Nonlinear Dyn (2013) 71:663–670
DOI 10.1007/s11071-012-0498-8

ORIGINAL PAPER

Fractional diffusion equations for open quantum system

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Received: 24 May 2012 / Accepted: 6 June 2012 / Published online: 15 June 2012
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Abstract To describe non-local interactions of quantum systems with environment we consider a fractional generalization of the quantum Markovian equation. Quantum analogs of fractional Laplacian operator for coordinate and momentum spaces are suggested. In phase-space form of quantum mechanics we obtain fractional equations for Wigner distribution function, where fractional Laplacian operators of the Grünvald–Letnikov type are used.

Keywords Fractional dynamics · Open quantum systems · Fractional Laplacian

1 Dynamics of open quantum systems

The natural dynamical description of open quantum systems is in terms of the infinitesimal change of the system. The infinitesimal motion is described by some form of infinitesimal generator. The problem of the non-Hamiltonian dynamics is to derive an explicit form for this infinitesimal generator. It is concerned with the problem of determining the most general explicit form of this superoperator. The problem was investigated by V. Gorini, A. Kossakowski, E.C.G. Sudarshan [1, 2] and G. Lindblad [3, 4] for completely dissipative superoperators. Superoperator is an operator that acts on operators.

Lindblad has shown that there exists a one-to-one correspondence between the completely positive norm continuous semi-groups [5, 6] and completely dissipative generating superoperators. The structural theorem of Lindblad gives the most general form of a completely dissipative superoperator: *If $\mathcal{L}$ is a completely dissipative superoperator on a W^*-algebra $\mathcal{M}$, then there exist a completely positive superoperator $\mathcal{K}$ and a self-adjoint operator $H \in \mathcal{M}$ such that*

$$\mathcal{L} = -L_H^- + \mathcal{K} - L_{\mathcal{K}(I)}^+, \tag{1}$$

where L_A^- and L_A^+ are Lie and Jordan left multiplication superoperators [10]:

$$L_A^- B = \frac{1}{i\hbar}[A, B] = \frac{1}{i\hbar}(AB - BA), \tag{2}$$

$$L_A^+ B = \frac{1}{2}[A, B]_+ = \frac{1}{2}(AB + BA). \tag{3}$$

By the Kraus theorem [7, 8], completely positive superoperators $\mathcal{K}$ on a W^*-algebra $\mathcal{M}$ can be presented in the form

$$\mathcal{K}(A) = \sum_{k=1}^{\infty} V_k^* A V_k.$$

We can formulate the following statements:

1. *The superoperators $\Phi_t = \exp(t\mathcal{L})$ is completely positive if and only if $\mathcal{L}$ has the form* (1).

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2. *If $\mathcal{K}$ is a completely positive superoperator on a C^*-algebra $\mathcal{M}$, and $H \in \mathcal{M}$ is self-adjoint superoperator, then $\mathcal{L}$ defined by* (1) *is completely dissipative*.

From these statements and the Kraus theorem follows Lindblad's structural theorem in the form: *A ultraweakly continuous superoperator $\mathcal{L}$ on a W^*-algebra $\mathcal{M}$ is completely dissipative if and only if it is of the form*

$$\mathcal{L}(A) = -\frac{1}{i\hbar}[H, A] + \frac{1}{2\hbar}\sum_{k=1}^{\infty}\big(V_k^*[A, V_k] + \big[V_k^*, A\big]V_k\big), \tag{4}$$

where $H, V_k, V_k^*, V_k^* V_k \in \mathcal{M}$.

Note that the form of $\mathcal{L}$ is not uniquely determined by (4). The equation remains invariant under the changes

$$V_k \to V_k + a_k I,$$

$$H \to H + \frac{1}{2i\hbar}\sum_{k=1}^{\infty}\big(a_k^* V_k - a_k V_k^*\big),$$

where a_k are arbitrary complex numbers.

