big)ds = h^{(3)}_{38,j,k},$$

$$\int_o^1 \big(\psi_{1,j}(s)\nu'_o(s)\psi'_{1,k}(s)h'_{32}/\nu_o(s)^4\big)ds = h^{(3)}_{39,j,k},$$

$$\int_o^1 \big(\psi_{1,j}(s)\nu'_o(s)\psi'_{1,k}(s)h'_{31}/\nu_o(s)^4\big)ds = h^{(3)}_{40,j,k},$$

$$\int_o^1 \big(\psi_{1,j}(s)\nu'_o(s)\psi'_{1,k}(s)h'_{32}/\nu_o(s)^2\big)ds = h^{(3)}_{40,j,k},$$

$$\int_o^1 \big(\psi_{1,j}(s)\nu'_o(s)\psi'_{1,k}(s)h'_{32}/\nu_o(s)^2\big)ds = h^{(3)}_{41,j,k},$$

$$\int_o^1 \big(\psi_{1,j}(s)\nu''_o(s)\psi'_{1,k}(s)^2\psi''_{1,k}(s)/\nu_o(s)^5\big)ds = g_{10,j,k},$$

$$\int_o^1 \big(\psi_{1,j}(s)\big(\nu'''_o(s)\big)\psi'_{1,k}(s)^3/\nu_o(s)^5\big)ds = g_{11,j,k},$$

$$\int_o^1 \big(\psi_{1,j}(s)\psi'_{1,k}(s)^2\psi''''_{1,k}(s)/\nu_o(s)^4\big)ds = g_{12,j,k},$$

$$\int_o^1 \big(\psi_{1,j}(s)\nu'_o(s)^2\psi'_{1,k}(s)h''_{31}/\nu_o(s)^5\big)ds = h^{(3)}_{42,j,k},$$

$$\int_o^1 \big(\psi_{1,j}(s)\nu'_o(s)\psi''_{1,k}(s)h''_{32}/\nu_o(s)^4\big)ds = h^{(3)}_{43,j,k},$$

$$\int_o^1 \big(\psi_{1,j}(s)\nu'_o(s)\nu''_o(s)\psi'_{1,k}(s)h'_{31}/\nu_o(s)^5\big)ds = h^{(3)}_{44,j,k},$$

$$\int_o^1 \big(\psi_{1,j}(s)\nu''_o(s)\psi''_{1,k}(s)h'_{32}/\nu_o(s)^4\big)ds = h^{(3)}_{44,j,k},$$

$$\int_o^1 \big(\psi_{1,j}(s)\nu''_o(s)\psi''_{1,k}(s)h'_{32}/\nu_o(s)^4\big)ds = h^{(3)}_{45,j,k},$$

$$\int_o^1 \big(\psi_{1,j}(s)\psi'_{1,k}(s)^2\psi''_{1,k}(s)/\nu_o(s)^4\big)ds = g_{13,j,k},$$

$$\int_o^1 \big(\psi_{1,j}(s)\nu'_o(s)\psi'_{1,k}(s)h'_{32}/\nu_o(s)^3\big)ds = h^{(3)}_{46,j,k},$$

$$\int_o^1 \big(\psi_{1,j}(s)\nu'_o(s)^3\psi'_{1,k}(s)h'_{31}/\nu_o(s)^6\big)ds = h^{(3)}_{47,j,k},$$

$$\int_o^1 \big(\psi_{1,j}(s)\nu'_o(s)^3\psi'_{1,k}(s)^3/\nu_o(s)^7\big)ds = g_{14,j,k},$$

$$\int_o^1 \big(\psi_{1,j}(s)\nu'_o(s)\psi'_{1,k}(s)^3/\nu_o(s)^3\big)ds = g_{15,j,k},$$

$$\int_o^1 \big(\psi_{1,j}(s)\psi_{1,k}(s)\psi'_{1,k}(s)^2/\nu_o(s)^2\big)ds = g_{16,j,k},$$

$$\int_o^1 \big(\psi_{1,j}(s)\nu'_o(s)\psi'_{1,k}(s)^3/\nu_o(s)^4\big)ds = g_{17,j,k},$$

$$\int_o^1 \big(\psi_{1,j}(s)\nu'_o(s)\psi'_{1,k}(s)h'_{31}/\nu_o(s)^3\big)ds = h^{(3)}_{48,j,k}.$$

