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Stochastic fractional optimal control of quasi-integrable Hamiltonian system with fractional derivative damping

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Abstract A stochastic fractional optimal control strategy for quasi-integrable Hamiltonian systems with fractional derivative damping is proposed. First, equations of the controlled system are reduced to a set of partially averaged Itô stochastic differential equations for the energy processes by applying the stochastic averaging method for quasi-integrable Hamiltonian systems and a stochastic fractional optimal control problem (FOCP) of the partially averaged system for quasi-integrable Hamiltonian system with fractional derivative damping is formulated. Then the dynamical programming equation for the ergodic control of the partially averaged system is established by using the stochastic dynamical programming principle and solved to yield the fractional optimal control law. Finally, an example is given to illustrate the application and effectiveness of the proposed control design procedure.

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1 Introduction

Fractional calculus is the generalization of the classical calculus and it has growing applications in various fields of science and engineering, e.g., viscoelasticity [1–9], biology [10], electronics [11], diffusion [12], signal processing [13], control [14–28]. Applications in mechanics mainly involve fractional derivative, which is an adequate tool to model the frequency-dependent damping behavior of many materials. Preliminary investigation in this field traces back to the work by Gemant [1], the first to propose the fractional derivative model of the viscoelastic material. Bagley and Torvik [2–4] have shown that fractional derivative models can describe the frequency-dependent damping behavior of some materials very well. Koeller [5] has considered a fractional derivative model to describe creep and relaxation in viscoelastic materials. Makris and Constantinou [6] have proposed a fractional-derivative Maxwell model for viscous dampers and validated their model using experimental results. Coronado et al. [7] have used the fractional derivative to model the frequency-dependent viscoelastic isolators in the study of a passive isolation system. It has been shown that the fractional derivative model leads to better approximation of dissipated energy and requires less parameters and its fitting converges quickly. Rossikhin and Shitikova [8, 9] have provided extensive reviews of applications of fractional calculus in dynamic problems of solid mechanics. Various approaches have been proposed to find the response of a fractionally damped system, including the Laplace transform [3, 4], the Fourier transform [29–31], numerical methods [32, 33], eigenvector expansion [34], the averaging method [35, 36], Adomian decomposition method [37], and variational iteration method [38]. Padovan and Sawicki [39] have established the harmonic solution of periodically excited nonlinear systems with fractional derivative damping by employing a diophantine version of the fractional operational powers and using the energy constrained Lindstedt–Poincare perturbation procedure.

Actually, dynamic systems, including those containing fractional derivative damping, are often subjected to random loading. It is necessary to study the stochastic response of fractionally damped materials and structures. Spanos and Zeldin [40] proposed a frequency-domain approach for the random vibration of fractionally damped systems. Agrawal presented an analytical scheme for stochastic dynamic systems with fractional derivative damping using the eigenvector expansion method and the properties of the Laplace transforms of convolution integrals [41], and a general analytical technique for stochastic analysis of a fractionally damped beam using the normal mode and Laplace transform techniques [42]. Huang and Jin [43] studied the response and stability of a SDOF strongly nonlinear stochastic system with light fractional derivative damping using the stochastic averaging method. Chen and Zhu [44] investigated the first passage failure of Duffing oscillator with small fractional derivative damping under combined harmonic and white noise excitations using the stochastic averaging method. Furthermore, an analytical scheme to determine the statistical behavior of a stochastic system including two terms of fractional derivative with real, arbitrary, fractional orders has been proposed by Huang et al. [45], who obtained the Green's functions based on the Laplace transform and the weighted generalized Mittag–Leffler function. Recently, Di Paola et al. [46] computed the stochastic response of a SDOF structural systems with fractional derivative damping subjected to stationary and non-stationary inputs through an appropriate change of variable and a discretization of the fractional derivative operator.

A FOCP is an optimal control problem in which the performance index and/or the differential equations governing the dynamics of the system contain at least one fractional derivative operator. The publications on FOCPs are limited compared to those on the integer order optimal control problems. Fractional order control was first introduced by Tustin et al. [14] for the position control of massive objects in the 1950s, and some other researches were fulfilled around the 1960s by Manabe [15]. Podlubny [16] proposed a generalization of the PID controllers, namely the $\mathrm{P}I^{\lambda}D^{\mu}$ controllers, involving an integrator of order λ and differentiator of order μ (where λ and μ are assumed to be real numbers), and Monje et al. [17] suggested the tuning rules for $\mathrm{P}I^{\lambda}D^{\mu}$ controllers. The fractional order [PD] controller as well as the fractional order [PI] controller have been proposed [18, 19], and their tuning rules were investigated [20]. Agrawal [21, 22] combined the calculus of variations and concept of fractional derivatives to develop Euler–Lagrange equations for the FOCPs and used a variational virtual work based formulation to develop a numerical scheme in which the solution was approximated by using some approximating functions over the entire domain. In [23], a similar approach was used to formulate a FOCP, and a direct numerical scheme was proposed to solve the two-point boundary value problems. Agrawal [24] also presented a quadratic formulation for a class of FOCPs in which the fractional dynamics were defined in terms of Caputo derivatives. In this formulation, the calculus of variations, the Lagrange multiplier technique, and the formula for fractional integration by parts were used to obtain the Euler–Lagrange equations for the FOCPs. Baleanu et al. [25] developed a central difference numerical scheme for the solution of FOCPs. Tricaud et al. [26] introduced a new formulation for solving a wide class of fractional optimal control problems using the Ousaloup recursive approximation. In [27], a new method for optimal tuning of fractional controllers was proposed by using genetic algorithms. Jumarie [28] derived a Hamilton–Jacobi equation and a Lagrangian variational approach for the optimal control of nonrandom fractional dynamics with fractional cost function using the variational calculus of fractional order.

So far, to the authors' knowledge, no work on the fractional optimal control of stochastic dynamic system with fractional derivative damping is available.

This subject forms the primary objective of this paper. The major contribution of this paper is to propose a fractional optimal control strategy for stochastic quasi-integrable Hamiltonian systems with fractional derivative damping. The key idea is to apply the stochastic averaging method to original systems and derive the fractional optimal control law by establishing and solving the dynamical programming equation. Finally, an example is treated for illustration.

2 Formulation of the problem

Consider an n degree-of-freedom (DOF) controlled, stochastically excited and dissipated quasi-Hamiltonian system with fractional derivative damping. The equations of the system are of the form

$$\begin{aligned}\dot{Q}_i &= \frac{\partial H}{\partial P_i}\\ \dot{P}_i &= -\frac{\partial H}{\partial Q_i} - \varepsilon c_{ij}(\mathbf{Q},\mathbf{P})D_j^{\alpha_j}(Q_j) + \varepsilon u_i\\ &\quad + \varepsilon^{1/2} f_{ik}(\mathbf{Q},\mathbf{P})W_k(t)\\ &\quad i,j=1,2,\ldots,n;\ k=1,2,\ldots,m\end{aligned} \tag{1}$$

where Q_i and P_i are generalized displacements and momenta, respectively; ε is a small positive parameter; $H = H(\mathbf{Q},\mathbf{P})$ is twice differentiable Hamiltonian generally representing total energy of system (1); $\varepsilon c_{ij}(\mathbf{Q},\mathbf{P})$ are differentiable functions representing coefficients of fractional dampings; $D_j^{\alpha_j}(Q_j)$ are fractional derivative dampings; $\varepsilon u_i = \varepsilon u_i(\mathbf{Q},\mathbf{P})$ denote fractional feedback control forces to be determined; $\varepsilon^{1/2} f_{ik}(\mathbf{Q},\mathbf{P})$ are twice differentiable functions representing amplitudes of stochastic excitations; both $c_{ij}(\mathbf{Q},\mathbf{P})$ and $f_{ik}(\mathbf{Q},\mathbf{P})$ are bounded; $W_k(t)$ are Gaussian white noises in the sense of Stratonovich with correlation functions $E[W_k(t)W_l(t+\tau)] = 2D_{kl}\delta(\tau)$.

