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The conservation laws and self-consistent sources for a super-Yang hierarchy

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Abstract In the present paper, a super-extension of the Yang hierarchy is proposed by super-matrix Lie algebras, and the super-Yang hierarchy with self-consistent sources is established. Furthermore, we establish infinitely many conservation laws of the super-integrable hierarchy. The methods presented by us can be generalized to other nonlinear equation hierarchies with self-consistent sources.

Keywords Conservation law · Self-consistent sources · Super-Yang hierarchy · Fermi variables

1 Introduction

As soliton theory has been applied in almost all the natural sciences especially in the physics branches, such as fluid mechanics, plasma physics, nonlinear optics, classical field theory, and quantum field theory, the study on integrable systems has attracted the interest of many mathematicians and physicists. Conservation laws are an important feature of integrable systems in soliton theory. An infinite number of conservation laws for the KdV equation was first discovered by Miura, Gardner, and Kruscal in 1968 [1]. Then a lot of methods have been developed to find those conservation laws, mainly due to the contribution of Wadati et al. [2–4].

Super-integrable systems are an important part in integrable systems. A lot of scholars and experts have done research on the topic, and have achieved extensive results. In 1990, Hu presented the super-trace identity in his Ph.D. thesis [5]. Then the super-trace identity was mentioned and applied to establish the super-Hamiltonian structure of super-integrable systems, but he did not give its rigorous proof [6, 7]. Recently, Ma gave a systematic proof of the super-trace identity and the expression of its constant γ [8]. For application, he also obtained the super-Hamiltonian structures of the super-AKNS hierarchy and the super-Dirac hierarchy. Then many super-integrable hierarchies and their super-Hamiltonian structures are presented [9–14].

Soliton equations with self-consistent sources are also an important topic in recent research of soliton theory. Physically, the sources may result in solitary waves with a nonconstant velocity and, therefore, lead to a variety of dynamics of physical models. They are usually used to describe interactions between different solitary waves. Very recently, self-consistent sources for a super-CKdV equation hierarchy and super G-J hierarchy are presented [15–17]. In this paper, the self-consistent sources of super-Yang hierarchy is pre-

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sented based on the theory of self-consistent sources. In fact, the self-consistent sources of the soliton hierarchy is just constraint flow and good supplements to related research in the field of noncommuting soliton equations [18–21]. Finally, we obtain infinitely many conservation laws for the hierarchy. In the calculation, it is worth noting that odd variables in the spectral problem are Fermi variables. The operation between extended Fermi variables constitute a Grassmann algebra.

2 A super-soliton hierarchy with self-consistent sources

In the following, we consider a set of super-Lie algebras G,

$$e_1 = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad e_2 = \frac{1}{2}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$
$$e_3 = \frac{1}{2}\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{1}$$
$$e_4 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}, \qquad e_5 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

along with the communicative operation

$$[e_1, e_2] = e_3, \qquad [e_1, e_3] = e_2, \qquad [e_2, e_3] = -e_1,$$
$$[e_1, e_4] = [e_2, e_5] = [e_3, e_5] = \frac{e_4}{2},$$
$$[e_5, e_1] = [e_2, e_4] = [e_4, e_3] = \frac{e_5}{2},$$
$$[e_4, e_5]_+ = [e_5, e_4]_+ = \frac{e_1}{2},$$
$$[e_4, e_4]_+ = -\frac{e_2 + e_3}{2},$$
$$[e_5, e_5]_+ = \frac{e_2 - e_3}{2}.$$

Consider an auxiliary linear problem

$$\begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix}_x = U \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix}, \quad U(u, \lambda) = R_1 + \sum_{i=1}^{5} u_i e_i(\lambda),$$
$$\begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix}_{t_n} = V_n(u, \lambda) \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix} \tag{2}$$

where $u = (u_1, \ldots, u_s)^{\mathrm{T}}, \phi_i = \phi(x, t)$ are field variables on $x \in R, t \in R$; $e_i = e_i(\lambda) \in \tilde{sl}(3)$ $(i = 1, 2, \ldots, 5)$; R_1 is a pseudoregular element.

