Nonlinear Dyn (2012) 70:1549–1561
DOI 10.1007/s11071-012-0555-3

ORIGINAL PAPER

Synchronization between integer-order chaotic systems and a class of fractional-order chaotic system based on fuzzy sliding mode control

Diyi Chen · Runfan Zhang · Julien Clinton Sprott · Xiaoyi Ma

Received: 30 April 2012 / Accepted: 24 July 2012 / Published online: 11 August 2012

Abstract In this paper, we focus on the synchronization between integer-order chaotic systems and a class of fractional-order chaotic system using the stability theory of fractional-order systems. A new fuzzy sliding mode method is proposed to accomplish this end for different initial conditions and number of dimensions. Furthermore, three examples are presented to illustrate the effectiveness of the proposed scheme, which are the synchronization between a fractional-order *Lü* chaotic system and an integer-order Liu chaotic system, the synchronization between a fractional-order hyperchaotic system based on Chen's system and an integer-order hyperchaotic system based upon the Lorenz system, and the synchronization between a fractional-order hyperchaotic system based on Chen's system, and an integer-order Liu chaotic system. Finally, numerical results are presented and are in agreement with theoretical analysis.

D. Chen · R. Zhang · X. Ma (✉)
Department of Electrical Engineering, Northwest A&F University, Yangling Shaanxi 712100, P.R. China
e-mail: ieee307@163.com

D. Chen
e-mail: diyichen@nwsuaf.edu.cn

J. Clinton Sprott
Department of Physics, University of Wisconsin, Madison, WI 53706, USA
e-mail: sprott@physics.wisc.edu

Keywords Chaos synchronization · Integer-order chaotic system · Fractional-order chaotic system · Fuzzy sliding mode control

1 Introduction

Fractional calculus is a much older classical mathematical notion with the same 300-year history as integer calculus. In recent years, it has found application in many areas of physics [1] and engineering [2]. At the same time, control and synchronization of fractional-order chaotic systems have made great contributions. Some papers discuss the synchronization of general fractional-order chaotic systems [3–5], while others consider special classes of fractional-order chaotic systems [6–8].

Chaos synchronization is the concept of closeness of the frequencies between different periodic oscillations generated by two chaotic systems, which is fist proposed by Pecora and Carroll [9]. And it is widely explored in a various field's chemical, ecological, and physical system [10–12]. Chaos synchronization is a very active topic in nonlinear science and has been extensively studied in the past decades. Therefore, various synchronization scheme such as sliding mode control [13–16], linear feedback control [17, 18], adaptive control theory [19, 20], back-stepping control [21], active control [22–24], fuzzy control [25, 26], and fuzzy sliding mode control [27] have been successfully applied to chaos synchronization.

However, there are few previous papers considering synchronization between integer-order chaotic systems and fractional-order chaotic systems with different structure and dimensions [28, 29]. Obviously, it is more difficult to achieve synchronization between integer-order chaotic systems and fractional-order chaotic systems. On the other hand, it is more interesting and more valuable for the application of fractional-order nonlinear systems.

Motivated by the above discussion, fuzzy sliding mode control is utilized here to realize the synchronization between integer-order chaotic systems and a class of fractional-order chaotic system because its robustness and stability. There are three advantages of our approach. First, based on fuzzy theory and sliding mode control (SMC), a new method for chaos synchronization between integer-order chaotic systems and a class of fractional-order chaotic system is presented. Second, the performance of the system in the sense of removing chattering is improved with the utilization of fuzzy logic. Last, two chaotic systems are synchronized with different structure and dimension.

The rest of the paper is outlined as follows. Section 2 introduces the integer-order chaotic systems and a class of fractional-order chaotic systems. Section 3 proposes a compensation controller and vector controller based on fuzzy sliding mode control theory. Furthermore, the controller design scheme and the stability analysis of the closed loop system are included in this section. Section 4 provides results of numerical simulations, and Sect. 5 gives brief comments and conclusions.

2 System description

Consider the n-dimensional, integer-order chaotic drive system

$$\frac{d^d x}{dt^d} = f(x) \tag{1}$$

where $x \in R^n$, $f : R^n \to R^n$ are differentiable functions.

Then consider the n-dimensional, fractional-order chaotic response system

$$\begin{cases} \dfrac{d^q y_i}{dt^q} = a(y_j - y_i) \\ \dfrac{d^q y}{dt^q} = g(y) \end{cases} \tag{2}$$

where $y \in R^n$, $g : R^{n-1} \to R^{n-1}$ are differentiable functions. The dimensions $q = (q_1, q_2, \dots, q_n)^{\mathrm{T}}$ $(0 < q_i < 1)$ may be equal or not, and the response system (2) is an integer-order system if $q_i = 1$ $(i \in [1, n])$. The constant a is positive.

3 Problem formulation and control design

System (1) represents the drive system, and the controller $u(t) \in R^n$ is added into the response system (2) according to

$$\frac{d^q y}{dt^q} = g(y) + u(t) \tag{3}$$

We define the synchronization errors as $e = y - x$. The aim is to choose suitable control signals $u(t) \in R^n$ such that the states of the master and response systems are synchronized (i.e., $\lim_{t\to\infty} \|e\| = 0$, where $\|\cdot\|$ is the Euclidean norm).

Now let the controller $u(t)$ be

$$u(t) = u_1(t) + u_2(t) \tag{4}$$

where $u_2(t) \in R^{n-1}$ is a vector control function that will be designed later. The $u_1(t) \in R^n$ is a compensation controller, and $u_1 = \frac{d^q x}{dt^q} - g(x)$. Using Eq. (4), the response system (3) can be rewritten as

$$\begin{aligned} \frac{d^q e}{dt^q} &= g(y) + \frac{d^q x}{dt^q} - g(x) + u_2 - \frac{d^q x}{dt^q} \\ &= g(y) - g(x) + u_2 \end{aligned} \tag{5}$$

To control the chaotic systems easily, the modified compensation controller u_1 can be represented as

$$u_1 = \frac{d^q x}{dt^q} - g(y - e) \tag{6}$$

and the modified error dynamics (5) can be represented as

$$\frac{d^q e}{dt^q} = h(e, y) + u_2 \tag{7}$$

where $h(e, y) = g(y) - g(e - y)$.