The Hamiltonian system associated with system (1) is

$$\dot{q}_i = \frac{\partial H}{\partial p_i},\quad \dot{p}_i = -\frac{\partial H}{\partial q_i},\quad i=1,2,\ldots,n \tag{2}$$

Assume that system (2) is integrable, i.e., it has n independent integrals of the motion $H_1, H_2, \ldots, H_n$ which are in involution. Then system (1) is a quasi-integrable Hamiltonian system.

In order to solve the problem analytically, the following Riemann–Liouville definition for fractional derivative is adopted:

$$\begin{aligned}&D_j^{\alpha_j}\big(Q_j(t)\big)\\ &= \frac{1}{\Gamma(n-\alpha_j)}\left(\frac{d}{dt}\right)^n \int_0^t \frac{Q_j(\tau)}{(t-\tau)^{\alpha_j-n+1}}\,d\tau\\ &\quad (n-1)\le \alpha_j < n\end{aligned} \tag{3}$$

where n is integer and $\Gamma(\bullet)$ is gamma function, in the present work, the value of fractional order α_j is restricted to 0∼1, the case most relevant to structural damping.

Assume that the Hamiltonian H is separable, i.e.,

$$H = \sum_{i=1}^{n} H_i(q_i,p_i),\quad H_i(q_i,p_i) = p_i^2/2 + V_i(q_i) \tag{4}$$

where $H_i(q_i,p_i)$ represent the energies of the ith oscillator consisting of the kinetic energies ${p_i}^2/2$ and the following potential energies $V_i(q_i)$:

$$V_i(q_i) = \int_0^{q_i} g_i(x)\,dx \tag{5}$$

where $g_i(x)$ denote linear or nonlinear restoring forces of the ith oscillator. Then Eq. (2) can be rewritten into the following form:

$$\dot{q}_i = p_i,\quad \dot{p}_i = -g_i(q_i),\quad i=1,2,\ldots,n \tag{6}$$

System (6) has a family of periodic solutions surrounding the origin of the phase plane (q_i,p_i) if $V_i(q_i)\ge 0$ is symmetric with respect to $q_i = 0$, and with minimum at $q_i = 0$. The periodic solution of system (6) can be written as [47]

$$\begin{aligned}q_i(t) &= a_i\cos\theta_i(t)\\ p_i(t) &= -a_i\upsilon_i(a_i,\theta_i)\sin\theta_i(t)\\ \theta_i(t) &= \phi_i(t) + \gamma_i\end{aligned} \tag{7}$$

where

$$\upsilon_i(a_i,\theta_i) = \frac{d\phi_i}{dt} = \sqrt{\frac{2[V_i(a_i) - V_i(a_i\cos\theta_i)]}{{a_i}^2\sin^2\theta_i}} \tag{8}$$

in which a_i is related to H_i as follows:

$$V_i(a_i) = V_i(-a_i) = H_i \tag{9}$$

cos and sin are the so-called generalized harmonic functions. a_i, $\upsilon_i(a_i,\theta_i)$, and θ_i are the amplitude, instantaneous frequency, and phase angle, respectively, of system (6). Expand $\upsilon_i(a_i,\theta_i)$ into a Fourier series as follows:

$$\upsilon_i(a_i,\theta_i)=c_0^i(a_i)+\sum_{r=1}^{\infty}c_r^i(a_i)\cos r\theta_i \tag{10}$$

Integrating Eq. (10) with respect to θ_i from 0 to 2π leads to the following approximate average frequency:

$$\omega_i(a_i)=\frac{1}{2\pi}\int_0^{2\pi}\upsilon_i(a_i,\theta_i)\,d\theta_i=c_0^i(a_i) \tag{11}$$

of the ith oscillator. Thus, in average, $\omega_i(a_i)$ can be used to approximate $\upsilon_i(a_i,\theta_i)$ and

$$\theta_i(t)\approx\omega_i(a_i)t+\gamma_i \tag{12}$$

Since ε is small, the sample motion of system (1) is periodically random process and can be written as

$$\begin{aligned}&Q_i(t)=A_i\cos\Theta_i(t)\\&P_i(t)=\dot{Q}_i(t)=-A_i\upsilon_i(A_i,\Theta_i)\sin\Theta_i(t)\\&\Theta_i(t)=\Phi_i(t)+\Gamma_i(t)\end{aligned} \tag{13}$$

where

$$\begin{aligned}\upsilon_i(A_i,\Theta_i)&=\frac{d\Phi_i}{dt}=\sqrt{\frac{2[V_i(A_i)-V_i(A_i\cos\Theta_i)]}{{A_i}^2\sin^2\Theta_i}}\\&=e_0^i(A_i)+\sum_{r=1}^{\infty}e_r^i(A_i)\cos r\Theta_i\end{aligned} \tag{14}$$

in which $A_i,\Theta_i,\Phi_i,\Gamma_i$ are all random processes, A_i are related to H_i in a similar way as Eq. (9). The average frequency can be obtained in a way similar to that in Eq. (11). Treating Eq. (13) as a generalized van der Pol transformation from Q_i,P_i to A_i,Γ_i yields the following stochastic differential equation for A_i and Γ_i:

$$\begin{aligned}\frac{dA_i}{dt}&=\varepsilon\big(F_i^{(11)}(\mathbf{A},\boldsymbol{\Theta})+F_i^{(12)}(\mathbf{A},\boldsymbol{\Theta},\mathbf{u})\big)\\&\quad+\varepsilon^{1/2}G_{ik}^{(1)}W_k(t)\\\frac{d\Gamma_i}{dt}&=\varepsilon\big(F_i^{(21)}(\mathbf{A},\boldsymbol{\Theta})+F_i^{(22)}(\mathbf{A},\boldsymbol{\Theta},\mathbf{u})\big)\\&\quad+\varepsilon^{1/2}G_{ik}^{(2)}W_k(t)\end{aligned} \tag{15}$$

where $\mathbf{A}=[A_1,A_2,\ldots,A_n]^T$, $\boldsymbol{\Theta}=[\Theta_1,\Theta_2,\ldots,\Theta_n]^T$, $\mathbf{u}=[u_1,u_2,\ldots,u_n]^T$, and

$$\begin{aligned}F_i^{(11)}&=\frac{A_i\upsilon_i\sin\Theta_i}{g_i(A_i)}c_{ij}(\mathbf{A},\boldsymbol{\Theta})D_j^{\alpha_j}(A_j\cos\Theta_j)\\F_i^{(21)}&=\frac{\upsilon_i\cos\Theta_i}{g_i(A_i)}c_{ij}(\mathbf{A},\boldsymbol{\Theta})D_j^{\alpha_j}(A_j\cos\Theta_j)\\F_i^{(12)}&=-\frac{A_i\upsilon_i\sin\Theta_i}{g_i(A_i)}u_i\\F_i^{(22)}&=-\frac{\upsilon_i\cos\Theta_i}{g_i(A_i)}u_i\\G_{ik}^{(1)}&=-\frac{A_i\upsilon_i\sin\Theta_i}{g_i(A_i)}f_{ik}\\G_{ik}^{(2)}&=-\frac{\upsilon_i\cos\Theta_i}{g_i(A_i)}f_{ik}\end{aligned} \tag{16}$$