The compatibility of Eq. (2) gives rise to the following zero curvature equations:

$$U_{nt} - V_x + [U_n, V_n] = 0, \quad n = 1, 2, \ldots. \tag{3}$$

If an equation

$$u_t = K(u) \tag{4}$$

can be worked out through Eq. (3), Eq. (3) is called a zero curvature representation of the super-evolution Eq. (4). If there is a super-Hamiltonian operator J and a functional J such that

$$u_t = K(u) = \frac{\delta H_{n+1}}{\delta u}, \tag{5}$$

where

$$\frac{\delta H_n}{\delta u} = L \frac{\delta H_{n-1}}{\delta u} = \cdots = L^n \frac{\delta H_0}{\delta u}, \quad n = 1, 2, \ldots,$$
$$\frac{\delta}{\delta u} = \left(\frac{\delta}{\delta u_1}, \ldots, \frac{\delta}{\delta u_5} \right)^{\mathrm{T}} \tag{6}$$

then Eq. (4) is called a super-Hamiltonian. If so, we say that Eq. (4) has a super-Hamiltonian structure.

Based on Eq. (2), we consider a new auxiliary linear problem. For N distinct λ_j, $j = 1, \ldots, N$, the systems of (2) become the following:

$$\begin{pmatrix} \phi_{1j} \\ \phi_{2j} \\ \phi_{3j} \end{pmatrix}_x = U(u, \lambda_j) \begin{pmatrix} \phi_{1j} \\ \phi_{2j} \\ \phi_{3j} \end{pmatrix} = \sum_{i=1}^{5} u_i e_i(\lambda) \begin{pmatrix} \phi_{1j} \\ \phi_{2j} \\ \phi_{3j} \end{pmatrix},$$
$$\begin{pmatrix} \phi_{1j} \\ \phi_{2j} \\ \phi_{3j} \end{pmatrix}_{t_n} = V_n(u, \lambda_j) \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix} \tag{7}$$
$$= \left[\sum_{m=0}^{n} V_m(u) \lambda_j^{n-m} + \triangle_n(u, \lambda_j) \right] \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix}.$$

Based on the result in [22], we show that the following equation

$$\frac{\delta H_k}{\delta u} + \sum_{j=1}^{N} \alpha_j \frac{\delta \lambda_j}{\delta u} = 0 \tag{8}$$

holds true, where α_j are constants, and Eq. (8) determines a finite-dimensional invariant set for the flows (6).

For (7), we may know that

$$\frac{\delta \lambda_j}{\delta u_i} = \frac{1}{3}\,\mathrm{Str}\left(\Psi_j \frac{\partial U(u,\lambda_j)}{\delta u_i}\right) = \frac{1}{3}\,\mathrm{Str}(\Psi_j e_i \lambda_j),\quad i = 1,\dots,5 \tag{9}$$

where Str denotes the super-trace of a super-matrix and

$$\Psi_j = \begin{pmatrix} \phi_{1j}\phi_{2j} & -\phi_{1j}^2 & \phi_{1j}\phi_{3j} \\ \phi_{2j}^2 & -\phi_{1j}\phi_{2j} & \phi_{2j}\phi_{3j} \\ \phi_{2j}\phi_{3j} & -\phi_{1j}\phi_{3j} & 0 \end{pmatrix}. \tag{10}$$

From Eq. (8) and Eq. (9), a kind of super-Hamiltonian soliton equation hierarchies with self-consistent sources is presented as follows:

$$u_t = J\frac{\delta H_{n+1}}{\delta u_i} + J\sum_{j=1}^{N}\frac{\delta \lambda_j}{\delta u_i}. \tag{11}$$

3 The super-Yang hierarchy with self-consistent sources

As we know, the basis of Lie algebra A_1 can be as follows:

$$e_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\qquad e_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},\qquad e_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{12}$$