Two steps are required to design a sliding mode controller. First, we construct a sliding surface that represents a desired system dynamic. Then we develop a switching control law such that a sliding mode exists on every point of the sliding surface, and any states outside the surface are driven to reach the surface in a finite time [30]. As a choice for the sliding surface, we take

$$\begin{cases} s_j = e_j + \dfrac{d^{-q}}{dt^{-q}}(k_1 e_j + a e_i) \\ s_r = e_r + \dfrac{d^{-q}}{dt^{-q}} k_p e_r \\ \dots \end{cases} \quad (8)$$

where $r \in [1, n]$, $r \notin (i, j)$, and $k_1, k_p (p \in [2, n-1])$ is a positive constant vector. For the sliding mode method, the sliding surface and its derivative must satisfy

$$s(t) = 0, \qquad \dot{s}(t) = 0 \quad (9)$$

Consider

$$\dot{s}(t) = D^{1-q}\big(D^q s(t)\big) = 0 \quad \Rightarrow \quad D^q s(t) = 0 \quad (10)$$

from which it follows that

$$\begin{cases} \dfrac{d^q}{dt^q} s_j = \dfrac{d^q}{dt^q} e_j + (k_1 e_j + a e_i) = 0 \\ \dfrac{d^q}{dt^q} s_r = \dfrac{d^q}{dt^q} e_r + k_p e_r = 0 \\ \dots \end{cases} \quad (11)$$

and

$$\begin{cases} \dfrac{d^q}{dt^q} e_i = a(e_j - e_i) \\ \dfrac{d^q}{dt^q} e_j = -(k_1 e_j + a e_i) \\ \dfrac{d^q}{dt^q} e_r = -k_p e_r \\ \dots \end{cases} \quad (12)$$

In accordance with active control design procedure, the nonlinear part of the error dynamics is eliminated by the following choice of the input vector:

$$\begin{cases} \dfrac{d^q y_i}{dt^q} = a(y_j - y_i) + u_{1i} \\ \dfrac{d^q y_j}{dt^q} = g(y) + u_{1j} + u_{2j} \\ \dfrac{d^q y}{dt^q} = g(y) + u_{1r} + u_{2r} \\ \dots \end{cases} \quad (13)$$

$$\begin{cases} u_{2j} = -h_j(e, y) - (k_1 e_j + a e_i) + k_f u_{fl(j)} \\ u_{2r} = -h_r(e, y) - k_p e_r + k_f u_{fl(r)} \\ \dots \end{cases} \quad (14)$$

where k_f is the normalization factor of the output variable, and u_{fl} is the output of the fuzzy logic, which is determined by the normalized s and $\dot{s}$.

In the above vector, a fuzzy inference engine is used for reaching phase instead of sign function. One major feature of fuzzy logic is its ability to express the amount of ambiguity in human thinking. The fuzzy control rules can be represented as the mapping of the input linguistic variables s and $\dot{s}$ to the output linguistic variable u_{fl} as follows:

$$\begin{aligned} u_{fl} &= FL(s, \dot{s}) = [u_{fl(1)}, u_{fl(2)}, \dots, u_{fl(6)}]^{\mathrm{T}} \\ &= \big[FL(s_1, \dot{s}_1), FL(s_2, \dot{s}_2), \dots, FL(s_n, \dot{s}_n)\big]^{\mathrm{T}} \end{aligned} \quad (15)$$

The membership function of input linguistic for each set of variables s_i and $\dot{s}_i$, and the membership functions of the output linguistic variable $u_{fl(i)}$ ($i = 1, 2, \dots, 6$), are shown in Fig. 1, respectively. Here, $u_{fl(i)}$ is denoted as:

$$u_{fl(i)} = FL(s_i, \dot{s}_i) \quad (16)$$

As usual, the dynamical behavior of a FLC scheme is governed by a set of linguistic rules derived from expert knowledge. By referencing these rules, the inference mechanism of the FLC is able to instruct an appropriate fuzzy control action in response to any change in the input signal. Suppose the rules of fuzzy controller are based on SMC, and then it is called the FSMC [31].

As is described above, our proposed FLC has two inputs and one output. These are s, $\dot{s}$, and the control signal, respectively. Linguistic variables which imply inputs and output have been classified as [32]: NB, NM, NS, ZE, PS, PM, PB. As is shown in Fig. 1, inputs are all normalized in the interval of $[-3, 3]$ and output is normalized in the interval of $[-1, 1]$, all with equal span. The linguistic labels used to describe the fuzzy sets were "Negative Big" (NB), "Negative Medium" (NM), "Negative Small" (NS), "Zero" (ZE), "Positive Small" (PS), "Positive Medium" (PM) and "Positive Big" (PB). It is possible to assign a set of decision rules as shown in Table 1. These rules contain the input/output relationships that define the control strategy. Each control input has seven fuzzy sets so that there are at most 49 fuzzy rules.

According to function (2) and control (6) and (14), the error is given by

$$\frac{d^q e}{dt^q} = Ae + k_f u_{fl} \quad (17)$$

Theorem *Consider the error function* (14). *The error between response system* (3) *and drive system* (1) *can*

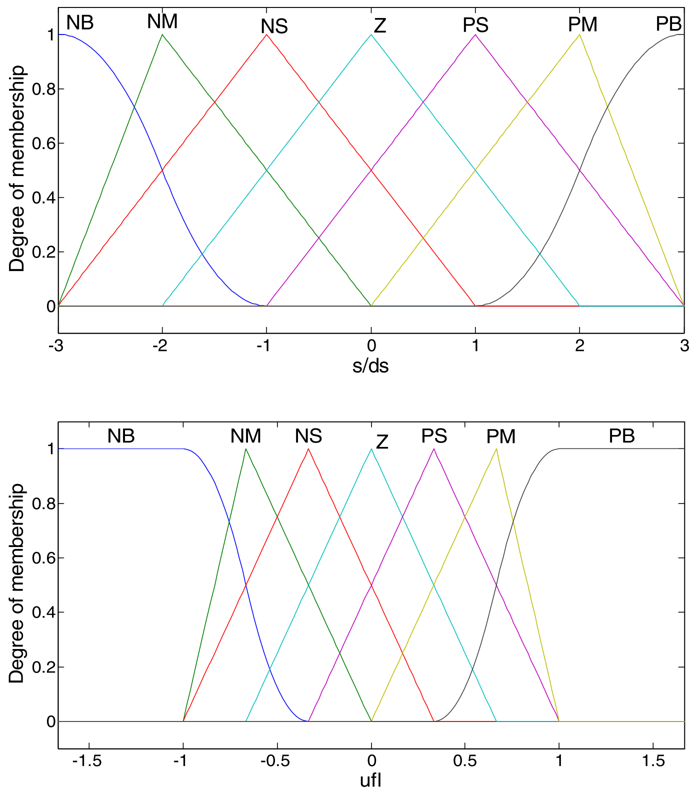


Fig. 1 The membership function for inputs s, $\dot{s}$, and output u_{fl}

Table 1 Rule base of FSMC

	u_{fl}	S						
		PB	PM	PS	ZE	NS	NM	NB
$\dot{S}$	PB	NB	NB	NB	NB	NM	NS	ZE
	PM	NB	NV	NB	NM	NS	ZE	PS
	PS	NB	NM	NS	ZE	PS	PM	PB
	ZE	NB	NM	NS	ZE	PS	PM	PB
	NS	NM	NS	ZE	PS	PM	PB	PB
	NM	NS	ZE	PS	PM	PB	PB	PB
	NB	ZE	PS	PM	PB	PB	PB	PB

be determined if and only if $|\arg(\lambda_i)| > q\pi/2$ is satisfied for all eigenvalues λ_i of matrix A. Besides, this error system is stable if and only if $|\arg(\lambda_i)| \geq q\pi/2$ is satisfied for all eigenvalues λ_i of matrix A and those critical eigenvalues which satisfies the condition $|\arg(\lambda_i)| = q\pi/2$, have geometric multiplicity one.