Appendix 4: Boundary-value problems

$$\begin{aligned}
&h''''_{21}(s) - 4\nu'_o h'''_{21}(s)/\nu_o + 1/2\big(2\nu_o^6\lambda_a^2\alpha_2^2 \\
&\quad - 2\nu_o^7\lambda_a^2\alpha_2^2 + 2\nu_o^5\bar{\omega}_R^2 \\
&\quad + 16\nu_o^3\lambda_b^{(2)^2}\nu_o'^2\alpha_2^2 - 6\nu_o^4\lambda_b^{(2)^2}\nu''_o\alpha_2^2 \\
&\quad + 2\nu_o^5\omega_k^2\big)h''_{21}(s)/\big(\nu_o^5\alpha_2^2\lambda_b^{(2)^2}\big) \\
&\quad + 1/2\big(-4\nu_o^4\bar{\omega}_R^2\nu'_o - 16\nu_o^2\lambda_b^{(2)^2}\nu_o'^3\alpha_2^2 \\
&\quad - 2\nu_o^4\lambda_b^{(2)^2}\nu'''_o\alpha_2^2 - 4\nu_o^4\nu'_o\omega_k^2 - 2\nu_o^5\lambda_a^2\nu'_o\alpha_2^2 \\
&\quad + 14\nu_o^3\lambda_b^{(2)^2}\nu'_o\nu''_o\alpha_2^2\big)h'_{21}(s)/\big(\nu_o^5\alpha_2^2\lambda_b^{(2)^2}\big) \\
&\quad - \nu_o^2 h_{21}(s)\omega_k^2/\lambda_b^{(2)^2}
\end{aligned}$$