As a corollary of Lindblad's structural theorem, a generating superoperator $\mathcal{L}$ of a completely positive unity-preserving semi-group $\{\Phi_t = \exp(t\mathcal{L}) | t \geq 0\}$ on $\mathcal{M}$ can be written by (4). Using $A_t = \Phi_t(A)$, we obtain the equation

$$\frac{d}{dt}A_t = \mathcal{L}A_t,$$

where $\mathcal{L}$ is defined by (4). This is the quantum Markovian equation (the Lindblad equation) for the quantum observable A.

The Lindblad theorem gives the explicit form of the generators of norm continuous quantum dynamical semi-groups $\Phi_t = \exp(t\mathcal{L})$ on the W^*-algebras. One can make the following more general statement for C^*-algebra: *Let $\mathcal{L}$ be a bounded real superoperator on a C^*-algebra $\mathcal{M}$ such that $\mathcal{L} = L_H + R_{H^*} + \mathcal{K}$, where $H \in \mathcal{M}$, H^* is adjoint of H, and $\mathcal{K}$ is completely positive superoperator on $\mathcal{M}$, the superoperators L_H and R_H are left and right multiplications on H. Then $\Phi_t = \exp(t\mathcal{L})$ are completely positive superoperators on $\mathcal{M}$.*

The Lindblad theorem gives the general explicit forms of equations of motion, when we introduce the following restrictions in the class of quantum non-Hamiltonian systems: (1) $\mathcal{L}$ and Λ are bounded superoperators. (2) $\mathcal{L}$ and Λ are completely dissipative superoperators. The Lindblad result has been extended by E.B. Davies [9] to a class of quantum dynamical semi-group with unbounded generating superoperators.

2 Quantum Markovian fractional equations

The structural theorem of Lindblad can be formulated in the form: *Let $\mathcal{L}$ be a completely dissipative superoperator on a W^*-algebra $\mathcal{M}$. Then the Liouville superoperator Λ on the quantum states ρ is an adjoint superoperator of $\mathcal{L}$ if and only if Λ is of the form*

$$\Lambda\rho = \frac{1}{i\hbar}[H, \rho] + \frac{1}{2\hbar}\sum_{k=1}^{\infty}\big(\big[V_k\rho, V_k^*\big] + \big[V_k, \rho V_k^*\big]\big),$$

where $H, V_k, V_k^*, V_k^* V_k \in \mathcal{M}$, and ρ is a density matrix operator.

This expression gives an explicit form of the most general time evolution equation with bounded completely dissipative Liouville superoperator. In the theory of open quantum systems, the quantum Liouville equation has the form of Markovian master equation for the density operator $\rho_s(t)$ in the Schrödinger dynamical representation:

$$\frac{d\rho_s(t)}{dt} = \Lambda\rho_s(t), \qquad \rho_s(t) = S_t\rho_s(0).$$

Here S_t denotes the dynamical semi-group [5, 6, 10] describing the evolution of the non-Hamiltonian quantum system in Schrödinger representation. The superoperator Λ is the infinitesimal generator of the dynamical semi-group S_t. Using the Lindblad theorem, which gives the most general form of a bounded completely dissipative generator Λ, we obtain the explicit form of the most general master equation of Markovian type:

$$\frac{d\rho_s(t)}{dt} = \frac{1}{i\hbar}\big[H, \rho_s(t)\big] + \frac{1}{2\hbar}\sum_{k=1}^{\infty}\big(\big[V_k\rho_s(t), V_k^*\big] + \big[V_k, \rho_s(t)V_k^*\big]\big), \tag{5}$$

where H is the Hamiltonian operator of the system and V_k, V_k^* are bounded operators on a Hilbert space

$\mathcal{H}$. We make the assumption that the general form (5) of the master equation with a bounded generator is also valid for an unbounded generator. To study the n-dimensional case, we consider the operators H, V_k, V_k^* as functions of the observables P_k and Q_l of the n-dimensional quantum system with $[Q_k, P_l] = i\hbar I \delta_{kl}$, where I is the identity operator.