And corresponding loop algebra $\tilde{A}_1$ is $\{e_1(n), e_2(n), e_3(n) | n \in \mathbb{Z}\}$ and satisfies $x(n) = x \otimes \lambda^n, x \in A_1$. By using the Tu scheme [23], we can obtain many evolution equations hierarchy. Based on the above basis, we consider the following spectral problem:

$$\varphi_x = U\varphi,\qquad U = e_2(1) + qe_2(0) + re_1(0) + se_3(0) = \begin{pmatrix} s & \lambda + q + r \\ -\lambda - q + r & -s \end{pmatrix}, \tag{13}$$

$$\lambda_t = 0.$$

Take $V^{(n)} = \sum_{m=0}^{n}(a_m e_3(n-m) + b_m e_2(n-m) + c_m e_1(n-m) + b_{n+1}e_2(0))$, the Yang hierarchy are obtained as follows [23]:

$$u_t = \begin{pmatrix} q \\ r \\ s \end{pmatrix}_t = J\begin{pmatrix} -2b_{n+1} \\ 2c_{n+1} \\ 2a_{n+1} \end{pmatrix} = J\frac{\delta H_n}{\delta u}, \tag{14}$$

where

$$J = \begin{pmatrix} -\partial & -s & r \\ s & 0 & -1 \\ -r & 1 & 0 \end{pmatrix},$$

$$L = \begin{pmatrix} 0 & 2\partial^{-1}qs - \partial^{-1}r\partial & -2\partial^{-1}qr - \partial^{-1}s\partial \\ -r & -q & \frac{1}{2}\partial \\ -s & -\frac{1}{2}\partial & -q \end{pmatrix},$$

$$H_n = \int \frac{2b_{n+2}}{n+1}\,\mathrm{d}x,\quad n \geq 0.$$

In the following, we will generalize Lie loop algebra $\tilde{A}_1$ to the Lie loop super-algebra $G_2 = \{\sum_{i=1}^{n}\lambda_i e_i, i = 1,2,\dots,5\}$

$$\begin{cases} e_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\ e_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}, \\ e_5 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad [e_1, e_2] = -2e_3, \\ [e_1, e_3] = -2e_2, \quad [e_2, e_3] = -2e_1, \\ [e_1, e_5] = [e_2, e_5] = [e_3, e_4] = e_4, \\ [e_1, e_4] = [e_4, e_2] = [e_5, e_3] = e_5, \\ [e_4, e_4]_+ = -(e_1 + e_2), \quad [e_5, e_5]_+ = e_1 - e_2, \\ [e_4, e_5]_+ = [e_5, e_4]_+ = e_3, \end{cases} \tag{15}$$

where e_1, e_2, e_3 are even elements and e_4, e_5 are odd elements, and $[\cdot,\cdot]$ and $[\cdot,\cdot]_+$ denote the commutator and the anticommutator. Consider the following super isospectral problem [9]:

$$\varphi_x = U\varphi,\qquad U = e_2(1) + qe_2(0) + re_1(0) + se_3(0) + \alpha e_4(0) + \beta e_5(0), \tag{16}$$

$$\lambda_t = 0.$$

Denote

$$V = \sum_{m\geq 0} a_m e_3(-m) + b_m e_2(-m) + c_m e_1(-m) + \rho_m e_4(-m) + \delta_m e_5(-m) = ae_3 + be_2 + ce_1 + \rho e_4 + \delta e_5, \quad (17)$$