$$+ 1/2(-96\lambda_{\mathrm{b}}^{(2)^2}\nu_{\mathrm{o}}'^2\psi_{1,\mathrm{k}}'(s)\alpha_2^2\nu_{\mathrm{o}}^2h_{32}''(s)$$
$$- 112\lambda_{\mathrm{b}}^{(2)^2}\nu_{\mathrm{o}}''\nu_{\mathrm{o}}'\psi_{1,\mathrm{k}}'(s)\alpha_2^2\nu_{\mathrm{o}}^2h_{32}'(s)$$
$$- 56\lambda_{\mathrm{b}}^{(2)^2}\nu_{\mathrm{o}}''\nu_{\mathrm{o}}'\psi_{1,\mathrm{k}}'(s)\alpha_2^2\nu_{\mathrm{o}}^2h_{31}'(s)$$
$$+ 28\lambda_{\mathrm{b}}^{(2)^2}\nu_{\mathrm{o}}'\psi_{1,\mathrm{k}}'(s)\alpha_2^2\nu_{\mathrm{o}}^3h_{32}'''(s)$$
$$+ \psi_{1,\mathrm{k}}''(s)\Gamma_{\mathrm{k,k}}\omega_{\mathrm{k}}\nu_{\mathrm{o}}^5$$
$$+ 64\lambda_{\mathrm{b}}^{(2)^2}\nu_{\mathrm{o}}'\psi_{1,\mathrm{k}}''(s)\alpha_2^2\nu_{\mathrm{o}}^3h_{32}''(s)$$
$$- 128\lambda_{\mathrm{b}}^{(2)^2}\nu_{\mathrm{o}}'^2\psi_{1,\mathrm{k}}''(s)\alpha_2^2\nu_{\mathrm{o}}^2h_{32}'(s)$$
$$+ 160\lambda_{\mathrm{b}}^{(2)^2}\nu_{\mathrm{o}}'^3\psi_{1,\mathrm{k}}'(s)\alpha_2^2\nu_{\mathrm{o}}h_{32}'(s)$$
$$- 4\lambda_{\mathrm{a}}^2\nu_{\mathrm{o}}'\psi_{1,\mathrm{k}}'(s)\alpha_2^2\nu_{\mathrm{o}}^5h_{32}'(s)$$
$$+ 12\lambda_{\mathrm{a}}^2\nu_{\mathrm{o}}'\psi_{1,\mathrm{k}}'(s)\alpha_2^2\nu_{\mathrm{o}}^4h_{32}'(s)$$
$$+ 36\lambda_{\mathrm{b}}^{(2)^2}\nu_{\mathrm{o}}''\psi_{1,\mathrm{k}}''(s)\alpha_2^2\nu_{\mathrm{o}}^3h_{32}'(s)$$
$$- 8\bar{\omega}_{\mathrm{R}}^2\psi_{1,\mathrm{k}}'(s)\nu_{\mathrm{o}}^4h_{32}''(s)$$
$$+ 20\nu_{\mathrm{o}}'\psi_{1,\mathrm{k}}'(s)^3\omega_{\mathrm{k}}^2\nu_{\mathrm{o}}^2 + 36\bar{\omega}_{\mathrm{R}}^2\nu_{\mathrm{o}}'\psi_{1,\mathrm{k}}'(s)^3\nu_{\mathrm{o}}^2$$
$$+ 180\lambda_{\mathrm{b}}^{(2)^2}\nu_{\mathrm{o}}'^3\psi_{1,\mathrm{k}}'(s)^3\alpha_2^2 - 15\psi_{1,\mathrm{k}}'(s)^2\omega_{\mathrm{k}}^2\psi_{1,\mathrm{k}}''(s)\nu_{\mathrm{o}}^3$$
$$- 16\lambda_{\mathrm{b}}^{(2)^2}\psi_{1,\mathrm{k}}'''(s)\alpha_2^2\nu_{\mathrm{o}}^4h_{32}''(s)$$
$$- 8\psi_{1,\mathrm{k}}''(s)\omega_{\mathrm{k}}^2\nu_{\mathrm{o}}^4h_{32}'(s) - 12\lambda_{\mathrm{b}}^{(2)^2}\psi_{1,\mathrm{k}}''(s)\alpha_2^2\nu_{\mathrm{o}}^4h_{32}'''(s)$$
$$- 8\lambda_{\mathrm{b}}^{(2)^2}\psi_{1,\mathrm{k}}''''(s)\alpha_2^2\nu_{\mathrm{o}}^4h_{32}'(s)$$
$$- 4\lambda_{\mathrm{b}}^{(2)^2}\psi_{1,\mathrm{k}}'(s)\alpha_2^2\nu_{\mathrm{o}}^4h_{32}''''(s)$$
$$+ 4\lambda_{\mathrm{a}}^2\psi_{1,\mathrm{k}}'(s)\alpha_2^2\nu_{\mathrm{o}}^6h_{32}''(s) - 4\psi_{1,\mathrm{k}}''(s)h_{31}'(s)\omega_{\mathrm{k}}^2\nu_{\mathrm{o}}^4$$
$$- 4\psi_{1,\mathrm{k}}'(s)\omega_{\mathrm{k}}^2\nu_{\mathrm{o}}^4h_{31}''(s) - 4\bar{\omega}_{\mathrm{R}}^2\psi_{1,\mathrm{k}}''(s)\nu_{\mathrm{o}}^4h_{31}'(s)$$
$$+ 48\lambda_{\mathrm{b}}^{(2)^2}\nu_{\mathrm{o}}'\psi_{1,\mathrm{k}}'''(s)\alpha_2^2\nu_{\mathrm{o}}^3h_{32}'(s)$$
$$- 4\lambda_{\mathrm{a}}^2\psi_{1,\mathrm{k}}'(s)\alpha_2^2\nu_{\mathrm{o}}^5h_{32}''(s)$$
$$+ 28\lambda_{\mathrm{b}}^{(2)^2}\nu_{\mathrm{o}}''\psi_{1,\mathrm{k}}'(s)\alpha_2^2\nu_{\mathrm{o}}^3h_{32}''(s)$$
$$+ 6\lambda_{\mathrm{a}}^2\nu_{\mathrm{o}}'\psi_{1,\mathrm{k}}'(s)\alpha_2^2\nu_{\mathrm{o}}^4h_{31}'(s)$$
$$- 2\lambda_{\mathrm{a}}^2\nu_{\mathrm{o}}'\psi_{1,\mathrm{k}}'(s)\alpha_2^2\nu_{\mathrm{o}}^5h_{31}'(s)$$
$$+ 80\lambda_{\mathrm{b}}^{(2)^2}\nu_{\mathrm{o}}'^3\psi_{1,\mathrm{k}}'(s)\alpha_2^2\nu_{\mathrm{o}}h_{31}'(s)$$
$$+ 4\bar{\omega}_{\mathrm{R}}^2\psi_{1,\mathrm{k}}'(s)\alpha_2^2\nu_{\mathrm{o}}^6h_{32}(s)$$