According to the stochastic averaging principle [48–50], in the case that system (2) is nonresonant, $\mathbf{A}$ converges weakly to an n-dimensional diffusion Markov process as $\varepsilon\to0$ in a time interval $0\le t\le T$, where $T\sim O(\varepsilon^{-1})$. This limiting diffusion process is governed by the following partially averaged Itô stochastic differential equations:

$$\begin{aligned}dA_i&=\big[m_i(\mathbf{A})+\varepsilon\big\langle F_i^{(12)}(\mathbf{A},\boldsymbol{\Theta},\mathbf{u})\big\rangle_\Theta\big]dt\\&\quad+\sigma_{ik}^{(1)}(\mathbf{A})dB_k(t)\end{aligned} \tag{17}$$

where $B_k(t)$ are standard Wiener processes and

$$\begin{aligned}m_i(\mathbf{A})&=\varepsilon\Big\langle F_i^{(11)}+D_{kl}\frac{\partial G_{ik}^{(1)}}{\partial A_j}G_{jl}^{(1)}\\&\quad+D_{kl}\frac{\partial G_{ik}^{(1)}}{\partial\Gamma_j}G_{jl}^{(2)}\Big\rangle_\Theta\\b_{ij}(\mathbf{A})&=\sigma_{ik}^{(1)}(\mathbf{A})\sigma_{jk}^{(1)}(\mathbf{A})=\varepsilon\big\langle2G_{ik}^{(1)}D_{kl}G_{jl}^{(1)}\big\rangle_\Theta\end{aligned} \tag{18}$$

in which $\langle\bullet\rangle_\Theta$ represents the averaging with respect to Θ.

It is seen from Eq. (15) that the time rates of A_i,Γ_i are of the order of ε, which means that A_i and Γ_i are slowly varying processes. Thus, the following approximate relation can be obtained by using Eq. (12):

$$\Theta_i(t-\tau) \approx \Theta_i(t) - \omega_i(A_i)\tau \tag{19}$$

By using the approximate relation in Eq. (19) and the following asymptotic integrals:

$$\begin{aligned}
\int_0^t \frac{\cos(\omega\tau)}{\tau^q}\,d\tau &= \omega^{(q-1)}\int_0^s \frac{\cos(u)}{u^q}\,du \\
&= \omega^{(q-1)}\left(\Gamma(1-q)\sin\left(\frac{q\pi}{2}\right) + \frac{\sin(s)}{s^q} + O\left(s^{(-q-1)}\right)\right) \\
\int_0^t \frac{\sin(\omega\tau)}{\tau^q}\,d\tau &= \omega^{(q-1)}\int_0^s \frac{\sin(u)}{u^q}\,du \\
&= \omega^{(q-1)}\left(\Gamma(1-q)\cos\left(\frac{q\pi}{2}\right) - \frac{\cos(s)}{s^q} + O\left(s^{(-q-1)}\right)\right) \\
&(u=\omega\tau,\ s=\omega t)
\end{aligned} \tag{20}$$

$\langle F_i^{(11)}\rangle_\Theta$ in Eq. (18) can be obtained as follows:

$$\begin{aligned}
\langle F_i^{(11)}\rangle_\Theta \approx -\frac{A_j}{g_i(A_i)} \times \frac{1}{2\pi\omega_j^{1-\alpha_j}} &\times \int_0^{2\pi} \frac{d}{dt}\left[A_i\upsilon_i \sin\Theta_i c_{ij}(\mathbf{A},\boldsymbol{\Theta})\right] \\
&\times \left[\cos\Theta_j \sin(\alpha_j\pi/2) + \sin\Theta_j \cos(\alpha_j\pi/2)\right] d\boldsymbol{\Theta}
\end{aligned} \tag{21}$$

Substituting Eq. (21) into Eq. (18) and completing the averaging with respect to $\boldsymbol{\Theta}$ yield the explicit expression for $m_i(\mathbf{A})$ and $b_{ij}(\mathbf{A})$.

Since A_i is related to H_i in a similar way as Eq. (9), the averaged *Itô* stochastic differential equations for H_i can be obtained from Eq. (17) by using the *Itô* differential rule as follows:

$$\begin{aligned}
dH_i &= \left[\bar{m}_i(\mathbf{H}) + \varepsilon\langle \bar{F}_i^{(12)}(\mathbf{H},\boldsymbol{\Theta},\mathbf{u})\rangle_\Theta\right]dt + \bar{\sigma}_{ik}^{(1)}(\mathbf{H})dB_k(t) \\
&i,j=1,2,\ldots,n;\ k=1,2,\ldots,m
\end{aligned} \tag{22}$$

where

$$\begin{aligned}
\bar{m}_i(\mathbf{H}) &= \left[g_i(A_i)m_i(\mathbf{A}) + \frac{1}{2}\frac{d[g_i(A_i)]}{dA_i}b_{ii}(\mathbf{A})\right]_{A_i=V_i^{-1}(H_i)} \\
\bar{F}_i^{(12)}(\mathbf{H},\boldsymbol{\Theta},\mathbf{u}) &= \frac{\partial H_i}{\partial P_i}u_i \\
\bar{b}_{ij}(\mathbf{H}) &= \bar{\sigma}_{ik}^{(1)}(\mathbf{H})\bar{\sigma}_{jk}^{(1)}(\mathbf{H}) \\
&= \left[g_i(A_i)g_j(A_j)b_{ij}(\mathbf{A})\right]_{A_i=V_i^{-1}(H_i)}
\end{aligned} \tag{23}$$

Since the control force $\mathbf{u}$ is unknown so far, the averaging of term $\bar{F}_i^{(12)}$ will be completed later. In the following, Eq. (22) is used to replace system (1) to design a fractional feedback control law for system (1).

It is seen that the dimension of averaged Eq. (22) is only half of that of original Eq. (1), and the former equation contains only slowly varying process, while the later equation contains both rapidly and slowly varying processes, thus averaged Eq. (22) is much simpler than original Eq. (1).

The objective of control is to minimize the response of uncontrolled system (1), which can be expressed in terms of minimizing a performance index. For semi-infinite time-interval ergodic control, the fractional order performance index is of the form

$$J = \lim_{T\to\infty}\frac{1}{T}\int_0^T \left\langle L\left(\mathbf{H}, \left(|\mathbf{u}|^\beta \operatorname{sign}(\mathbf{u})\right)\right)\right\rangle_\Theta dt \tag{24}$$

where $|\mathbf{u}|^\beta \operatorname{sign}(\mathbf{u}) = (|u_1|^\beta \operatorname{sign}(u_1), |u_2|^\beta \operatorname{sign}(u_2), \ldots, |u_n|^\beta \operatorname{sign}(u_n))^T$; $L(\mathbf{H}, (|\mathbf{u}|^\beta \operatorname{sign}(\mathbf{u})))$ is cost function, β is arbitrary positive real number. Equations (22) and (24) constitute a stochastic optimal control problem of the partially averaged system for quasi-integrable Hamiltonian system with fractional derivative damping.