Proof According to functions (2), (6), (14), and (17), A is given by the matrix

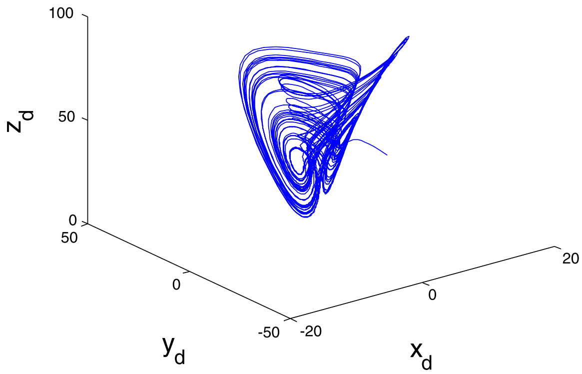
$$A = \begin{bmatrix} -a & a & 0 & \cdots & 0 & \cdots & 0 \\ -a & -k_1 & 0 & \cdots & 0 & \cdots & 0 \\ 0 & 0 & -k_2 & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & -k_r & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & -k_{n-1} \end{bmatrix} \tag{18}$$

whose eigenvalues are

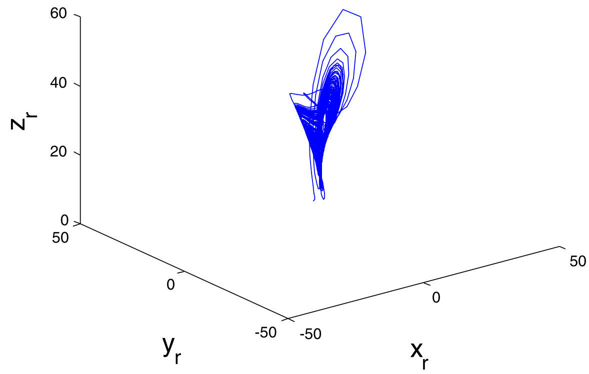
$$\begin{cases} \lambda_i = -\frac{1}{2}\left(a + k_1 + \sqrt{-3a^2 - 2ak_1 - k_1^2}\right) \\ \lambda_j = -\frac{1}{2}\left(a + k_1 - \sqrt{-3a^2 - 2ak_1 - k_1^2}\right) \\ \lambda_r = -k_r \\ \ldots \end{cases} \tag{19}$$

Thus, all eigenvalues of matrix A satisfy $\mathrm{Re}(\lambda) < 0$, which implies $|\arg\lambda| > \frac{\pi}{2} > \frac{q\pi}{2}$. According to the stability theory of fractional-order systems [33–35], the equilibrium point $e = 0$ in function (17) is asymptotically stable:

$$\lim_{t\to+\infty} e = \lim_{t\to+\infty} (y - x) = 0 \tag{20}$$



a. Integer-order Liu chaotic system.



b. Fractional-order *Lü* chaotic system

Fig. 2 Chaotic attractors of integer-order Liu chaotic system and fractional-order *Lü* chaotic system

Remark 1 If chaotic orders in drive system (10) meet $\alpha_i = 1$, like the system $\dot{x} = Ax + f(x)$, then the synchronization between a fractional-order system and an integer-order system can be achieved using the controller (4).

Remark 2 Most system parameters change stochastically within a certain range. As we can see, the fuzzy logic output (16), which vary with system parameters, that is, when the system parameter change, the controller (4) changes in certain regular way as well. Especially for the system with stochastic parameters, the controller is extremely effective.

4 Numerical simulation

This section of the paper presents three illustrative examples to verify and demonstrate the effectiveness of the proposed control scheme. In Case I, a three-dimensional integer-order system is synchronized with a fractional-order system having a different structure. In Case II, a four-dimensional integer-order system is synchronized with a fractional-order system. In Case III, a three-dimensional integer-order system is synchronized with a four-dimensional fractional-order system. The numerical simulation results were carried out in MATLAB using the Caputo version and a predictor-corrector algorithm with a fixed step size of 0.01, which is the generalization of Adams–Bashforth–Moulton one.

Case I Synchronization between a fractional-order *Lü* chaotic system and an integer-order *Lü* chaotic system.

The integer-order Liu chaotic system [36] is described by

$$\begin{cases} \dfrac{dx_d}{dt} = a(y_d - x_d) \\ \dfrac{dy_d}{dt} = bx_d - kx_d z_d \\ \dfrac{dz_d}{dt} = -cz_d + hx_d^2 \end{cases} \tag{21}$$

The Liu system exhibits chaotic behavior for the parameters $(a, b, c, k, h) = (10, 40, 2.5, 1, 4)$ with initial conditions $[x_d, y_d, z_d]^{\mathrm{T}} = [10, 0, 30]^{\mathrm{T}}$ and a chaotic attractor as shown in Fig. 2(a).

The fractional-order *Lü* chaotic system [37, 38] is

$$\begin{cases} \dfrac{d^{q_{r1}}x_r}{dt^{q_{r1}}} = a(y_r - x_r) \\ \dfrac{d^{q_{r2}}y_r}{dt^{q_{r2}}} = -x_r z_r + cy_r \\ \dfrac{d^{q_{r3}}z_r}{dt^{q_{r3}}} = x_r y_r - bz_r \end{cases} \tag{22}$$

and exhibits chaotic behavior for $q_{r1} = q_{r2} = q_{r3} = 0.90$ and $(a, b, c) = (35, 3, 28)$ with initial conditions $[x_r, y_r, z_r]^{\mathrm{T}} = [0, 3, 9]^{\mathrm{T}}$ and a chaotic attractor as shown in Fig. 2(b).