$$+ 24\bar{\omega}_{\mathrm{R}}^2\nu_{\mathrm{o}}'\psi_{1,\mathrm{k}}'(s)\nu_{\mathrm{o}}^3h_{32}'(s)$$
$$+ 24\nu_{\mathrm{o}}'\psi_{1,\mathrm{k}}'(s)\omega_{\mathrm{k}}^2\nu_{\mathrm{o}}^3h_{32}'(s)$$
$$+ 4\lambda_{\mathrm{a}}^2\psi_{1,\mathrm{k}}''(s)\alpha_2^2\nu_{\mathrm{o}}^6h_{32}'(s)$$
$$- 8\lambda_{\mathrm{a}}^2\psi_{1,\mathrm{k}}''(s)\alpha_2^2\nu_{\mathrm{o}}^5h_{32}'(s) + 2\lambda_{\mathrm{a}}^2\psi_{1,\mathrm{k}}''(s)\alpha_2^2\nu_{\mathrm{o}}^6h_{31}'(s)$$
$$- 27\bar{\omega}_{\mathrm{R}}^2\psi_{1,\mathrm{k}}'(s)^2\psi_{1,\mathrm{k}}''(s)\nu_{\mathrm{o}}^3 - 12\lambda_{\mathrm{b}}^{(2)^2}\psi_{1,\mathrm{k}}''(s)^3\alpha_2^2\nu_{\mathrm{o}}^3$$
$$- 6\lambda_{\mathrm{b}}^{(2)^2}\psi_{1,\mathrm{k}}''(s)\alpha_2^2\nu_{\mathrm{o}}^4h_{31}'''(s)$$
$$- 12\lambda_{\mathrm{a}}^2\psi_{1,\mathrm{k}}'(s)^2\psi_{1,\mathrm{k}}''(s)\alpha_2^2\nu_{\mathrm{o}}^4$$
$$- 4\lambda_{\mathrm{a}}^2\psi_{1,\mathrm{k}}''(s)\alpha_2^2\nu_{\mathrm{o}}^5h_{31}'(s)$$
$$- 2\lambda_{\mathrm{b}}^{(2)^2}\psi_{1,\mathrm{k}}'(s)\alpha_2^2\nu_{\mathrm{o}}^4h_{31}''''(s)$$
$$- 2\lambda_{\mathrm{a}}^2\psi_{1,\mathrm{k}}'(s)\alpha_2^2\nu_{\mathrm{o}}^5h_{31}''(s)$$
$$+ 12\lambda_{\mathrm{b}}^{(2)^2}\nu_{\mathrm{o}}'''\psi_{1,\mathrm{k}}'(s)\alpha_2^2\nu_{\mathrm{o}}^3h_{32}'(s)$$
$$+ 78\lambda_{\mathrm{b}}^{(2)^2}\nu_{\mathrm{o}}'\psi_{1,\mathrm{k}}'(s)^2\psi_{1,\mathrm{k}}'''(s)\alpha_2^2\nu_{\mathrm{o}}^2$$
$$- 42\lambda_{\mathrm{b}}^{(2)^2}\psi_{1,\mathrm{k}}'(s)\psi_{1,\mathrm{k}}'''(s)\psi_{1,\mathrm{k}}''(s)\alpha_2^2\nu_{\mathrm{o}}^3$$
$$- 8\lambda_{\mathrm{b}}^{(2)^2}\psi_{1,\mathrm{k}}'''(s)\alpha_2^2\nu_{\mathrm{o}}^4h_{31}''(s)$$
$$+ 2\lambda_{\mathrm{a}}^2\psi_{1,\mathrm{k}}'(s)\alpha_2^2\nu_{\mathrm{o}}^6h_{31}''(s)$$
$$- 9\lambda_{\mathrm{b}}^{(2)^2}\psi_{1,\mathrm{k}}'(s)^2\psi_{1,\mathrm{k}}''''(s)\alpha_2^2\nu_{\mathrm{o}}^3$$
$$+ 9\lambda_{\mathrm{b}}^{(2)^2}\nu_{\mathrm{o}}'''\psi_{1,\mathrm{k}}'(s)^3\alpha_2^2\nu_{\mathrm{o}}^2$$
$$+ 14\lambda_{\mathrm{b}}^{(2)^2}\nu_{\mathrm{o}}''\psi_{1,\mathrm{k}}'(s)\alpha_2^2\nu_{\mathrm{o}}^3h_{31}''(s)$$
$$+ 18\lambda_{\mathrm{b}}^{(2)^2}\nu_{\mathrm{o}}''\psi_{1,\mathrm{k}}''(s)\alpha_2^2\nu_{\mathrm{o}}^3h_{31}'(s)$$
$$+ 32\lambda_{\mathrm{b}}^{(2)^2}\nu_{\mathrm{o}}'\psi_{1,\mathrm{k}}''(s)\alpha_2^2\nu_{\mathrm{o}}^3h_{31}''(s)$$
$$+ 6\lambda_{\mathrm{b}}^{(2)^2}\nu_{\mathrm{o}}'''\psi_{1,\mathrm{k}}'(s)\alpha_2^2\nu_{\mathrm{o}}^3h_{31}'(s)$$
$$+ 69\lambda_{\mathrm{b}}^{(2)^2}\nu_{\mathrm{o}}''\psi_{1,\mathrm{k}}'(s)^2\psi_{1,\mathrm{k}}''(s)\alpha_2^2\nu_{\mathrm{o}}^2$$
$$- 48\lambda_{\mathrm{b}}^{(2)^2}\nu_{\mathrm{o}}'^2\psi_{1,\mathrm{k}}'(s)\alpha_2^2\nu_{\mathrm{o}}^2h_{31}''(s)$$
$$+ 132\lambda_{\mathrm{b}}^{(2)^2}\nu_{\mathrm{o}}'\psi_{1,\mathrm{k}}'(s)\psi_{1,\mathrm{k}}''(s)^2\alpha_2^2\nu_{\mathrm{o}}^2$$
$$- 105\lambda_{\mathrm{b}}^{(2)^2}\nu_{\mathrm{o}}''\nu_{\mathrm{o}}'\psi_{1,\mathrm{k}}'(s)^3\alpha_2^2\nu_{\mathrm{o}}$$
$$+ 24\lambda_{\mathrm{b}}^{(2)^2}\nu_{\mathrm{o}}'\psi_{1,\mathrm{k}}'''(s)\alpha_2^2\nu_{\mathrm{o}}^3h_{31}'(s)$$
$$- 64\lambda_{\mathrm{b}}^{(2)^2}\nu_{\mathrm{o}}'^2\psi_{1,\mathrm{k}}''(s)\alpha_2^2\nu_{\mathrm{o}}^2h_{31}'(s)$$
$$+ 14\lambda_{\mathrm{b}}^{(2)^2}\nu_{\mathrm{o}}'\psi_{1,\mathrm{k}}'(s)\alpha_2^2\nu_{\mathrm{o}}^3h_{31}'''(s)$$