It is easy to see that

$$\begin{aligned}&[V_k \rho_s, V_k^*] + [V_k, \rho_s V_k^*] \\ &\quad = V_k[\rho_s, V_k^*] + [V_k, V_k^*]\rho_s + [V_k, \rho_s]V_k^* \\ &\qquad + \rho_s[V_k, V_k^*].\end{aligned} \tag{6}$$

For simplicity, we assume that the operators V_k are self-adjoint ($V_k^* = V_k$) and

$$V_k = \lambda_k \sqrt{\frac{2}{\hbar}} A_k, \tag{7}$$

where λ_k are real numbers. Then $[V_k, V_k^*] = 0$ and (5) can be represented in the form

$$\frac{d\rho_s(t)}{dt} = \frac{1}{i\hbar}[H, \rho_s(t)] - \frac{1}{\hbar^2} \sum_{k=1}^{m} \lambda_k^2 [A_k, [A_k, \rho_s(t)]]. \tag{8}$$

Using the Lie multiplication superoperator $L_A^- B = (1/i\hbar)[A, B]$, we rewrite equation (8) as

$$\frac{d\rho_s(t)}{dt} = L_H^- \rho_s(t) + \sum_{k=1}^{\infty} \lambda_k^2 (L_{A_k}^-)^2 \rho_s(t). \tag{9}$$

To describe non-local properties of the interaction of quantum systems with environment we should generalize this equation:

$$\frac{d\rho_s(t)}{dt} = L_H^- \rho_s(t) + \sum_{k=1}^{\infty} \lambda_k(\alpha) (L_{A_k}^-)^\alpha \rho_s(t). \tag{10}$$

The transition from formula (9) to (10) implies the substitution of the second order Lie derivative by the fractional order Lie derivative. An argument supporting this substitution is related to the fact that in the classical case presence of the long-range interaction (non-locality) of power type leads to the appearance of a space fractional derivative. To realize this generalization we use the representation of the Grünvald–Letnikov derivative in the form of the infinite series, which is considered in Sect. 20.1 of [23, 24]:

$$D_x^\alpha = \sum_{j=1}^{\infty} (-1)^j b_j D_x^j, \tag{11}$$

with the coefficients

$$b_j = \sum_{\nu=j}^{\infty} (-1)^\nu \binom{\nu}{j} a_\nu, \tag{12}$$

where

$$\begin{aligned}&a_\nu = \frac{a^{(\nu)}(1)}{\nu!}, \qquad a(\xi) = \xi^\alpha, \\ &\binom{\alpha}{\beta} = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-\beta+1)\Gamma(\beta+1)}.\end{aligned} \tag{13}$$

We use the fact that the superoperators $L_{A_k}^-$ is a derivative of the first order on operator algebra $\mathcal{M}$. Therefore the fractional order Lie derivative should include the infinite series with respect of the superoperators $L_{A_k}^-$, such that

$$D_{\text{NonLocal}} = \sum_{j=1}^{\infty} (-1)^j b_j (L_{A_k}^-)^j. \tag{14}$$

If the coefficients b_j has the form (12) then we have a Lie derivative of fractional order α in the Grünvald–Letnikov form on operator algebra $\mathcal{M}$:

$$D_{\text{NonLocal}} = (L_{A_k}^-)^\alpha. \tag{15}$$

As a result, we have quantum Markovian equation for open quantum systems that are characterized by a power-law non-local interaction with environment. Therefore a generalization of (9) can be considered in the form

$$\frac{d\rho_s(t)}{dt} = L_H^- \rho_s(t) + \sum_{k=1}^{\infty} \lambda_k(\alpha) (L_{A_k}^-)^\alpha \rho_s(t). \tag{16}$$

Here we use dimensionless values. For the case $\alpha = 2$, (16) gives (9). Equation (16) can be called a quantum Markovian fractional equation. It allows us to take into account non-local properties of the interaction of open quantum system with an environment. Note that fractional dynamics of open quantum systems are considered also in [11–16]. Fractional dynamics is an application of fractional calculus [25] to describe processes with long-term memory, non-local and fractal properties.