solving the stationary zero curvature equation

$$V_x = [U, V], \quad (18)$$

we have

$$\begin{cases} a_{mx} = 2c_{m+1} - 2rb_m + 2qc_m + \beta\rho_m + \alpha\delta_m, \\ b_{mx} = -2ra_m + 2sc_m - \alpha\rho_m - \beta\delta_m, \\ c_{mx} = -2a_{m+1} - 2qa_m + 2sb_m - \alpha\rho_m + \beta\delta_m, \\ \rho_{mx} = \delta_{m+1} - \alpha a_m - \beta b_m - \beta c_m + s\rho_m + q\delta_m + r\delta_m, \\ \delta_{mx} = -\rho_{m+1} + \beta a_m + \alpha b_m - \alpha c_m - q\rho_m + r\rho_m - s\delta_m, \\ a_0 = c_0 = \rho_0 = \delta_0 = 0, b_0 = \mu = \text{constant} \neq 0, \\ a_1 = \mu s, b_1 = 0, c_1 = \mu r, \rho_1 = \mu\alpha, \delta_1 = \mu\beta, \\ a_2 = -\frac{1}{2}\mu r_x - \mu qs, \\ b_2 = \frac{1}{2}\mu r^2 + \frac{1}{2}\mu s^2 + \mu\alpha\beta, c_2 = \frac{1}{2}\mu s_x - \mu qr, \\ \rho_2 = -\mu\beta_x - \mu q\alpha, \delta_2 = \mu\alpha_x - \mu q\beta, \\ a_3 = \mu\left(-\frac{1}{4}s_{xx} + \frac{1}{2}q_x r + qr_x + q^2 s + \frac{1}{2}r^2 s + \frac{1}{2}s^3 + s\alpha\beta + \frac{1}{2}\alpha\beta_x - \frac{1}{2}\alpha_x\beta\right), \\ b_3 = \mu\left(\frac{1}{2}rs_x - \frac{1}{2}r_x s - qr^2 - qs^2 + \alpha\alpha_x + \beta\beta_x - 2q\alpha\beta\right), \\ c_3 = \mu\left(-\frac{1}{4}r_{xx} - \frac{1}{2}q_x s - qs_x + \frac{1}{2}r^3 + \frac{1}{2}rs^2 + r\alpha\beta + q^2 r - \frac{1}{2}\alpha\alpha_x + \frac{1}{2}\beta\beta_x\right), \\ \rho_3 = \mu\left(-\alpha_{xx} + q_x\beta + 2q\beta_x - \frac{1}{2}r_x\beta + \frac{1}{2}r^2\alpha + \frac{1}{2}s^2\alpha - \frac{1}{2}s_x\alpha + q^2\alpha - r\beta_x - s\alpha_x\right), \\ \delta_3 = \mu\left(-\beta_{xx} - q_x\alpha - 2q\alpha_x - \frac{1}{2}r_x\alpha + \frac{1}{2}r^2\beta + \frac{1}{2}s^2\beta + \frac{1}{2}s_x\beta + s\beta_x + q^2\beta - r\alpha_x\right). \end{cases} \quad (19)$$

From Eqs. (19), we can obtain the following recursions for

$-2b_{m+1}, 2c_{m+1}, \quad 2a_{m+1}, 2\delta_{m+1} \quad$ and $\quad -2\rho_{m+1}$,

$$\begin{pmatrix} -2b_{m+1} \\ 2c_{m+1} \\ 2a_{m+1} \\ 2\delta_{m+1} \\ -2\rho_{m+1} \end{pmatrix} = L \begin{pmatrix} -2b_m \\ 2c_m \\ 2a_m \\ 2\delta_m \\ -2\rho_m \end{pmatrix} \quad (20)$$

with

$$L = \begin{pmatrix} 0 & 2\partial^{-1}qs - \partial^{-1}r\partial & -2\partial^{-1}qr - \partial^{-1}s\partial & -\partial^{-1}\alpha\partial - \partial^{-1}q\beta & -\partial^{-1}\beta\partial + \partial^{-1}q\alpha \\ -r & -q & \frac{1}{2}\partial & -\frac{1}{2}\alpha & \frac{1}{2}\beta \\ -s & -\frac{1}{2}\partial & -q & \frac{1}{2}\beta & \frac{1}{2}\alpha \\ -\beta & \beta & \alpha & -(q+r) & -\partial + s \\ \alpha & \alpha & -\beta & \partial + s & -q + r \end{pmatrix}. \quad (21)$$

Denoting

$$V_+^{(n)} = \sum_{m=0}^{n} a_m g_3(n-m) + b_m e_2(n-m) + c_m e_1(n-m) + \rho_m e_4(n-m) + \delta_m e_5(n-m),$$
$$V_-^{(n)} = \lambda^n V - V_+^{(n)},$$

a direct calculation reads,

$$-V_{+x}^{(n)} + [U, V_+^n] = -2c_{n+1}e_3(0) + 2a_{n+1}e_1(0) - \delta_{n+1}e_4(0) + \rho_{n+1}e_5(0). \quad (22)$$