3 Fractional optimal control

Applying the stochastic dynamical programming principle [51] to system (22) with performance index (24), the following dynamical programming equation can be established:

$$\min_{\mathbf{u}}\Big\{\big\langle L\big(\mathbf{H},\big(|\mathbf{u}|^{\beta}\,\mathrm{sign}(\mathbf{u})\big)\big)\big\rangle_{\Theta} + \frac{\partial V}{\partial H_i}\big[\bar{m}_i(\mathbf{H}) + \varepsilon\big\langle \bar{F}_i^{(12)}(\mathbf{H},\boldsymbol{\Theta},\mathbf{u})\big\rangle_{\Theta}\big] + \frac{1}{2}\bar{b}_{ij}(\mathbf{H})\frac{\partial^2 V}{\partial H_i \partial H_j}\Big\} = \gamma \tag{25}$$

where $V = V(\mathbf{H})$ is value function and γ is a constant representing optimal average cost.

The optimal control law $\mathbf{u}^*$ is determined by minimizing the left-hand side of Eq. (25) with respect to $\mathbf{u}$, i.e.,

$$\left(\frac{\partial L}{\partial u_i} + \varepsilon\frac{\partial H_i}{\partial P_i}\frac{\partial V}{\partial H_i}\right)\bigg|_{\mathbf{u}=\mathbf{u}^*} = 0, \quad i = 1, 2, \ldots, n \tag{26}$$

To ensure that the obtained ${u_i}^*$ are indeed optimal, the following sufficient conditions should be satisfied:

$$\frac{\partial^2 L(\mathbf{H},(|\mathbf{u}|^{\beta}\,\mathrm{sign}(\mathbf{u})))}{\partial {u_i}^2}\bigg|_{\mathbf{u}=\mathbf{u}^*} \geq 0, \quad i = 1, 2, \ldots, n \tag{27}$$

Let function L be of the form

$$L\big(\mathbf{H},\big(|\mathbf{u}|^{\beta}\,\mathrm{sign}(\mathbf{u})\big)\big) = f(\mathbf{H}) + \varepsilon\big(|\mathbf{u}|^{\beta}\,\mathrm{sign}(\mathbf{u})\big)^T\mathbf{R}\big(|\mathbf{u}|^{\beta}\,\mathrm{sign}(\mathbf{u})\big) \tag{28}$$

where $f(\mathbf{H})$ is convex function of $\mathbf{H}$, and $\mathbf{R}$ is a positive definite matrix. Then the fractional order optimal control forces are

$$u_i^* = -\left(\frac{R_{ij}^{-1}}{2\beta}\frac{\partial V}{\partial H_j}\right)^{\frac{1}{2\beta-1}}\left|\frac{\partial H_j}{\partial P_j}\right|^{\frac{1}{2\beta-1}}\mathrm{sign}\left(\frac{\partial H_j}{\partial P_j}\right) \tag{29}$$

where R_{ij}^{-1} are the elements of $\mathbf{R}^{-1}$. If $\mathbf{R}$ is a diagonal matrix with positive elements R_i, then Eq. (29) can be reduced to

$$u_i^* = -\left(\frac{1}{2\beta R_i}\frac{\partial V}{\partial H_i}\right)^{\frac{1}{2\beta-1}}\left|\frac{\partial H_i}{\partial P_i}\right|^{\frac{1}{2\beta-1}}\mathrm{sign}\left(\frac{\partial H_i}{\partial P_i}\right) \tag{30}$$

It is seen from Eqs. (29) and (30) that an extra parameter β is added in the fractional order optimal control, compared with the ordinary integer order control, thus the proposed fractional order optimal control is more flexible and gives an opportunity to better adjust the dynamical properties of a system.

For the case of diagonal $\mathbf{R}$, substituting Eqs. (28) and (30) into Eq. (25) and completing the averaging of the terms involving $\mathbf{u}$ lead to the following final dynamical programming equation:

$$\varepsilon\left[R_i\left(\frac{1}{2\beta R_i}\frac{\partial V}{\partial H_i}\right)^{\frac{2\beta}{2\beta-1}} - \left(\frac{1}{2\beta R_i}\right)^{\frac{1}{2\beta-1}}\left(\frac{\partial V}{\partial H_i}\right)^{\frac{2\beta}{2\beta-1}}\right] \times \left\langle\left(\frac{\partial H_i}{\partial P_i}\right)^{\frac{2\beta}{2\beta-1}}\right\rangle_{\Theta} + f(\mathbf{H}) + \bar{m}_i(\mathbf{H})\frac{\partial V}{\partial H_i} + \frac{1}{2}\bar{b}_{ij}(\mathbf{H})\frac{\partial^2 V}{\partial H_i \partial H_j} = \gamma \tag{31}$$

The stochastic optimal control force $\mathbf{u}^*$ can be obtained by solving Eq. (31) and then substituting the resultant $\partial V/\partial H_i$ into Eq. (30).

The responses of uncontrolled and controlled quasi-integrable Hamiltonian systems with fractional derivative damping can then be predicted by solving the Fokker–Planck–Kolmogorov (FPK) equations associated with the fully averaged *Itô* equation (22) with $\mathbf{u} = 0$ and $\mathbf{u} = \mathbf{u}^*$, respectively.

To evaluate the performance of the proposed control strategy, the control effectiveness K_i and control efficiency μ_i are introduced as follows:

$$K_i = \frac{E[Q_i^2]_u - E[Q_i^2]_c}{E[Q_i^2]_u}, \quad \mu_i = \frac{K_i}{E[u_i^2]/2D_{ii}}, \quad i = 1, 2, \ldots, n \tag{32}$$

where $E[{Q_i}^2]_u$ and $E[{Q_i}^2]_c$ denote the mean-square displacements of the uncontrolled and controlled system, respectively; $E[{u_i}^2]$ denote the mean-square control forces.

4 Example

Consider the stochastic FOCP of two coupled Rayleigh oscillators with fractional derivative dampings subject to Gaussian white noise excitations. The equations of motion of the system are of the form:

$$\begin{aligned}&\ddot{X}_1+\left(-\beta'_{10}+\beta'_{11}\dot{X}_1^2+\beta'_{12}\dot{X}_2^2\right)D^{\alpha_1}(X_1)\\&\quad+\omega_1^2X_1=W_1(t)+u_1\\&\ddot{X}_2+\left(-\beta'_{20}+\beta'_{21}\dot{X}_1^2+\beta'_{22}\dot{X}_2^2\right)D^{\alpha_2}(X_2)\\&\quad+\omega_2^2X_2=W_2(t)+u_2\end{aligned}\tag{33}$$

where ω_i $(i=1,2)$ denote the natural frequencies of the degenerated system; β'_{i0} and β'_{ij} $(i,j=1,2)$ are small damping coefficients; α_i represent the orders of the fractional derivative dampings; $W_i(t)$ are independent Gaussian white noises in the senses of Stratonovich with intensities $2D_i$.