For simplicity, we also assume that

$$A_k = P_k, \qquad A_{k+n} = Q_k, \quad (k=1,\dots,n), \qquad A_k = 0, \quad (k > 2n) \tag{17}$$

and

$$\lambda_k(\alpha) = \lambda_Q(\alpha), \qquad \lambda_{k+n}(\alpha) = \lambda_P(\alpha), \quad (k=1,\dots,n). \tag{18}$$

Then (9) can be represented in the form

$$\frac{d\rho_s(t)}{dt} = L_H^- \rho_s(t) + \lambda_Q(\alpha) \sum_{k=1}^{n} (L_{P_k}^-)^\alpha \rho_s(t) + \lambda_P(\alpha) \sum_{k=1}^{n} (L_{Q_k}^-)^\alpha \rho_s(t). \tag{19}$$

Using the Weyl symbols and the phase-space representation, we have classical limit of the superoperators

$$\Delta_Q^\alpha = \sum_{k=1}^{n} (L_{P_k}^-)^\alpha, \qquad \Delta_P^\alpha = \sum_{k=1}^{n} (L_{Q_k}^-)^\alpha \tag{20}$$

in the form

$$(\Delta_Q^2)_W = \sum_{k=1}^{n} \frac{\partial^2}{\partial q_k^2}, \qquad (\Delta_P^2)_W = \sum_{k=1}^{n} \frac{\partial^2}{\partial p_k^2}. \tag{21}$$

As a result, superoperators (20) can be considered as quantum analogs of fractional Laplacian operator in the coordinate and momentum spaces.

Note that quantum analogs of fractional Laplacian can be defined by using the generalized Taylor formula [17, 18] and the Weyl quantization [19].

3 Quantum Markovian fractional equations in the phase-space representation

The phase-space representation becomes transparent a transition from quantum to classical mechanics. This representation is suitable for considering quantum dynamics in situations where a good initial approximation comes from the classical dynamics and also for deriving classical limits of quantum processes.

The phase-space function $A(q,p)$ corresponding to the operator A is given by the equation

$$A(q,p) = \int dz e^{ipz/\hbar} \left\langle q - \frac{1}{2}z|A|q + \frac{1}{2}z \right\rangle. \tag{22}$$

The Weyl symbol $C(q,p)$ of the product of operators $C = AB$ in terms of the Weyl symbols of A and B is defined by the following relation:

$$C(q,p) = A(q,p)\left(\exp \frac{\hbar \mathcal{P}}{2i}\right) B(q,p), \tag{23}$$

where $\mathcal{P}$ is the Poisson bracket operator given by

$$\mathcal{P} = \frac{\overleftarrow{\partial}}{\partial p}\frac{\overrightarrow{\partial}}{\partial q} - \frac{\overleftarrow{\partial}}{\partial q}\frac{\overrightarrow{\partial}}{\partial p}. \tag{24}$$

Here the arrows indicate in which direction the derivatives act.

In general, a quantum system is described by a density operator ρ_s and the expectation value of an observable $A(Q,P)$ is

$$\langle A \rangle = \mathrm{Tr}[A\rho_s]. \tag{25}$$

In the phase-space representation, we have

$$\langle A \rangle = \int dp\, dq\, A(q,p) \rho_W(q,p), \tag{26}$$

where the function $A(q,p)$ is a symbol of the operator $A(Q,P)$, and $\rho_W(q,p)$ is the Wigner distribution function. The function $\rho_W(q,p)$ represents the Weyl symbol that is defined through the Fourier transform of the off-diagonal elements of the density operator (we consider the one-dimensional case) by

$$\rho_W(q,p) = \frac{1}{2\pi\hbar} \int dy \langle q - y|\rho_s|q + y\rangle e^{ipy/\hbar}, \tag{27}$$