Considering

$$\Delta_n = b_{n+1}e_2(0),\ V^{(n)} = V_+^{(n)} + \Delta_n. \quad (23)$$

Substituting Eq. (23) into a zero curvature equation $U_t - V_x^{(n)} + [U, V^{(n)}] = 0$, we have the following super-integrable system:

$$u_t = \begin{pmatrix} q \\ r \\ s \\ \alpha \\ \beta \end{pmatrix}_t = \begin{pmatrix} -\partial & -s & r & \frac{1}{2}\beta & -\frac{1}{2}\alpha \\ s & 0 & -1 & 0 & 0 \\ -r & 1 & 0 & 0 & 0 \\ -\frac{1}{2}\beta & 0 & 0 & \frac{1}{2} & 0 \\ \frac{1}{2}\alpha & 0 & 0 & 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} -2b_{n+1} \\ 2c_{n+1} \\ 2a_{n+1} \\ 2\delta_{n+1} \\ -2\rho_{n+1} \end{pmatrix} = J \begin{pmatrix} -2b_{n+1} \\ 2c_{n+1} \\ 2a_{n+1} \\ 2\delta_{n+1} \\ -2\rho_{n+1} \end{pmatrix} = JP_{n+1}, \quad (24)$$

where P_{n+1} meets as follows: $P_{n+1} = LP_n$. If taking $\alpha = \beta = 0$ in system (24), it can be reduced to the Yang hierarchy (14). Therefore, we call system (24) a super-Yang hierarchy.

Based on Lie super-algebra G_2 in (15) and the associated corresponding loop super-algebra $\widetilde{G}_2$, a direct calculation gives

$$\mathrm{ad}_a = \begin{pmatrix} 0 & 2a_3 & -2a_2 & -a_4 & a_5 \\ 2a_3 & 0 & -2a_1 & -a_4 & -a_5 \\ 2a_2 & -2a_1 & 0 & a_5 & a_4 \\ -a_5 & -a_5 & -a_4 & a_3 & a_1+a_2 \\ -a_4 & a_4 & a_5 & a_1-a_2 & -a_3 \end{pmatrix} \quad (25)$$

where $a = a_1e_1 + a_2e_2 + a_3e_3 + a_4e_4 + a_5e_5 \in \tilde{B}(0,1)$, $\mathrm{ad}_a b = [a,b]$, $a, b \in \tilde{B}(0,1)$, and the bracket $[\cdot,\cdot]$ is the Lie super-bracket of $\widetilde{G}_2$. If we define the compact super-trace as follows:

$$\mathrm{str}(c) = c_{11} + c_{22} - c_{33},\ c = ab, \quad a, b \in \tilde{B}(0,1).$$
$$\mathrm{str}(P) = p_{11} + p_{22} + p_{33} - p_{44} - p_{55}, \quad (26)$$

where $c = (c_{ij})_{3\times3}$, $P = (p_{ij})_{5\times5}$ and ab is matrix product of a and b, then we have

$$\mathrm{str}(\mathrm{ad}_a \mathrm{ad}_b) = 3\,\mathrm{str}(ab). \quad (27)$$

By using the super-trace identity in [8], we have

$$\frac{\delta}{\delta u}\int(-6b)\,\mathrm{d}x = \lambda^{-\gamma}\frac{\delta}{\delta\lambda}\lambda^{\gamma}(-6b, 6c, 6a, 6\delta, -6\rho)^{\mathrm{T}}. \quad (28)$$

Comparing the coefficient of λ^{-n-1} yields

$$\frac{\delta}{\delta u}\int(-6b_{n+1})\,\mathrm{d}x = (\gamma - n)(-6b_n, 6c_n, 6a_n, 6\delta_n, -6\rho_n)^{\mathrm{T}}. \quad (29)$$