$$
\begin{aligned}
&-4\lambda_{\mathrm{b}}^{(2)^2}\psi''''_{1,\mathrm{k}}(s)\alpha_2^2\nu_{\mathrm{o}}^4h'_{31}(s)\\
&-8\psi'_{1,\mathrm{k}}(s)\omega_{\mathrm{k}}^2\nu_{\mathrm{o}}^4h''_{32}(s)\\
&+9\lambda_{\mathrm{a}}^2\psi'_{1,\mathrm{k}}(s)^2\psi''_{1,\mathrm{k}}(s)\alpha_2^2\nu_{\mathrm{o}}^5\\
&-\psi_{1,\mathrm{k}}(s)\Gamma_{\mathrm{k,k}}\omega_{\mathrm{k}}\nu_{\mathrm{o}}^7\alpha_2^2-2\nu'_{\mathrm{o}}\psi'_{1,\mathrm{k}}(s)\Gamma_{\mathrm{k,k}}\omega_{\mathrm{k}}\nu_{\mathrm{o}}^4\\
&-8\bar{\omega}_{\mathrm{R}}^2\psi''_{1,\mathrm{k}}(s)\nu_{\mathrm{o}}^4h'_{32}(s)\\
&+12\lambda_{\mathrm{a}}^2\nu'_{\mathrm{o}}\psi'_{1,\mathrm{k}}(s)^3\alpha_2^2\nu_{\mathrm{o}}^3-6\lambda_{\mathrm{a}}^2\nu'_{\mathrm{o}}\psi'_{1,\mathrm{k}}(s)^3\alpha_2^2\nu_{\mathrm{o}}^4\\
&+3\psi_{1,\mathrm{k}}(s)\omega_{\mathrm{k}}^2\psi'_{1,\mathrm{k}}(s)^2\alpha_2^2\nu_{\mathrm{o}}^5\\
&+2\bar{\omega}_{\mathrm{R}}^2\psi'_{1,\mathrm{k}}(s)\alpha_2^2\nu_{\mathrm{o}}^6h_{31}(s)\\
&+12\bar{\omega}_{\mathrm{R}}^2\nu'_{\mathrm{o}}\psi'_{1,\mathrm{k}}(s)\nu_{\mathrm{o}}^3h'_{31}(s)\\
&+8h_{31}(s)\omega_{\mathrm{k}}^2\psi'_{1,\mathrm{k}}(s)\alpha_2^2\nu_{\mathrm{o}}^6\\
&+12\nu'_{\mathrm{o}}\psi'_{1,\mathrm{k}}(s)\omega_{\mathrm{k}}^2\nu_{\mathrm{o}}^3h'_{31}(s)\\
&-300\lambda_{\mathrm{b}}^{(2)^2}\nu_{\mathrm{o}}'^2\psi'_{1,\mathrm{k}}(s)^2\psi''_{1,\mathrm{k}}(s)\alpha_2^2\nu_{\mathrm{o}}\\
&-4\bar{\omega}_{\mathrm{R}}^2\psi'_{1,\mathrm{k}}(s)\nu_{\mathrm{o}}^4h''_{31}(s))/\left(\nu_{\mathrm{o}}^5\alpha_2^2\lambda_{\mathrm{b}}^{(2)^2}\right)=0, \qquad (79)
\end{aligned}
$$