Here, the controller parameters $K_1 = K_2 = 10$ and $k_f = 1$ are chosen, and the eigenvalues $(\lambda_1, \lambda_2, \lambda_3) = (-22.5 + 32.6917i, -22.5 - 32.6917i, -10)$ are lo-

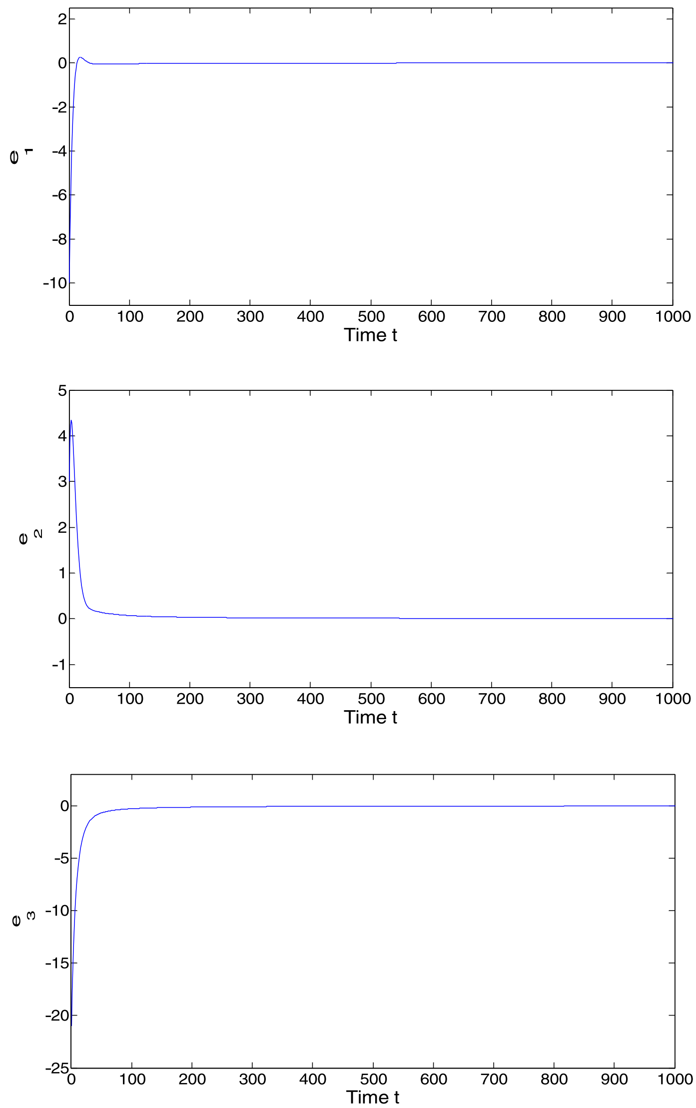


Fig. 3 Synchronization errors between integer-order Liu chaotic system and fractional-order *Lü* chaotic system ($e_1 = x_r - x_d$, $e_2 = y_r - y_d$, $e_3 = z_r - z_d$)

cated in the stable region. As described above, we can obtain the controller $u(t)$ for the response system (6) and (14) as follows:

i. Compensation controller

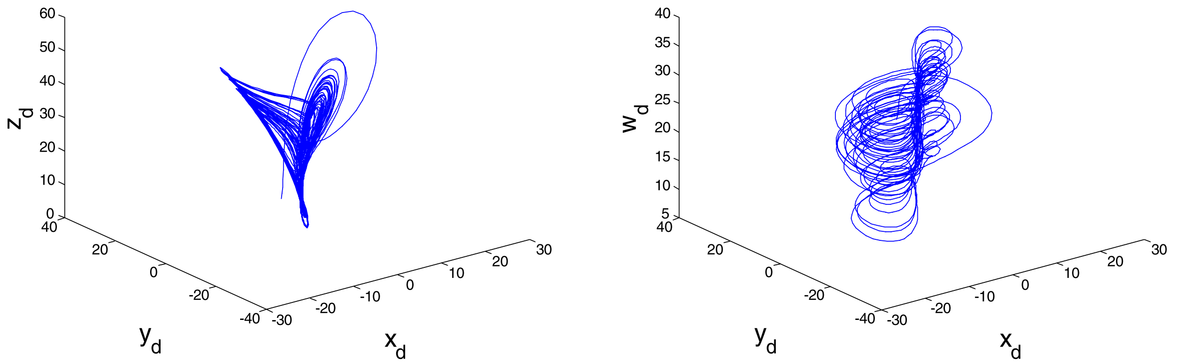
$$\begin{cases} u_{11} = 10(y_d - x_d) - 35\big[(y_r - e_2) - (x_r - e_1)\big] \\ u_{12} = 40x_d - x_d z_d + (x_r - e_1)(z_r - e_3) \\ \qquad - 28(y_r - e_2) \\ u_{13} = -2.5z_d + 4x_d^2 - (x_r - e_1)(y_r - e_2) \\ \qquad + 3(z_r - e_3) \end{cases} \tag{23}$$

ii. Vector controller

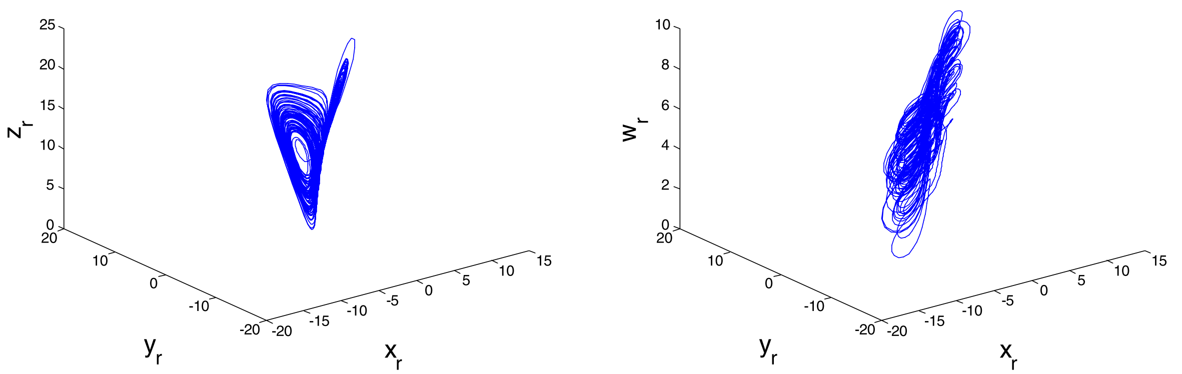
$$\begin{cases} u_{22} = -(x_r - e_1)(z_r - e_3) + 28(y_r - e_2) \\ \qquad + x_r z_r - 28y_r - (k_1 e_2 + a e_1) \\ \qquad + k_f u_{fl(1)} \\ u_{23} = (x_r - e_1)(y_r - e_2) - 3(z_r - e_3) \\ \qquad - x_r y_r + 3z_r - k_2 e_3 + k_f u_{fl(2)} \end{cases} \tag{24}$$

The synchronization errors are shown in Fig. 3, which demonstrates that the proposed method is successful in synchronizing the two systems.