Assume that $\omega_1/\omega_2\neq r/s$, where r and s are prime integers. Then the Hamiltonian system associated with system (33) is integrable and nonresonant. Let $X_1=Q_1$, $\dot{X}_1=P_1$, $X_2=Q_2$, $\dot{X}_2=P_2$, the Hamiltonian is

$$\begin{aligned}&H=H_1+H_2,\qquad H_i=\frac{1}{2}P_i^2+\frac{1}{2}\omega_i^2Q_i^2,\\&\quad i=1,2\end{aligned}\tag{34}$$

By using the procedure of stochastic averaging given in Sect. 2, the following partially averaged *Itô* stochastic differential equation for H_i can be obtained:

$$\begin{aligned}dH_1&=\left[\bar{m}_1(H_1,H_2)+\left\langle u_1\frac{\partial H_1}{\partial P_1}\right\rangle_\Theta\right]dt\\&\quad+\bar{\sigma}_{11}\,dB_1(t)\\dH_2&=\left[\bar{m}_2(H_1,H_2)+\left\langle u_2\frac{\partial H_2}{\partial P_2}\right\rangle_\Theta\right]dt\\&\quad+\bar{\sigma}_{22}\,dB_2(t)\end{aligned}\tag{35}$$

where

$$\begin{aligned}&\bar{m}_1(H_1,H_2)\\&\quad=\frac{1}{\omega_1^{1-\alpha_1}}\Bigg[\beta'_{10}H_1\sin(\pi\alpha_1/2)\\&\qquad-\frac{3\beta'_{11}H_1^2\sin(\pi\alpha_1/2)}{2}\\&\qquad-\beta'_{12}H_1H_2\sin(\pi\alpha_1/2)\Bigg]+D_1\\&\bar{m}_2(H_1,H_2)\\&\quad=\frac{1}{\omega_2^{1-\alpha_2}}\Bigg[\beta'_{20}H_2\sin(\pi\alpha_2/2)\\&\qquad-\frac{3\beta'_{22}H_2^2\sin(\pi\alpha_2/2)}{2}\\&\qquad-\beta'_{21}H_1H_2\sin(\pi\alpha_2/2)\Bigg]+D_2\\&\bar{b}_{11}=\bar{\sigma}_{11}^2=2D_1H_1,\qquad\bar{b}_{22}=\bar{\sigma}_{22}^2=2D_2H_2\end{aligned}\tag{36}$$

Following the procedure described in the last section, the expressions for fractional order optimal control forces can be obtained, i.e.,

$$u_i^*=-\left(\frac{1}{2\beta R_i}\frac{\partial V}{\partial H_i}\right)^{\frac{1}{2\beta-1}}|P_i|^{\frac{1}{2\beta-1}}\operatorname{sign}(P_i)\tag{37}$$

and the final dynamical programming equation is

$$\begin{aligned}&\frac{1}{2}\bar{b}_{11}\frac{\partial^2V}{\partial H_1^2}+\frac{1}{2}\bar{b}_{22}\frac{\partial^2V}{\partial H_2^2}+\bar{m}_1\frac{\partial V}{\partial H_1}+\bar{m}_2\frac{\partial V}{\partial H_2}\\&\quad+co_1g_1(H_1,H_2)+co_2g_2(H_1,H_2)\\&\quad+f(H_1,H_2)=\gamma\end{aligned}\tag{38}$$

where

$$\begin{aligned}co_1&=\left[R_1\left(\frac{1}{2\beta R_1}\right)^{\frac{2\beta}{2\beta-1}}-\left(\frac{1}{2\beta R_1}\right)^{\frac{1}{2\beta-1}}\right]\\&\quad\times\left(\frac{1}{2\pi}\int_0^{2\pi}\left(2\sin^2\Theta\right)^{\frac{\beta}{2\beta-1}}d\Theta\right)\\co_2&=\left[R_2\left(\frac{1}{2\beta R_2}\right)^{\frac{2\beta}{2\beta-1}}-\left(\frac{1}{2\beta R_2}\right)^{\frac{1}{2\beta-1}}\right]\\&\quad\times\left(\frac{1}{2\pi}\int_0^{2\pi}\left(2\sin^2\Theta\right)^{\frac{\beta}{2\beta-1}}d\Theta\right)\\g_1(H_1,H_2)&=\left(H_1^{1/2}\frac{\partial V}{\partial H_1}\right)^s\\g_2(H_1,H_2)&=\left(H_2^{1/2}\frac{\partial V}{\partial H_2}\right)^s,\quad s=\frac{2\beta}{2\beta-1}\end{aligned}\tag{39}$$

Assume that $V(H_1,H_2)$ is of the following form:

$$V=c_1H_1+c_2H_2+c_3H_1^2+c_5H_2^2\tag{40}$$

where c_i $(i=1,2,3,5)$ are functions of H_1, H_2. Then g_i (H_1,H_2) $(i=1,2)$ are converted into the following form:

$$\begin{aligned} g_1(H_1,H_2) &= \left(c_1H_1^{1/2}+2c_3H_1^{3/2}\right)^s \\ &= Z_1^s = g'_1(Z_1), \\ Z_1 &= c_1H_1^{1/2}+2c_3H_1^{3/2} \\ g_2(H_1,H_2) &= \left(c_2H_2^{1/2}+2c_5H_2^{3/2}\right)^s \\ &= Z_2^s = g'_2(Z_2), \\ Z_2 &= c_2H_2^{1/2}+2c_5H_2^{3/2} \end{aligned} \tag{41}$$

Expand $g'_i(Z_i)$ into the following Taylor series at Z_{0i}, the values of which range from Z_i-0.05 to Z_i+0.05,

$$g'_i(Z_i) = \sum_{k=0}^{\infty} \frac{g'^{(k)}_i(Z_{0i})}{k!}(Z-Z_{0i})^k \tag{42}$$

Since the value of s is between 1 and 2 for the case $\beta \geq 1$, the first three terms of Eq. (42) are taken. Thus, $g_i(H_1, H_2)$ can be approximated as follows:

$$\begin{aligned} g_1(H_1,H_2) &= \frac{(s^2-3s+2)}{2}Z_{01}^s \\ &\quad + (2s-s^2)Z_{01}^{s-1}\left(c_1H_1^{1/2}+2c_3H_1^{3/2}\right) \\ &\quad + \frac{s(s-1)Z_{01}^{s-2}}{2} \\ &\quad \times \left(c_1^2H_1+4C_3^2H_1^3+4c_1c_3H_1^2\right) \\ g_2(H_1,H_2) &= \frac{(s^2-3s+2)}{2}Z_{02}^s \\ &\quad + (2s-s^2)Z_{02}^{s-1}\left(c_2H_2^{1/2}+2c_5H_2^{3/2}\right) \\ &\quad + \frac{s(s-1)Z_{02}^{s-2}}{2} \\ &\quad \times \left(c_2^2H_2+4C_5^2H_2^3+4c_2c_5H_2^2\right) \end{aligned} \tag{43}$$

It is seen that in order to satisfy Eq. (38), $f(H_1, H_2)$ should be of the following form:

$$\begin{aligned} & f(H_1,H_2) \\ &= l_0+l_{11}H_1+l_{12}H_2+l_{21}H_1^2 \\ &\quad + l_{22}H_1H_2+l_{23}H_2^2+l_{31}H_1^3+l_{32}H_1^2H_2 \\ &\quad + l_{33}H_1H_2^2+l_{34}H_2^3+f_{11}H_1^{1/2} \\ &\quad + f_{12}H_2^{1/2}+f_{21}H_1^{3/2}+f_{22}H_2^{3/2} \end{aligned} \tag{44}$$

where $l_{11}, l_{12}, l_{31}, l_{34}$ are given, while the other coefficients are to be determined later.