Since $\mathrm{str}(\mathrm{ad}_V \mathrm{ad}_V) = 6\mu^2 = \text{constant} \neq 0$, we can obtain $\gamma = 0$ from the computation formula in [8]. Therefore, we conclude that

$$P_{n+1} = \frac{\delta\mathcal{H}_n}{\delta u}, \quad \mathcal{H}_n = \int \frac{2b_{n+2}}{n+1}\,\mathrm{d}x. \quad (30)$$

Hence, the super-Yang hierarchy (24) has the following super-Hamiltonian structure:

$$u_t = JP_{n+1} = J\frac{\delta\mathcal{H}_n}{\delta u}, \quad n \geq 0. \quad (31)$$

For the integrable system (14) and super-integrable system (24), $n = 2$ $\mu = 1$, we can get the nonlinear evolution equations and super-integrable couplings (also see [9]).

Next, we will construct the super-Yang hierarchy with self-consistent sources. Consider the linear system

$$\begin{pmatrix}\phi_{1j}\\ \phi_{2j}\\ \phi_{3j}\end{pmatrix}_x = U\begin{pmatrix}\phi_{1j}\\ \phi_{2j}\\ \phi_{3j}\end{pmatrix}, \qquad \begin{pmatrix}\phi_{1j}\\ \phi_{2j}\\ \phi_{3j}\end{pmatrix}_t = V\begin{pmatrix}\phi_{1j}\\ \phi_{2j}\\ \phi_{3j}\end{pmatrix} \tag{32}$$

From Eqs. (8)–(9), we have the following $\frac{\delta\lambda_j}{\delta u}$:

$$\sum_{j=1}^{N}\frac{\delta\lambda_j}{\delta u} = \sum_{j=1}^{N}\begin{pmatrix}\mathrm{Str}(\Psi_j\frac{\delta U}{\delta q})\\ \mathrm{Str}(\Psi_j\frac{\delta U}{\delta s})\\ \mathrm{Str}(\Psi_j\frac{\delta U}{\delta r})\\ \mathrm{Str}(\Psi_j\frac{\delta U}{\delta \alpha})\\ \mathrm{Str}(\Psi_j\frac{\delta U}{\delta \beta})\end{pmatrix} = \begin{pmatrix}\langle\Psi_1,\Psi_1\rangle+\langle\Psi_2,\Psi_2\rangle\\ 2\langle\Psi_1,\Psi_2\rangle\\ 2\langle\Psi_1,\Psi_2\rangle\\ -2\langle\Psi_2,\Psi_3\rangle\\ 2\langle\Psi_1,\Psi_3\rangle\end{pmatrix}$$

where $\Psi_1 = (\phi_{i1},\ldots,\phi_{N1})^{\mathrm{T}}$, $(i=1,2,3)$. And $\langle\cdot,\cdot\rangle$ stand for standard inner product.

According to (11), the integrable super-Yang hierarchy with self-consistent sources is proposed

$$u_t = \begin{pmatrix}q\\ r\\ s\\ \alpha\\ \beta\end{pmatrix}_t = J\begin{pmatrix}-2b_{n+1}\\ 2c_{n+1}\\ 2a_{n+1}\\ 2\delta n+1\\ 2\rho n+1\end{pmatrix} + J\begin{pmatrix}\langle\Psi_1,\Psi_1\rangle+\langle\Psi_2,\Psi_2\rangle\\ 2\langle\Psi_1,\Psi_2\rangle\\ 2\langle\Psi_1,\Psi_2\rangle\\ -2\langle\Psi_2,\Psi_3\rangle\\ 2\langle\Psi_1,\Psi_3\rangle\end{pmatrix} = J\frac{\delta H_n}{\delta u} + J\begin{pmatrix}\langle\Psi_1,\Psi_1\rangle+<\Psi_2,\Psi_2\rangle\\ 2\langle\Psi_1,\Psi_2\rangle\\ 2\langle\Psi_1,\Psi_2\rangle\\ -2\langle\Psi_2,\Psi_3\rangle\\ 2\langle\Psi_1,\Psi_3\rangle\end{pmatrix} \tag{33}$$