$$
\begin{aligned}
&h''''_{23}(s)-4\nu'_{\mathrm{o}}h'''_{23}(s)/\nu_{\mathrm{o}}\\
&+1/2\big(-2\nu_{\mathrm{o}}^7\lambda_{\mathrm{a}}^2\alpha_2^2-6\nu_{\mathrm{o}}^4\lambda_{\mathrm{b}}^{(2)^2}\nu''_{\mathrm{o}}\alpha_2^2\\
&+18\nu_{\mathrm{o}}^5\omega_{\mathrm{k}}^2+2\nu_{\mathrm{o}}^6\lambda_{\mathrm{a}}^2\alpha_2^2+16\nu_{\mathrm{o}}^3\lambda_{\mathrm{b}}^{(2)^2}\nu_{\mathrm{o}}'^2\alpha_2^2\\
&+2\nu_{\mathrm{o}}^5\bar{\omega}_{\mathrm{R}}^2\big)h''_{23}(s)/\left(\nu_{\mathrm{o}}^5\alpha_2^2\lambda_{\mathrm{b}}{}^{(2)^2}\right)\\
&+1/2\big(-2\nu_{\mathrm{o}}^5\lambda_{\mathrm{a}}^2\nu'_{\mathrm{o}}\alpha_2^2-16\nu_{\mathrm{o}}^2\lambda_{\mathrm{b}}^{(2)^2}\nu_{\mathrm{o}}'^3\alpha_2^2\\
&-4\nu_{\mathrm{o}}^4\bar{\omega}_{\mathrm{R}}^2\nu'_{\mathrm{o}}+14\nu_{\mathrm{o}}^3\lambda_{\mathrm{b}}^{(2)^2}\nu'_{\mathrm{o}}\nu''_{\mathrm{o}}\alpha_2^2-36\nu_{\mathrm{o}}^4\nu'_{\mathrm{o}}\omega_{\mathrm{k}}^2\\
&-2\nu_{\mathrm{o}}^4\lambda_{\mathrm{b}}^{(2)^2}\nu'''_{\mathrm{o}}\alpha_2^2\big)h'_{23}(s)/\left(\nu_{\mathrm{o}}^5\alpha_2^2\lambda_{\mathrm{b}}{}^{(2)^2}\right)\\
&-9\nu_{\mathrm{o}}^2h_{23}(s)\omega_{\mathrm{k}}^2/\lambda_{\mathrm{b}}^{(2)^2}\\
&+1/2(-56\lambda_{\mathrm{b}}^{(2)^2}\nu''_{\mathrm{o}}\nu'_{\mathrm{o}}\psi'_{1,\mathrm{k}}(s)\alpha_2^2\nu_{\mathrm{o}}^2h'_{31}(s)\\
&+2\lambda_{\mathrm{a}}^2\psi''_{1,\mathrm{k}}(s)\alpha_2^2\nu_{\mathrm{o}}^6h'_{31}(s)\\
&-6\lambda_{\mathrm{b}}^{(2)^2}\psi''_{1,\mathrm{k}}(s)\alpha_2^2\nu_{\mathrm{o}}^4h'''_{31}(s)\\
&-4\lambda_{\mathrm{a}}^2\psi'_{1,\mathrm{k}}(s)^2\psi''_{1,\mathrm{k}}(s)\alpha_2^2\nu_{\mathrm{o}}^4\\
&-4\lambda_{\mathrm{a}}^2\psi''_{1,\mathrm{k}}(s)\alpha_2^2\nu_{\mathrm{o}}^5h'_{31}(s)\\
&-2\lambda_{\mathrm{b}}^{(2)^2}\psi'_{1,\mathrm{k}}(s)\alpha_2^2\nu_{\mathrm{o}}^4h''''_{31}(s)\\
&-2\lambda_{\mathrm{a}}^2\psi'_{1,\mathrm{k}}(s)\alpha_2^2\nu_{\mathrm{o}}^5h''_{31}(s)\\
&+2\lambda_{\mathrm{a}}^2\psi'_{1,\mathrm{k}}(s)\alpha_2^2\nu_{\mathrm{o}}^6h''_{31}(s)\\
&-8\lambda_{\mathrm{b}}^{(2)^2}\psi'''_{1,\mathrm{k}}(s)\alpha_2^2\nu_{\mathrm{o}}^4h''_{31}(s)\\
&+3\lambda_{\mathrm{b}}^{(2)^2}\nu'''_{\mathrm{o}}\psi'_{1,\mathrm{k}}(s)^3\alpha_2^2\nu_{\mathrm{o}}^2\\
&-3\lambda_{\mathrm{b}}^{(2)^2}\psi'_{1,\mathrm{k}}(s)^2\psi''''_{1,\mathrm{k}}(s)\alpha_2^2\nu_{\mathrm{o}}^3\\
&-4\lambda_{\mathrm{b}}^{(2)^2}\psi''''_{1,\mathrm{k}}(s)\alpha_2^2\nu_{\mathrm{o}}^4h'_{31}(s)\\
&+3\lambda_{\mathrm{a}}^2\psi'_{1,\mathrm{k}}(s)^2\psi''_{1,\mathrm{k}}(s)\alpha_2^2\nu_{\mathrm{o}}^5\\