Case II Synchronization between a fractional-order hyperchaotic system based on Chen's system and an



a. Integer-order hyperchaotic system based upon Lorenz system.



b. Fractional-order hyperchaotic system based upon Chen system.

Fig. 4 Chaotic attractors of integer-order hyperchaotic based upon Lorena system and fractional-order hyperchaotic system based upon Chen system

integer-order hyperchaotic system based on the Lorenz system.

The integer-order hyper-chaotic system based on the Lorenz system [39] is given by

$$\begin{cases} \dfrac{dx_d}{dt} = a(y_d - x_d) \\ \dfrac{dy_d}{dt} = bx_d + y_d - x_d z_d - w_d \\ \dfrac{dz_d}{dt} = x_d y_d - cz_d \\ \dfrac{dw}{dt} = ky_d z_d \end{cases} \tag{25}$$

This system exhibits chaotic behavior for the parameters $(a, b, c, k) = (10, 28, 8/3, 0.03)$ with initial conditions $[x_d, y_d, z_d, w_d]^{\mathrm{T}} = [5, 15, 2, 20]^{\mathrm{T}}$ and a chaotic attractor as shown in Fig. 4(a).

The fractional-order hyperchaotic system based on Chen's system [40] is

$$\begin{cases} \dfrac{d^{q_{r1}} x_d}{dt^{q_{r1}}} = a(y_r - x_r) \\ \dfrac{dy_d^{q_{r2}}}{dt^{q_{r2}}} = dx_r - x_r z_r + cy_r - w_r \\ \dfrac{d^{q_{r3}} z_d}{dt^{q_{r3}}} = x_r y_r - bz_r \\ \dfrac{d^{q_{r4}} w}{dt^{q_{r4}}} = x_r + k \end{cases} \tag{26}$$

This system exhibits chaos for $q_{r1}=q_{r2}=q_{r3}=0.90$ and $(a,b,c,d,k)=(36,3,28,-16,0.5)$ with initial conditions $[x_r,y_r,z_r,w_r]^{\mathrm{T}}=[0,1,9,7]^{\mathrm{T}}$ and a chaotic attractor as shown in Fig. 4(b).

The controller parameters $K_1=K_2=K_3=10$ and $k_f=1$ are chosen, and the eigenvalues $(\lambda_1,\lambda_2,\lambda_3,\lambda_4)=(-23+33.5708i,-23-33.5708i,-10,-10)$ are located in the stable region. As before, we can obtain the controller $u(t)$ for the response system (6) and (14) as follows:

i. Compensation controller

$$\begin{cases} u_{11}=10(y_d-x_d)-36\big[(y_r-e_2)-(x_r-e_1)\big] \\ u_{12}=28x_d+y_d-x_dz_d-w_d+16(x_r-e_1) \\ \qquad +(x_r-e_1)(z_r-e_3)-28(y_r-e_2) \\ \qquad +(w_r-e_4) \\ u_{13}=x_dy_d-\frac{8}{3}z_d-(x_r-e_1)(y_r-e_2) \\ \qquad +3(z_r-e_3) \\ u_{14}=0.03y_dz_d-(x_r-e_1)-0.5 \end{cases} \tag{27}$$

ii. Vector controller

$$\begin{cases} u_{22}=-16(x_r-e_1)-(x_r-e_1)(z_r-e_3) \\ \qquad +28(y_r-e_2)-(w_r-e_4) \\ \qquad +16x_r+x_rz_r-28y_r+w_r \\ \qquad -(k_1e_2+ae_1)+k_fu_{fl(1)} \\ u_{23}=(x_r-e_1)(y_r-e_2)-3(z_r-e_3) \\ \qquad -x_ry_r+3z_r-k_2e_3+k_fu_{fl(2)} \\ u_{24}=(x_r-e_1)-x_r-k_3e_4+k_fu_{fl(3)} \end{cases} \tag{28}$$

The synchronization errors are shown in Fig. 5, which demonstrates that the proposed method is successful in synchronizing the two systems.

Case III Synchronization between a fractional-order hyperchaotic system based on Chen's system and an integer-order Liu chaotic system.

The Liu chaotic system [36] is described by

$$\begin{cases} \frac{dx_d}{dt}=a(y_d-x_d) \\ \frac{dy_d}{dt}=bx_d-kx_dz_d \\ \frac{dz_d}{dt}=-cz_d+hx_d^2 \end{cases} \tag{29}$$

and exhibits chaotic behavior for the parameters $(a,b,c,k,h)=(10,40,2.5,1,4)$ with initial conditions $[x_d,y_d,z_d]^{\mathrm{T}}=[5,10,15]^{\mathrm{T}}$ and a chaotic attractor as shown in Fig. 6(a).

The fractional-order hyperchaotic system based on Chen's system [40] is

$$\begin{cases} \frac{d^{q_{r1}}x_d}{dt^{q_{r1}}}=a(y_r-x_r) \\ \frac{dy_d^{q_{r2}}}{dt^{q_{r2}}}=dx_r-x_rz_r+cy_r-w_r \\ \frac{d^{q_{r3}}z_d}{dt^{q_{r3}}}=x_ry_r-bz_r \\ \frac{d^{q_{r4}}w}{dt^{q_{r4}}}=x_r+k \end{cases} \tag{30}$$

and exhibits chaotic behavior for $q_{r1}=q_{r2}=q_{r3}=0.90$ and $(a,b,c,d,k)=(36,3,28,-16,0.5)$ with initial conditions $[x_r,y_r,z_r,w_r]^{\mathrm{T}}=[0,3,9,17]^{\mathrm{T}}$ and a chaotic attractor as shown in Fig. 6(b).