For $n=1$, $\mu=1$, we obtain the super-Yang equation with self-consistent sources

$$\begin{cases} q_t = rr_x + ss_x + 4\alpha_x\beta + 4\alpha\beta_x + 2sqr - 2qs \\ \quad - \partial\sum_{j=1}^{N}(\phi_{1j}^2+\phi_{2j}^2) - 2s\sum_{j=1}^{N}\phi_{1j}\phi_{2j} \\ \quad + 2r\sum_{j=1}^{N}\phi_{1j}\phi_{2j} - \beta\sum_{j=1}^{N}\phi_{2j}\phi_{3j} \\ \quad - \alpha\sum_{j=1}^{N}\phi_{1j}\phi_{3j}, \\ r_t = -sr^2 - s^3 + r_x + 2qs - s\sum_{j=1}^{N}(\phi_{1j}^2+\phi_{2j}^2) \\ \quad - 2\sum_{j=1}^{N}\phi_{1j}\phi_{2j}, \\ s_t = -r^3 - rs^2 + s^2 - 2qr - r\sum_{j=1}^{N}(\phi_{1j}^2+\phi_{2j}^2) \\ \quad + 2\sum_{j=1}^{N}\phi_{1j}\phi_{2j}, \\ \alpha_t = \frac{1}{2}(r^2+s^2) + \alpha_x - q\beta - \frac{1}{2}\beta\sum_{j=1}^{N}(\phi_{1j}^2+\phi_{2j}^2) \\ \quad - \sum_{j=1}^{N}\phi_{2j}\phi_{3j}, \\ \beta_t = -\frac{1}{2}\alpha(r^2+s^2) + \beta_x + q\alpha \\ \quad - \frac{1}{2}\alpha\sum_{j=1}^{N}(\phi_{1j}^2+\phi_{2j}^2)\sum_{j=1}^{N}\phi_{1j}\phi_{2j}. \end{cases}$$

4 Conservation laws for the super-Yang hierarchy

In what follows, we will construct conservation laws of the super-Yang hierarchy. We introduce the variables

$$K = \frac{\psi_2}{\psi_1}, \qquad G = \frac{\psi_3}{\psi_1}. \tag{34}$$

From Eq. (7) and Eq. (12), we have

$$\begin{aligned} K_x &= (-\lambda - q + r) - 2sK + \beta G \\ &\quad - (\lambda + q + r)K^2 - 2KG, \\ G_x &= \beta - 2K - sG - (\lambda + q + r)KG - 2G^2. \end{aligned} \tag{35}$$

Expand K, G in the power of λ^{-1}

$$K = \sum_{j=1}^{\infty} k_j \lambda^{-j}, \qquad G = \sum_{j=1}^{\infty} g_j \lambda^{-j}. \tag{36}$$

Substituting Eq. (30) into Eq. (29) and comparing the coefficients of the same power of λ, we obtain

$$\begin{aligned} & k_0 = i, \qquad g_0 = 0, \qquad k_1 = -ir - s, \\ & g_1 = -\beta i - 2, \\ & k_2 = \frac{1}{2}r_x - \frac{1}{2}s_x - \frac{i}{2}s^2 + \frac{1}{2}(\alpha^2 - \beta^2) + i\alpha\beta \\ &\quad + \frac{i}{2}r^2 + iqr + qs + rs, \\ & g_2 = \beta_x - i\alpha_x i\alpha s + \alpha q + i\beta q + 2r, \\ & k_3 = \frac{1}{2}u_{1xx} - \frac{1}{2}u_{2xx} - \frac{1}{8}u_1^3 + \frac{1}{8}u_1^2 u_2 \\ &\quad + \frac{1}{8}u_2^2 u_1 - \frac{1}{8}u_2^3 - u_4 u_{4x}, \\ & g_3 = 4u_{4xx} - \frac{1}{2}u_1^2 u_4 + \frac{1}{2}u_2^2 u_4 - u_2 u_{3x} - \frac{1}{2}u_3 u_{2x} \\ &\quad + \frac{1}{2}u_3 u_{1x} + u_1 u_{3x}, \ldots, \end{aligned} \tag{37}$$