Substituting Eqs. (43) and (44) into Eq. (38) yields the coefficients in Eq. (40) and partial coefficients in Eq. (44) as follows:

$$\begin{aligned} c_1 &= \frac{-\beta_{10}-\sqrt{\beta_{10}{}^2-2s(s-1)Z_{01}^{s-2}co_1(4D_1c_3+l_{11})}}{s(s-1)Z_{01}^{s-2}co_1} \\ c_2 &= \frac{-\beta_{20}-\sqrt{\beta_{20}{}^2-2s(s-1)Z_{02}^{s-2}co_2(4D_2c_5+l_{12})}}{s(s-1)Z_{02}^{s-2}co_2} \\ c_3 &= \frac{3\beta_{11}-\sqrt{9\beta_{11}{}^2-8s(s-1)Z_{01}^{s-2}co_1l_{31}}}{4s(s-1)Z_{01}^{s-2}co_1} \\ c_5 &= \frac{3\beta_{22}-\sqrt{9\beta_{22}{}^2-8s(s-1)Z_{02}^{s-2}co_2l_{34}}}{4s(s-1)Z_{02}^{s-2}co_2} \\ l_0 &= \gamma-\Bigg(c_1D_1+c_2D_2+co_1\frac{(s^2-3s+2)}{2}Z_{01}^s \\ &\quad + co_2\frac{(s^2-3s+2)}{2}Z_{02}^s\Bigg) \\ l_{21} &= -2c_3\beta_{10}+\frac{c_1}{2}\sqrt{9\beta_{11}{}^2-8s(s-1)Z_{01}^{s-2}co_1l_{31}} \\ l_{23} &= -2c_5\beta_{20}+\frac{c_2}{2}\sqrt{9\beta_{22}{}^2-8s(s-1)Z_{02}^{s-2}co_2l_{34}} \\ l_{22} &= c_1\beta_{12}+c_2\beta_{21}, \qquad l_{33} = 2c_5\beta_{21} \\ \beta_{10} &= \frac{\beta'_{10}\sin(\pi\alpha_1/2)}{\omega_1^{1-\alpha_1}}, \qquad \beta_{11} = \frac{\beta'_{11}\sin(\pi\alpha_1/2)}{\omega_1^{1-\alpha_1}} \\ \beta_{12} &= \frac{\beta'_{12}\sin(\pi\alpha_1/2)}{\omega_1^{1-\alpha_1}}, \qquad \beta_{20} = \frac{\beta'_{20}\sin(\pi\alpha_2/2)}{\omega_2^{1-\alpha_2}} \\ \beta_{22} &= \frac{\beta'_{22}\sin(\pi\alpha_2/2)}{\omega_2^{1-\alpha_2}}, \qquad \beta_{21} = \frac{\beta'_{21}\sin(\pi\alpha_2/2)}{\omega_2^{1-\alpha_2}} \\ f_{11} &= -co_1\left(2s-s^2\right)Z_{01}^{s-1}c_1 \\ f_{12} &= -co_2\left(2s-s^2\right)Z_{02}^{s-1}c_2 \\ f_{21} &= -2co_1\left(2s-s^2\right)Z_{01}^{s-1}c_3 \\ f_{12} &= -2co_2\left(2s-s^2\right)Z_{02}^{s-1}c_5 \end{aligned} \tag{45}$$

The uncontrolled and controlled stationary FPK equations associated with the fully averaged *Itô* Eq. (35) are of the following form:

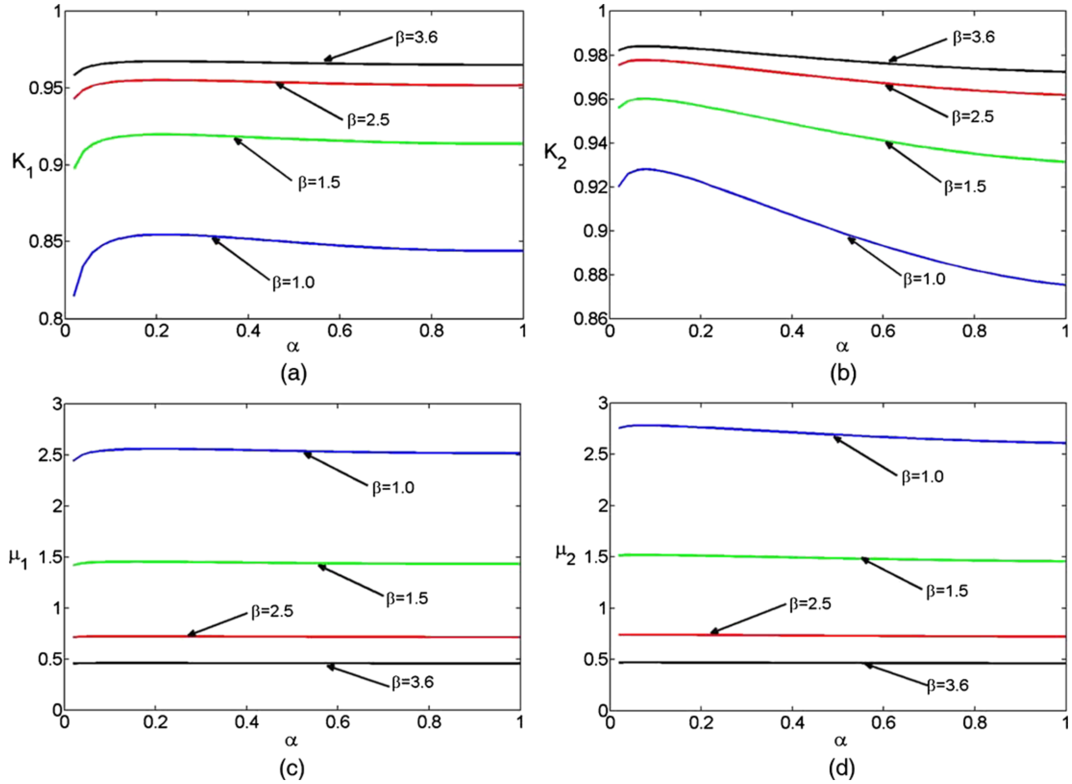

Fig. 1 (**a**) Control effectiveness K_1 for displacement of the first DOF, (**b**) control effectiveness K_2 for displacement of the second DOF, (**c**) control efficiency μ_1 for displacement of the first DOF, (**d**) control efficiency μ_2 for displacement of the second DOF, of system (33) versus fractional derivative order $\alpha_1 = \alpha_2 = \alpha$ for different β. The parameters are: $\omega_1 = 1.0$, $\omega_2 = \sqrt{2}$, $\beta'_{10} = \beta'_{20} = 0.02$, $\beta'_{11} = \beta'_{12} = \beta'_{21} = \beta'_{22} = 0.05$, $D_1 = D_2 = 0.05$, $R_1 = R_2 = 2.0$, $l_{11} = l_{12} = l_{31} = l_{34} = 0.5$

$$-\frac{\partial}{\partial H_i}\left[\bar{m}_i p^u\right] + \frac{1}{2}\frac{\partial^2}{\partial H_i{}^2}\left[\bar{b}_{ii} p^u\right] = 0 \tag{46}$$

$$-\frac{\partial}{\partial H_i}\left[\bar{\bar{m}}_i p^c\right] + \frac{1}{2}\frac{\partial^2}{\partial H_i{}^2}\left[\bar{b}_{ii} p^c\right] = 0 \tag{47}$$

where $p^u(H_1, H_2)$ and $p^c(H_1, H_2)$ denote the stationary probability densities of the uncontrolled and controlled system, respectively, and

$$\begin{aligned}\bar{\bar{m}}_1(H_1, H_2) &= \bar{m}_1(H_1, H_2) - \left(\frac{1}{2\beta R_1}\right)^{\frac{1}{2\beta-1}} \\ &\quad\times \left(\frac{1}{2\pi}\int_0^{2\pi} (2\sin^2\Theta)^{\frac{\beta}{2\beta-1}}\, d\Theta\right) \\ &\quad\times \left(c_1 H_1^{\beta} + 2c_3 H_1^{\beta+1}\right)^{\frac{1}{2\beta-1}} \\ \bar{\bar{m}}_2(H_1, H_2) &= \bar{m}_2(H_1, H_2) - \left(\frac{1}{2\beta R_2}\right)^{\frac{1}{2\beta-1}} \\ &\quad\times \left(\frac{1}{2\pi}\int_0^{2\pi} (2\sin^2\Theta)^{\frac{\beta}{2\beta-1}}\, d\Theta\right) \\ &\quad\times \left(c_2 H_2^{\beta} + 2c_5 H_2^{\beta+1}\right)^{\frac{1}{2\beta-1}}\end{aligned} \tag{48}$$