&+24\lambda_{\mathrm{b}}^{(2)^2}\nu'_{\mathrm{o}}\psi'''_{1,\mathrm{k}}(s)\alpha_2^2\nu_{\mathrm{o}}^3h'_{31}(s)\\
&-100\lambda_{\mathrm{b}}^{(2)^2}\nu_{\mathrm{o}}'^2\psi'_{1,\mathrm{k}}(s)^2\psi''_{1,\mathrm{k}}(s)\alpha_2^2\nu_{\mathrm{o}}\\
&-64\lambda_{\mathrm{b}}^{(2)^2}\nu_{\mathrm{o}}'^2\psi''_{1,\mathrm{k}}(s)\alpha_2^2\nu_{\mathrm{o}}^2h'_{31}(s)\\
&+28\nu'_{\mathrm{o}}\psi'_{1,\mathrm{k}}(s)^3\omega_{\mathrm{k}}^2\nu_{\mathrm{o}}^2+12\bar{\omega}_{\mathrm{R}}^2\nu'_{\mathrm{o}}\psi'_{1,\mathrm{k}}(s)^3\nu_{\mathrm{o}}^2\\
&+60\lambda_{\mathrm{b}}^{(2)^2}\nu_{\mathrm{o}}'^3\psi'_{1,\mathrm{k}}(s)^3\alpha_2^2\\
&+14\lambda_{\mathrm{b}}^{(2)^2}\nu'_{\mathrm{o}}\psi'_{1,\mathrm{k}}(s)\alpha_2^2\nu_{\mathrm{o}}^3h'''_{31}(s)\\
&-35\lambda_{\mathrm{b}}^{(2)^2}\nu''_{\mathrm{o}}\nu'_{\mathrm{o}}\psi'_{1,\mathrm{k}}(s)^3\alpha_2^2\nu_{\mathrm{o}}\\
&+44\lambda_{\mathrm{b}}^{(2)^2}\nu'_{\mathrm{o}}\psi'_{1,\mathrm{k}}(s)\psi''_{1,\mathrm{k}}(s)^2\alpha_2^2\nu_{\mathrm{o}}^2\\
&+14\lambda_{\mathrm{b}}^{(2)^2}\nu''_{\mathrm{o}}\psi'_{1,\mathrm{k}}(s)\alpha_2^2\nu_{\mathrm{o}}^3h''_{31}(s)\\
&+6\lambda_{\mathrm{a}}^2\nu'_{\mathrm{o}}\psi'_{1,\mathrm{k}}(s)\alpha_2^2\nu_{\mathrm{o}}^4h'_{31}(s)\\
&-2\lambda_{\mathrm{a}}^2\nu'_{\mathrm{o}}\psi'_{1,\mathrm{k}}(s)\alpha_2^2\nu_{\mathrm{o}}^5h'_{31}(s)\\
&+80\lambda_{\mathrm{b}}^{(2)^2}\nu_{\mathrm{o}}'^3\psi'_{1,\mathrm{k}}(s)\alpha_2^2\nu_{\mathrm{o}}h'_{31}(s)\\
&+18\lambda_{\mathrm{b}}^{(2)^2}\nu''_{\mathrm{o}}\psi''_{1,\mathrm{k}}(s)\alpha_2^2\nu_{\mathrm{o}}^3h'_{31}(s)\\
&+32\lambda_{\mathrm{b}}^{(2)^2}\nu'_{\mathrm{o}}\psi''_{1,\mathrm{k}}(s)\alpha_2^2\nu_{\mathrm{o}}^3h''_{31}(s)\\
&+6\lambda_{\mathrm{b}}^{(2)^2}\nu'''_{\mathrm{o}}\psi'_{1,\mathrm{k}}(s)\alpha_2^2\nu_{\mathrm{o}}^3h'_{31}(s)\\
&+23\lambda_{\mathrm{b}}^{(2)^2}\nu''_{\mathrm{o}}\psi'_{1,\mathrm{k}}(s)^2\psi''_{1,\mathrm{k}}(s)\alpha_2^2\nu_{\mathrm{o}}^2\\
&-48\lambda_{\mathrm{b}}^{(2)^2}\nu_{\mathrm{o}}'^2\psi'_{1,\mathrm{k}}(s)\alpha_2^2\nu_{\mathrm{o}}^2h''_{31}(s)\\
&-20\psi'_{1,\mathrm{k}}(s)\omega_{\mathrm{k}}^2\nu_{\mathrm{o}}^4h''_{31}(s)\\
&-20\psi''_{1,\mathrm{k}}(s)h'_{31}(s)\omega_{\mathrm{k}}^2\nu_{\mathrm{o}}^4-4\lambda_{\mathrm{b}}^{(2)^2}\psi''_{1,\mathrm{k}}(s)^3\alpha_2^2\nu_{\mathrm{o}}^3\\
&-9\bar{\omega}_{\mathrm{R}}^2\psi'_{1,\mathrm{k}}(s)^2\psi''_{1,\mathrm{k}}(s)\nu_{\mathrm{o}}^3\\
&-4\bar{\omega}_{\mathrm{R}}^2\psi'_{1,\mathrm{k}}(s)\nu_{\mathrm{o}}^4h''_{31}(s)
\end{aligned}
$$