and a recursion formula for k_n and g_n

$$\begin{aligned} k_{nx} &= -2sk_n + \beta g_n - \sum_{l=1}^{n-1} k_l k_{n-1-l} \\ &\quad - (q + r)\sum_{l=1}^{n-1} k_l k_{n-l} - 2\sum_{l=0}^{n-1} k_l g_{n-l}, \\ g_{nx} &= -2k_n - sg_n - \sum_{l=1}^{n-1} k_l g_{n-1-l} \\ &\quad - (q + r)\sum_{l=1}^{n-1} k_l g_{n-l} - 2\sum_{l=0}^{n-1} g_l g_{n-l}. \end{aligned} \tag{38}$$

Because of

$$\frac{\partial}{\partial t}\left[s + (\lambda + q + r)K + 2G\right] = \frac{\partial}{\partial t}(a + bK + \rho G) \tag{39}$$

where

$$\begin{aligned} a &= m_0\left(\frac{1}{2}r_x + qs\right) + m_1 s, \\ b &= m_0\left[\frac{1}{2}(r^2 + s^2) + 2\beta\right], \\ \rho &= m_0\alpha\lambda - m_0(\beta_x + q\alpha) + m_1\alpha. \end{aligned}$$

Assume that $\sigma = s + (\lambda + q + r)K + 2G, \theta = a + bK + \rho G$. Then Eq. (33) can be written as $\sigma_t = \theta_x$, which is the right form of conservation laws. We expand σ and θ as series in powers of λ with the coefficients, which are called conserved densities and currents, respectively,

$$\sigma = s + \sum_{j=1}^{\infty} \sigma_j \lambda^{-j}, \qquad \theta = m_0\lambda + \sum_{j=1}^{\infty} \theta_j \lambda^{-j}$$

where m_0, m_1 are constants of integration. The first two conserved densities and currents read

$$\begin{aligned} \sigma_0 &= -ir - s + i(q + r), \\ \sigma_1 &= \frac{1}{2}(r_x - \alpha^2 - \beta^2) - \frac{i}{2}(r^2 + s_x + s^2), \\ \theta_0 &= m_0\left[\frac{i}{2}(r^2 + s^2) - \alpha^2\right], \\ \theta_1 &= m_0\left[-\frac{i}{2}(r^3 + rs^2) - \frac{1}{2}(s^3 + r^2 s) - i\alpha^2 s\right. \\ &\quad + 2\alpha^2 q + \alpha^2 r - i(\alpha\beta r + \alpha\alpha_x + 2\alpha\beta q + \beta\beta_x) \\ &\quad \left. + 2\alpha^2 q - \alpha\beta s + 2\alpha\beta_x\right] - m_1(\alpha^2 + i\alpha), \ldots. \end{aligned}$$

The recursion relation for σ_n and θ_n are

$$\begin{cases} \sigma_n = k_{n+1} + (q + r)k_n + 2g_n, \\ \theta_n = m_0\left[\left(\frac{1}{2}r^2 + \frac{1}{2}s^2 + 2\beta\right)k_n \right. \\ \qquad \left. + 2g_{n+1} - (\beta_x + 2q)g_n\right] \end{cases} \tag{40}$$

where f_n and g_n can be calculated from Eq. (40). The infinitely many conservation laws of Eq. (24) can be easily obtained from Eq. (29)–(34), respectively.

5 Remarks and conclusions

As there is little research on the topic of the integrable couplings of the self-consistent sources on super-integrable hierarchy, the super-Yang hierarchy with self-consistent sources enriched the content of self-consistent sources. Meanwhile, we also get the conservation laws of hierarchy. We must point out that the super-integrable hierarchies involve Fermi variables. In detail, the potential function in the coupling terms are Fermi variables, which satisfy the Grassmann algebra, namely $\alpha^2 = \beta^2 = 0$, $\alpha\beta + \beta\alpha = 0$ is used in the operations of the paper. We can also use this method to get more super-hierarchies with self-consistent sources.

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