The associated boundary conditions are

$$\begin{aligned}&p^u, p^c = \text{finite at } H_i = 0 \\ &p^u, p^c, \frac{\partial p^u}{\partial H_i}, \frac{\partial p^c}{\partial H_i} \to 0 \quad \text{as } H_i \to \infty \\ &\quad i = 1, 2\end{aligned} \tag{49}$$

The FPK Eqs. (46) and (47) with boundary conditions (49) can be solved numerically by using the combination of finite difference method and successive over relaxation method. Then the stationary joint probability density for the displacements and velocity of uncontrolled and controlled systems are as follows:

$$\begin{aligned}&p^u(Q_1, Q_2, P_1, P_2) \\ &\quad = \frac{\omega_1\omega_2}{4\pi^2} p^u(H_1, H_2)\big|_{H_i = \frac{1}{2}P_i^2 + \frac{1}{2}\omega_i^2 Q_i^2} \\ &p^c(Q_1, Q_2, P_1, P_2) \\ &\quad = \frac{\omega_1\omega_2}{4\pi^2} p^c(H_1, H_2)\big|_{H_i = \frac{1}{2}P_i^2 + \frac{1}{2}\omega_i^2 Q_i^2}\end{aligned} \tag{50}$$

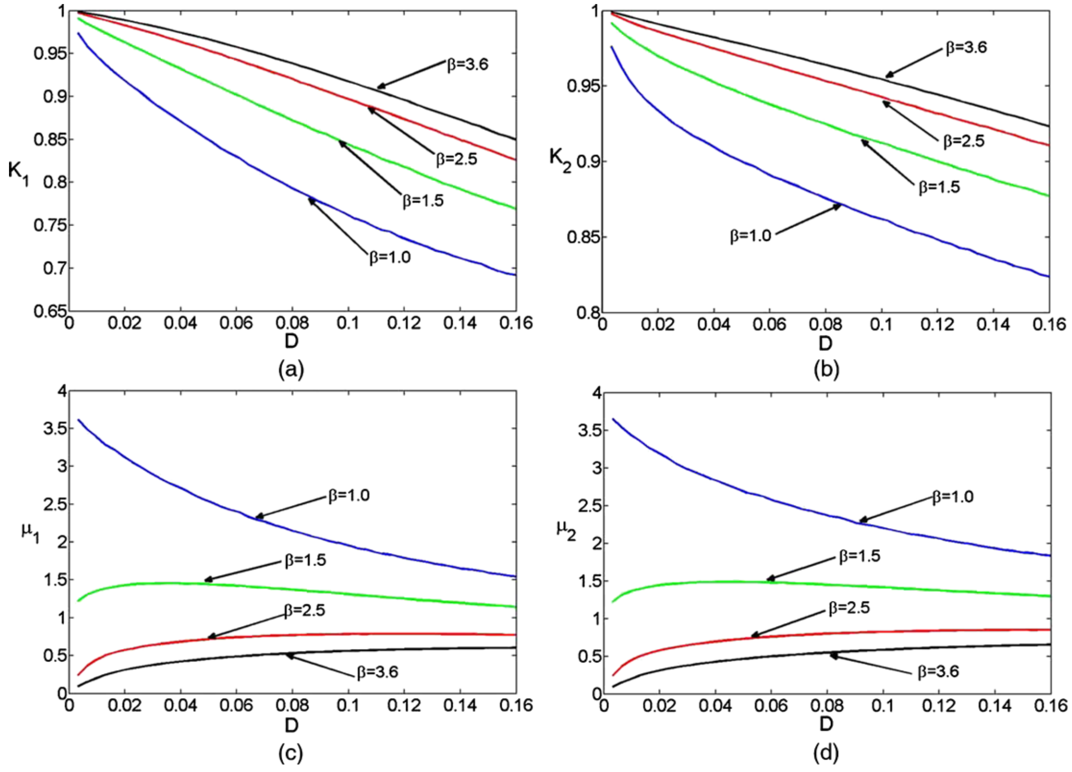

Fig. 2 (**a**) Control effectiveness K_1 for displacement of the first DOF, (**b**) control effectiveness K_2 for displacement of the second DOF, (**c**) control efficiency μ_1 for displacement of the first DOF, (**d**) control efficiency μ_2 for displacement of the second DOF of system (33) versus excitation intensity $D_1 = D_2 = D$ for different β. $\alpha_1 = \alpha_2 = 0.5$. The other parameters are the same as Fig. 1

Some numerical results for system (33) are shown in Figs. 1–4, Fig. 1 represents the control effectiveness K_i and control efficiency μ_i $(i = 1, 2)$ as functions of the order of fractional derivative damping $\alpha_1 = \alpha_2 = \alpha$ for different values of β, where $\beta = 1.0$ denotes the ordinary integer order control. It is seen that K_i and μ_i will first increase and then decrease as α increases, and the increasing interval is much smaller than the decreasing interval. The changing rates of K_i and μ_i with respect to α will decrease as β increases, while μ_1 in Fig. 1c and μ_2 in Fig. 1d are almost constants for different α values when β gets larger. Thus, both control effectiveness and control efficiency are robust to the change of the order of fractional derivative damping when β is large.

Figures 2a and 2b show that K_i will decreases as D increases, and it decreases faster for $\beta = 1.0$, while it decreases much slower for larger β, especially when D is small. It is seen from Figs. 2c and 2d that μ_i decreases faster as D increases for small β, while it will increases as D increase when β gets larger, and the increasing rate of μ_i as D increases will also increases as β gets larger.

K_i and μ_i as functions of β for different values of α are shown in Fig. 3. It is seen that K_i will increase while μ_i will decrease as β increases. When β is small, the changing rates of K_i and μ_i with respect to β are much larger than the case of larger β. Figure 3b shows that the changing rate of K_2 with respect to β will increases as α increases, while it is seen from Fig. 3a that the changing rate of K_1 with respect to β is not very sensitive to the change of α. Thus, as β increases, the changing rate of control effectiveness of the oscillator with larger natural frequency depends more on α compared with that of the oscillator with small natural frequency. Figures 3c and 3d show that the changing rates of μ_i with respect to β are almost the same for different values of α.