The controller parameters $K_1=K_2=K_3=7$, $k_f=1$ are chosen, and the eigenvalues are located in the stable region. As mentioned above, we can obtain the controller $u(t)$ for the response system (6) and (14) as follows:

i. Compensation controller

$$\begin{cases} u_{11}=10(y_d-x_d)-36\big[(y_r-e_2)-(x_r-e_1)\big] \\ u_{12}=40x_d-x_dz_d+16(x_r-e_1) \\ \qquad +(x_r-e_1)(z_r-e_3)-28(y_r-e_2) \\ u_{13}=-2.5z_d+4x_d^2-(x_r-e_1)(y_r-e_2) \\ \qquad +3(z_r-e_3) \\ u_{14}=-(x_r-e_1)-0.5 \end{cases} \tag{31}$$

ii. Vector controller

$$\begin{cases} u_{22}=-16(x_r-e_1)-(x_r-e_1)(z_r-e_3) \\ \qquad +28(y_r-e_2)+16x_r+x_rz_r \\ \qquad -28y_r+w_r-(k_1e_2+ae_1)+k_fu_{fl(1)} \\ u_{23}=(x_r-e_1)(y_r-e_2)-3(z_r-e_3) \\ \qquad -x_ry_r+3z_r-k_2e_3+k_fu_{fl(2)} \\ u_{24}=(x_r-e_1)-x_r-k_3e_4 \\ +k_fu_{fl(3)} \end{cases} \tag{32}$$

The synchronization errors are shown in Fig. 7, which demonstrates that the proposed method is successful in synchronizing the two systems.

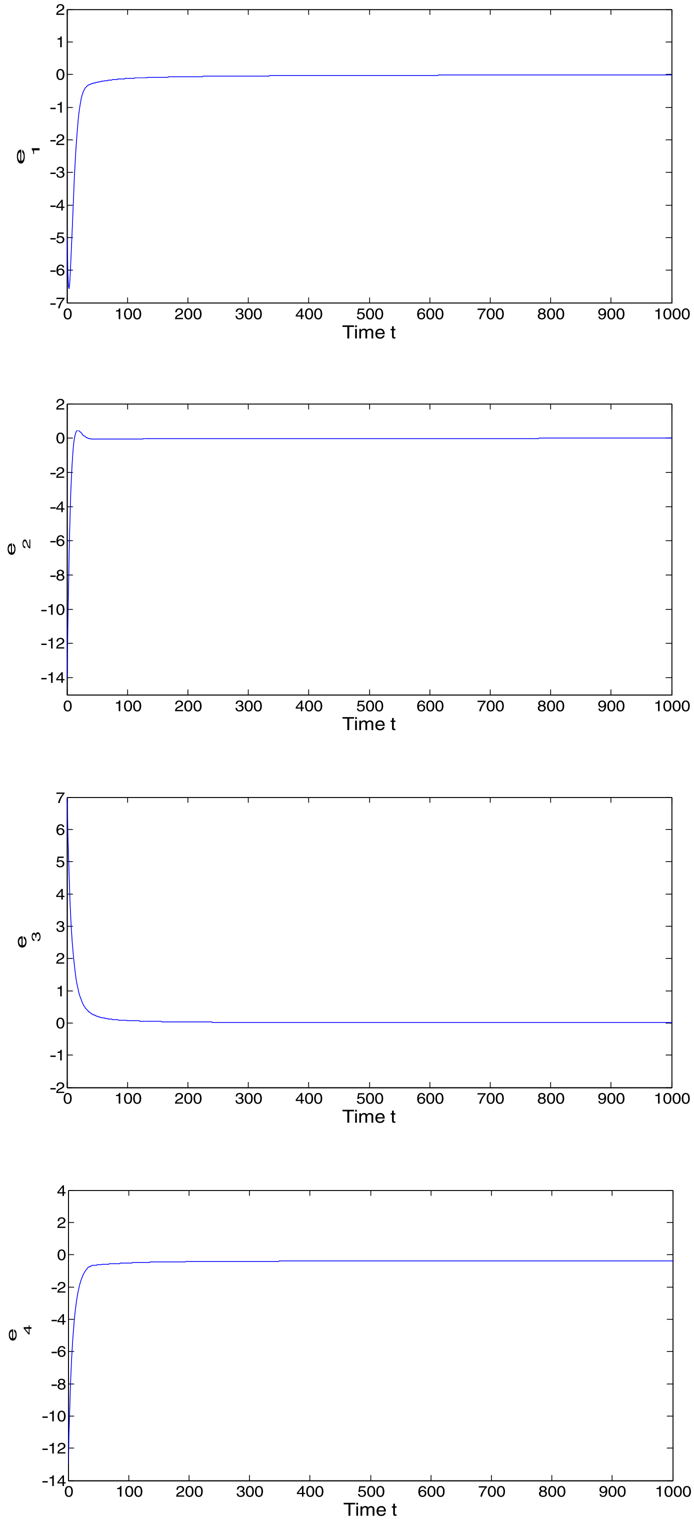
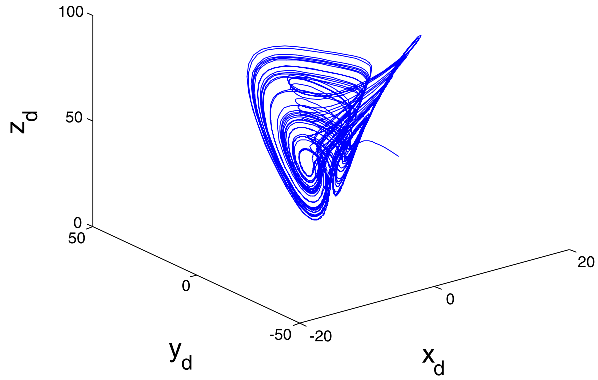
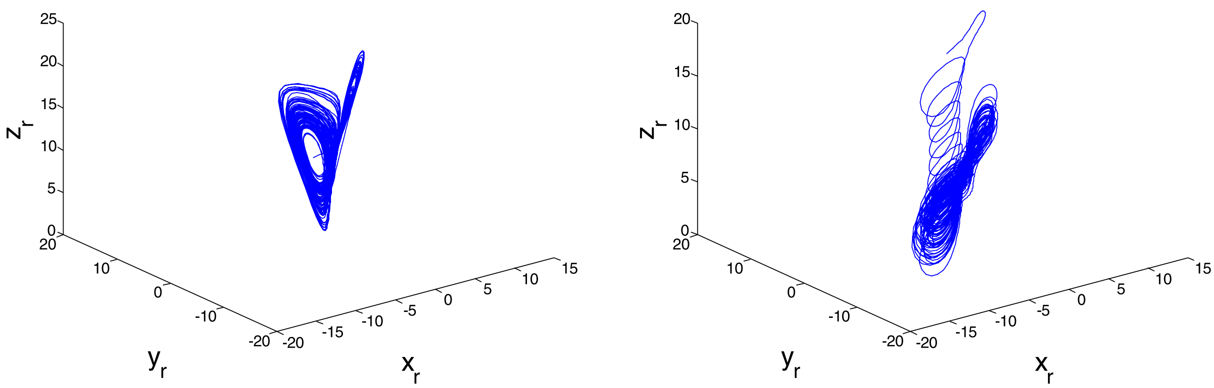


Fig. 5 Synchronization errors between integer-order hyperchaotic based upon Lorena system and fractional-order hyperchaotic system based upon Chen system



a. Integer-order Liu chaotic system.



b. Fractional-order hyperchaotic system based upon Chen system

Fig. 6 Chaotic attractors of integer-order Liu chaotic system and fractional-order hyperchaotic system based upon Chen system

5 Conclusions and discussion

In this paper, the synchronization between fractional-order chaotic systems and integer-order chaotic systems was achieved based on fuzzy sliding mode control. The proposed synchronization approach is theoretically rigorous and pervasive. Furthermore, three typical examples were shown: (1) the synchronization between different three-dimensional chaotic systems, (2) between different four-dimensional chaotic systems, and (3) between a three-dimensional chaotic system and a four-dimensional chaotic system. Numerical results illustrated the effectiveness of the proposed scheme. These theoretical and numerical results provide a bridge between integer-order chaotic system and fractional-order chaotic systems and lend theoretical support for fractional-order chaotic systems.