$$+26\lambda_{\mathrm{b}}^{(2)^2}\nu_{\mathrm{o}}'\psi_{1,\mathrm{k}}'(s)^2\psi_{1,\mathrm{k}}'''(s)\alpha_2^2\nu_{\mathrm{o}}^2$$
$$-14\lambda_{\mathrm{b}}^{(2)^2}\psi_{1,\mathrm{k}}'(s)\psi_{1,\mathrm{k}}'''(s)\psi_{1,\mathrm{k}}''(s)\alpha_2^2\nu_{\mathrm{o}}^3$$
$$-4\bar{\omega}_{\mathrm{R}}^2\psi_{1,\mathrm{k}}''(s)\nu_{\mathrm{o}}^4h_{31}'(s)$$
$$-21\psi_{1,\mathrm{k}}'(s)^2\omega_{\mathrm{k}}^2\psi_{1,\mathrm{k}}''(s)\nu_{\mathrm{o}}^3$$
$$+4\lambda_{\mathrm{a}}^2\nu_{\mathrm{o}}'\psi_{1,\mathrm{k}}'(s)^3\alpha_2^2\nu_{\mathrm{o}}^3-2\lambda_{\mathrm{a}}^2\nu_{\mathrm{o}}'\psi_{1,\mathrm{k}}'(s)^3\alpha_2^2\nu_{\mathrm{o}}^4$$
$$+\psi_{1,\mathrm{k}}(s)\omega_{\mathrm{k}}^2\psi_{1,\mathrm{k}}'(s)^2\alpha_2^2\nu_{\mathrm{o}}^5$$
$$+2\bar{\omega}_{\mathrm{R}}^2\psi_{1,\mathrm{k}}'(s)\alpha_2^2\nu_{\mathrm{o}}^6h_{31}(s)$$
$$+12\bar{\omega}_{\mathrm{R}}^2\nu_{\mathrm{o}}'\psi_{1,\mathrm{k}}'(s)\nu_{\mathrm{o}}^3h_{31}'(s)$$
$$+8h_{31}(s)\omega_{\mathrm{k}}^2\psi_{1,\mathrm{k}}'(s)\alpha_2^2\nu_{\mathrm{o}}^6$$
$$+60\nu_{\mathrm{o}}'\psi_{1,\mathrm{k}}'(s)\omega_{\mathrm{k}}^2\nu_{\mathrm{o}}^3h_{31}'(s))/\big(\nu_{\mathrm{o}}^5\alpha_2^2\lambda_{\mathrm{b}}^{(2)^2}\big)=0. \tag{80}$$

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