Figure 4 shows that K_i will decreases while μ_i will increases as R increases, and the changing rate of K_i

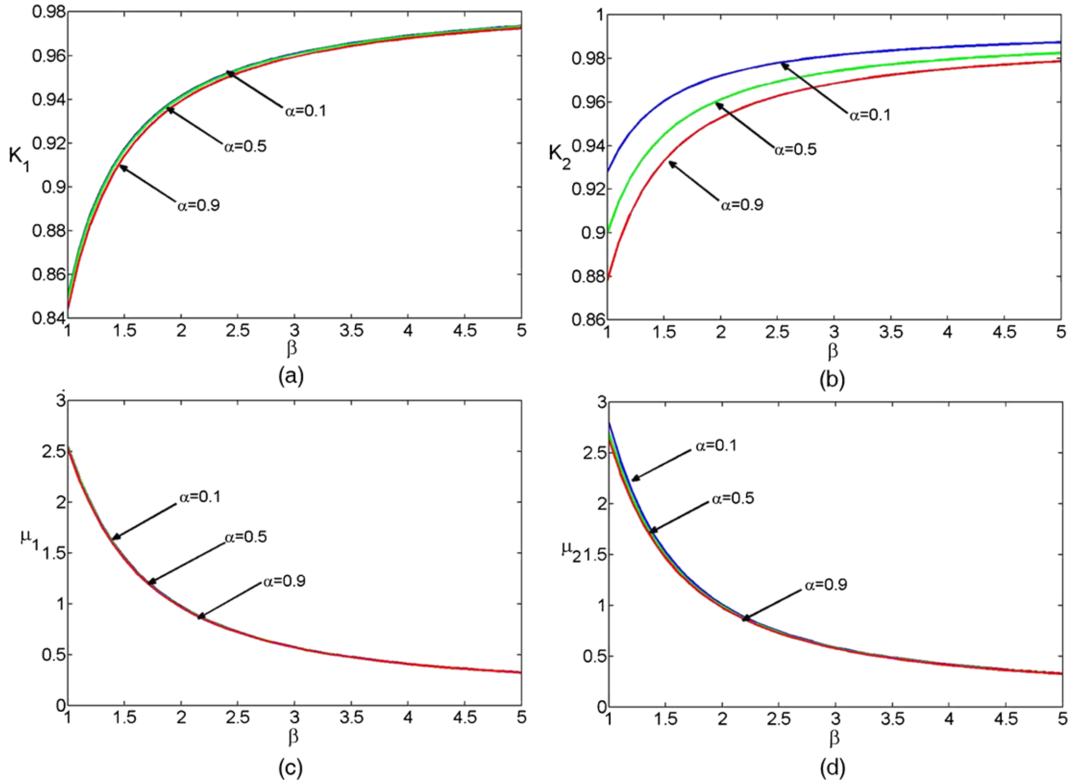

Fig. 3 (**a**) Control effectiveness K_1 for displacement of the first DOF, (**b**) control effectiveness K_2 for displacement of the second DOF, (**c**) control efficiency μ_1 for displacement of the first DOF, (**d**) control efficiency μ_2 for displacement of the second DOF of system (33) versus β for different order of fractional damping $\alpha_1 = \alpha_2 = \alpha$. The other parameters are the same as Fig. 1

and μ_i with respect to R will decrease as β increases. It can be concluded from Fig. 4 that the system will be more robust to the change of R for a fractional order control with large value of β.

In summary, as the order of fractional controller increases, control effectiveness will increases while control efficiency will decreases, and the controlled system with fractional order controller will be more robust to the changes of excitation intensity, fractional damping order, as well as parameter R in the feedback controller, compared with the one with integer order controller. Thus, the performance of fractional order control is better than that of integer order control, especially for the fractional order system.

5 Conclusion

In the present paper, a stochastic fractional optimal control strategy for quasi-integrable Hamiltonian system with fractional derivative damping has been proposed based on the stochastic averaging method for quasi-integrable Hamiltonian systems and stochastic dynamic programming principle. The partially averaged *Itô* stochastic differential equations for the controlled system were derived to replace the original system and the fractional optimal controller designed by establishing and solving the dynamic programming equation. The proposed control strategy has two-fold advantages. First, using the stochastic averaging method reduces the dimension of the controlled system, and the simplification of both the controlled system and the dynamical programming equation makes the dynamical programming equation having classical solution. Second, the proposed fractional control strategy makes the optimal control more flexible and robust, especially for fractional order systems. The proposed procedure has been applied to a two DOF coupled Rayleigh oscillators with fractional derivative damping. It has been shown that the proposed control

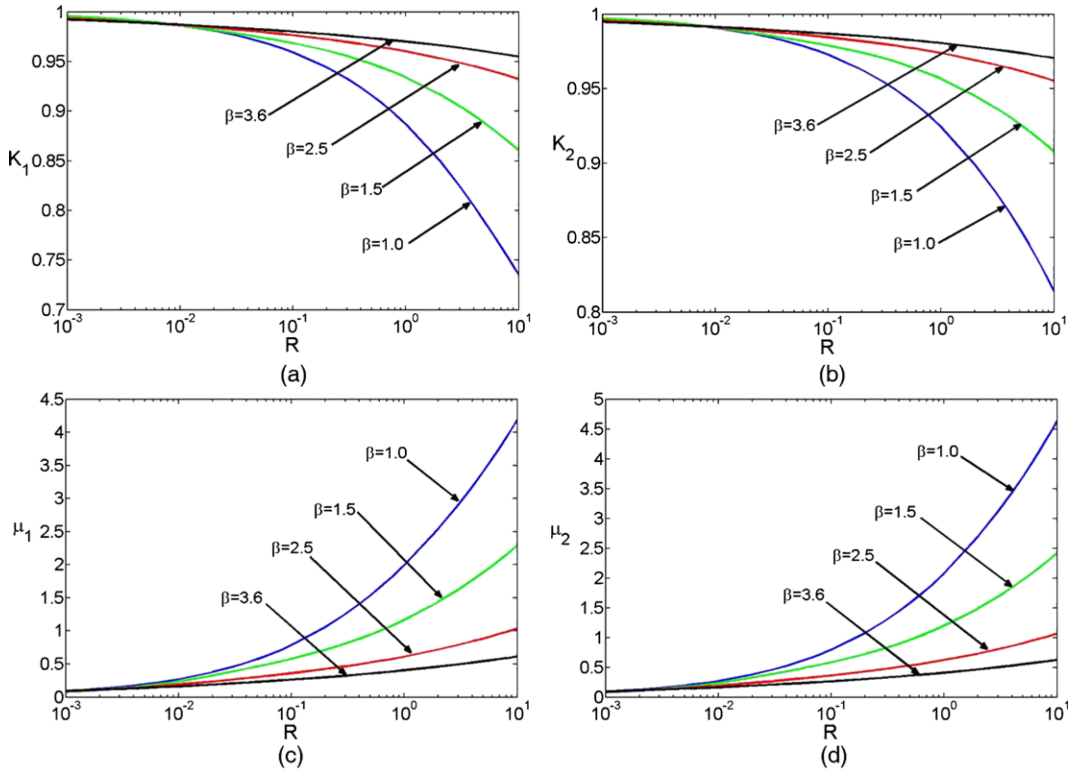

Fig. 4 (**a**) Control effectiveness K_1 for displacement of the first DOF, (**b**) control effectiveness K_2 for displacement of the second DOF, (**c**) control efficiency μ_1 for displacement of the first DOF, (**d**) control efficiency μ_2 for displacement of the second DOF of system (33) versus $R_1 = R_2 = R$ for different β. $\alpha_1 = \alpha_2 = 0.5$. The other parameters are the same as Fig. 1

strategy is effective and efficient, and the system will be more robust to the changes of parameters of system and controller by taking the fractional order control than integer order control.

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