More and better methods for the synchronization between integer-order chaotic systems and fractional-order chaotic systems should be studied. Moreover, this knowledge should be applied in engineering to fields such as communications, and its circuit design will be a subject of our future work.

Acknowledgements This wok was supported by the scientific research foundation of National Natural Science Foundation (51109180) and Personnel Special Fund of North West A&F University (RCZX-2009-01).

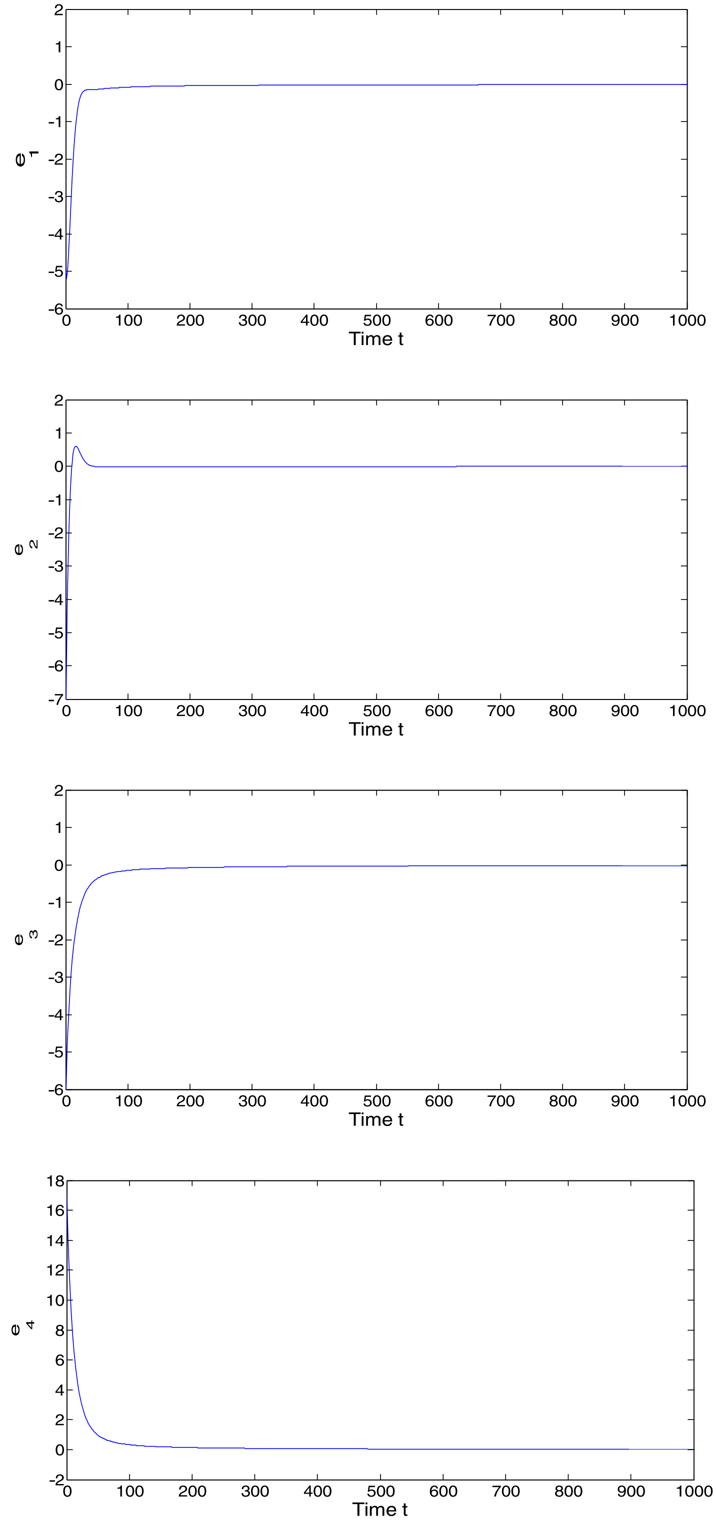


Fig. 7 Synchronization errors between integer-order Liu chaotic system and fractional-order hyperchaotic system based upon Chen system ($e_1 = x_r - x_d$, $e_2 = y_r - y_d$, $e_3 = z_r - z_d$, $e_4 = w_r - w_d$)