It is easy to see that (27) is a special case of (22) for the density operator, i.e., $\rho_W(q,p)$ is the phase-space function which corresponds to the operator $\rho_s/2\pi\hbar$. Note that any real-valued distribution function $\rho_W(q,p)$ can have negative values for some p and q. The Wigner distribution function $\rho_W(q,p)$ satisfies the following properties:

(1) $\rho_W(q,p)$ is real, but cannot be everywhere positive;
(2) $\rho_W(q,p)$ has a unit trace, i.e. $\int dp\, dq \rho_W(q,p) = \mathrm{Tr}[\rho_s] = 1$.
(3) $\rho_W(q,p)$ is translation invariant and invariant with respect to space and time reflections.

The Wigner distribution functions are useful to consider the connection between classical and quantum mechanics. It is known that the first of the Wigner distributions was introduced to study quantum corrections to classical statistical mechanics. The Wigner distribution function has found many applications in statistical mechanics and quantum theory, and also in areas such as quantum chemistry, density functional theory, quantum optics, quantum chaos.

For a Hamiltonian quantum system we have the following phase-space equation which determines the time evolution of the Wigner distribution function:

$$\begin{aligned} i\hbar\frac{\partial\rho_W(t,q,p)}{\partial t} &= H(q,p)\left(\exp\frac{\hbar\mathcal{P}}{2i}\right)\rho_W(t,q,p) \\ &\quad - \rho_W(t,q,p)\left(\exp\frac{\hbar\mathcal{P}}{2i}\right)H(q,p), \end{aligned} \tag{28}$$

where $H(q,p)$ is the Weyl symbol of the Hamiltonian operator H of the system. Note that if we take the $\hbar\to 0$ limit of this equation, we obtain the classical Liouville equation.

In the force free case ($H(q,p)=p^2/2m$), we have the equation

$$\frac{\partial\rho_W(t,q,p)}{\partial t} = -\frac{p}{m}\frac{\partial\rho_W(t,q,p)}{\partial q}. \tag{29}$$

The time evolution of the Wigner distribution function corresponding to the Lindblad master equation, can be obtained from (28) by adding in the right-hand side the Weyl symbol of the non-Hamiltonian part of equation, i.e., the sum of commutators. As a result, we have the following evolution equation for the Wigner distribution:

$$\begin{aligned} \frac{\partial\rho_W(t,q,p)}{\partial t} &= -\frac{2}{\hbar}H(q,p)\left(\sin\frac{\hbar\mathcal{P}}{2}\right)\rho_W(t,q,p) \\ &\quad + \frac{1}{2\hbar}\sum_k\left(\left(\exp\frac{\hbar\mathcal{P}}{2i}\right)\rho_W(t,q,p)\left(\exp\frac{\hbar\mathcal{P}}{2i}\right)V_k\right. \\ &\quad - V_k\left(\exp\frac{\hbar\mathcal{P}}{2i}\right)V_k\left(\exp\frac{\hbar\mathcal{P}}{2i}\right)\rho_W(t,q,p) \\ &\quad \left.- \rho_W(t,q,p)\left(\exp\frac{\hbar\mathcal{P}}{2i}\right)V_k\left(\exp\frac{\hbar\mathcal{P}}{2i}\right)V_k\right), \end{aligned} \tag{30}$$

where V_k are the Weyl symbol of the operators V_k.