References

1. Li, D., Zhou, Y., Zhou, C.S., Hu, B.B.: Fractional locking of spin-torque oscillator by injected ac current. Phys. Rev. B **83**, 174424 (2011)
2. Abdzadeh-Ziabari, H., Shayesteh, M.G.: Robust timing and frequency synchronization for OFDM systems. IEEE Trans. Veh. Technol. **60**, 3646–3656 (2011)
3. Urazhdin, S., Tabor, P., Tiberkevich, V., Slavin, A.: Fractional synchronization of spin-torque nano-oscillators. Phys. Rev. Lett. **105**, 104101 (2010)
4. Wang, S., Yu, Y.G., Diao, M.A.: Hybrid projective synchronization of chaotic fractional order systems with different dimensions. Physica A **389**, 4981–4988 (2010)
5. Tavazoei, M.S., Haeri, M.: Stabilization of unstable fixed points of fractional-order systems by fractional-order linear controllers and its applications in suppression of chaotic oscillations. J. Dyn. Syst. **132**, 021008 (2010)
6. Song, L., Yang, J.Y., Xu, S.Y.: Chaos synchronization for a class of nonlinear oscillators with fractional order. Nonlinear Anal. **72**, 2326–2336 (2010)
7. Cafagna, D., Grassi, G.: Observer-based synchronization for a class of fractional chaotic systems via a scalar signal: results involving the exact solution of the error dynamics. Int. J. Bifurc. Chaos **21**, 955–962 (2011)
8. Zheng, Y.G., Nian, Y.B., Wang, D.J.: Controlling fractional order chaotic systems based on Takagi-Sugeno fuzzy model and adaptive adjustment mechanism. Phys. Lett. A **375**, 125–129 (2010)
9. Pecora, L., Carroll, T.: Synchronization in chaotic systems. Phys. Rev. Lett. **64**, 821–824 (1990)
10. Gong, Y.B., Xie, Y.H., Lin, X., Hao, Y.H., Ma, X.G.: Ordering chaos and synchronization transitions by chemical delay and coupling on scale-free neuronal networks. Chaos Solitons Fractals **43**, 96–103 (2010)
11. Li, C.B., Wang, J., Hu, W.: Absolute term introduced to rebuild the chaotic attractor with constant Lyapunov exponent spectrum. Nonlinear Dyn. (2011). doi:10.1007/s11071-011-0239-4
12. Wang, W.X., Huang, L., Lai, Y.C., Chen, G.: Onset of synchronization in weighted scale-free networks. Chaos **19**, 013134 (2009)
13. Chen, D.Y., Zhao, W.L., Ma, X.Y., Zhang, R.F.: No-chattering sliding mode control chaos in Hindmarsh-Rose neurons with uncertain parameters. Comput. Math. Appl. **61**, 3161–3171 (2011)
14. Wang, C.C., Yau, H.T.: Nonlinear dynamic analysis and sliding mode control for a gyroscope system. Nonlinear Dyn. **66**, 53–65 (2011)
15. Li, H.Y., Hu, Y.A.: Robust sliding-mode backstepping design for synchronization control of cross-strict feedback hyperchaotic systems with unmatched uncertainties. Commun. Nonlinear Sci. Numer. Simul. **16**, 3904–3913 (2011)
16. Chen, D.Y., Liu, Y.X., Ma, X.Y., Zhang, R.F.: Control of a class of fractional-order chaotic systems via sliding mode. Nonlinear Dyn. **67**, 893–901 (2012)
17. Zeng, C.B., Yang, Q.G., Wang, J.W.: Chaos and mixed synchronization of a new fractional-order system with one saddle and two stable node-foci. Nonlinear Dyn. **65**, 457–466 (2011)
18. Chen, D.Y., Wu, C., Liu, C.F., Ma, X.Y., You, Y.J., Zhang, R.F.: Synchronization and circuit simulation of a new double-wing chaos. Nonlinear Dyn. **67**, 1481–1504 (2012)
19. Sharma, B.B., Kar, I.N.: Observer based synchronization scheme for a class of chaotic systems using contraction theory. Nonlinear Dyn. **63**, 429–445 (2011)
20. Li, S.Y., Ge, Z.M.: Pragmatical adaptive synchronization of different chaotic systems with all uncertain parameters via nonlinear control. Nonlinear Dyn. **64**, 77–87 (2011)
21. Chen, D.Y., Shi, L., Chen, H.T., Ma, X.Y.: Analysis and control of a hyperchaotic system with only one nonlinear term. Nonlinear Dyn. **67**, 1745–1752 (2012)
22. Zhang, R.X., Yang, S.P.: Robust chaos synchronization of fractional-order chaotic systems with unknown parameters and uncertain perturbations. Nonlinear Dyn. (2011). doi:10.1007/s11071-011-0320-z
23. Yuan, L.G., Yang, Q.G.: Parameter identification and synchronization of fractional-order chaotic systems. Commun. Nonlinear Sci. Numer. Simul. **17**, 305–316 (2012)
24. Odibat, Z.: A note on phase synchronization in coupled chaotic fractional order systems. Nonlinear Anal., Real World Appl. **13**, 779–789 (2012)
25. Wen, J., Jiang, C.S.: Adaptive fuzzy control for a class of chaotic systems with nonaffine inputs. Commun. Nonlinear Sci. Numer. Simul. **16**, 475–492 (2011)
26. Lin, T.C., Lee, T.Y.: Chaos synchronization of uncertain fractional-order chaotic systems with time delay based on adaptive fuzzy sliding mode control. IEEE Trans. Fuzzy Syst. **19**, 623–635 (2011)
27. Yau, H.T.: Chaos synchronization of two uncertain chaotic nonlinear gyros using fuzzy sliding mode control. Mech. Syst. Signal Process. **22**, 408–418 (2008)
28. Chen, D.Y., Zhang, R.F., Sprott, J.C., Chen, H.T., Ma, X.Y.: Synchronization between integer-order chaotic systems and a class of fractional-order chaotic systems via sliding mode control. Chaos **22**, 023130 (2012)
29. Yang, L.X., He, W.S., Liu, X.J.: Synchronization between a fractional-order system and an integer order system. Comput. Math. Appl. **62**, 4708–4716 (2011)
30. Asheghan, M.M., Beheshti, M.T.H., Tavazoei, M.S.: Robust synchronization of perturbed Chen's fractional-order chaotic systems. Commun. Nonlinear Sci. Numer. Simul. **16**, 1044–1051 (2011)
31. Kuo, C.L., Li, T.H.S., Guo, N.R.: Design of a novel fuzzy sliding-mode control for magnetic ball levitation system. J. Intell. Robot. Syst. **42**, 295–316 (2005)
32. Roopaei, M., Jahromi, M., Zolghadri, M.: Chattering-free fuzzy sliding mode control in MIMO uncertain systems. Nonlinear Anal. **71**, 4430–4437 (2009)
33. Lin, W.: Global enistence theory and chaos control of fractional differential equations. J. Math. Anal. Appl. **332**, 709–726 (2007)
34. Lü, J.G.: Chaotic dynamics of the fractional-order Lü system and its synchronization. Phys. Lett. A **360**, 171–185 (2006)
35. Tang, Y., Wang, Z.D., Fang, J.A.: Pinning control of fractional-order weighted complex networks. Chaos **19**, 013112 (2009)
36. Bhalekar, S., Daftardar-Gejji, V.: Fractional ordered Liu system with time-delay. Commun. Nonlinear Sci. Numer. Simul. **15**, 2178–2191 (2010)

37. Lü, J., Chen, G.: A new chaotic attractor coined. Int. J. Bifurc. Chaos **12**(3), 659–661 (2002)
38. Lü, J.G.: Chaotic dynamics of the fractional-order *Lü* system and its synchronization. Phys. Lett. A **354**, 305–311 (2006)
39. Gao, T.G., Chen, G.R., Chen, Z.Q.: The generation and circuit implementation of a new hyper-chaos based upon Lorenz system. Phys. Lett. A **361**, 78–86 (2007)
40. Wu, X.J., Lu, H.T., Shen, S.L.: Synchronization of a new fractional-order hyperchaotic system. Phys. Lett. A **373**, 2329–2337 (2009)