Let us consider a phase-space representation of quantum Markovian fractional equation. If the operators V_k are taken of the form (7), (17), then the fractional equation for the Wigner distribution has the form

$$\begin{aligned} \frac{\partial\rho_W(t,q,p)}{\partial t} &= -\frac{2}{\hbar}H(q,p)\left(\sin\frac{\hbar\mathcal{P}}{2}\right)\rho_W(t,q,p) \\ &\quad + \lambda_Q(\alpha)\sum_{k=1}^{n} D^{\alpha}_{q_k}\rho_W(t,q,p) \\ &\quad + \lambda_P(\alpha)\sum_{k=1}^{n} D^{\alpha}_{p_k}\rho_W(t,q,p), \end{aligned} \tag{31}$$

where $D^{\alpha}_{q_k}$ and $D^{\alpha}_{p_k}$ are fractional derivatives of the Grünvald–Letnikov type.

The first term on the right-hand side generates the evolution in phase space of a Hamiltonian system and gives the Poisson bracket and the higher derivatives containing the quantum contribution. The following terms represent the contribution from the non-Hamiltonian terms.

For Hamiltonian system with $H(q,p)=T(p)+U(q)$, where $U(q)$ is an analytic function, (31) takes the form

$$\begin{aligned} \frac{\partial\rho_W}{\partial t} &= -\frac{p}{m}\frac{\partial\rho_W}{\partial q} + \frac{\partial U}{\partial q}\frac{\partial\rho_W}{\partial p} \\ &\quad + \sum_{n=1}^{\infty}\frac{(-1)^n(\hbar)^{2n}}{2^{2n}(2n+1)!}\frac{\partial^{2n+1}U(q)}{\partial q^{2n+1}}\frac{\partial^{2n+1}\rho_W}{\partial p^{2n+1}} \\ &\quad + N(\rho_W), \end{aligned} \tag{32}$$

where we have introduced the notation

$$\begin{aligned} N_\alpha(\rho_W) &= \lambda_Q(\alpha)\sum_{k=1}^{n} D^{\alpha}_{q_k}\rho_W(t,q,p) \\ &\quad + \lambda_P(\alpha)\sum_{k=1}^{n} D^{\alpha}_{p_k}\rho_W(t,q,p). \end{aligned} \tag{33}$$

In the term (33) we have fractional Laplacian of the Grünvald–Letnikov type. For $\alpha=2$, we have the usual term

$$N_2(\rho_W) = \lambda_Q(2) \sum_{k=1}^{n} \frac{\partial^2 \rho_W}{\partial q_k^2} + \lambda_P(2) \sum_{k=1}^{n} \frac{\partial^2 \rho_W}{\partial p_k^2}. \quad (34)$$

In this case, we have the Laplacian of the integer order.

The following remarks should be noted.

(1) If (32) had only the first two terms on the right-hand side, $\rho_W(t, q, p)$ would evolve along the classical flow in phase space.
(2) The infinite sum in (32) together with the first two terms make up the unitary part of the evolution. Hence, up to corrections of order $\hbar^2$, unitary evolution corresponds to approximately classical evolution of the Wigner function.
(3) The higher corrections can often be assumed as negligible and give structures on small scales. There are, however, important examples where they cannot be neglected, e.g., in chaotic systems. From (32) it is clear that, as a consequence of the quantum correction terms with higher derivatives, the Wigner function of a nonlinear system does not follow the classical Liouville flow. The higher derivative terms are generated by the nonlinearities in the potential $U(q)$.
(4) The term (33) containing $\lambda_Q(\alpha)$ and $\lambda_P(\alpha)$ are the anomalous diffusive [20, 22] terms and produce an expansion of the volume elements. The anomalous diffusion terms are responsible for the destruction of interference, by erasing the structure of the Wigner function on small scales.
(5) We have two well-known limits in which (32) can go over into a classical equation:
   (a) U is at most quadratic in q;
   (b) the limit $\hbar \to 0$.

The anomalous diffusion terms, allows us to get a third classical limit. In the limit of large $\lambda_P(\alpha)$ the diffusive smoothing becomes so effective that it damps out all the momentum-derivatives in the infinite sum and (32) approaches the Liouville equation with anomalous diffusion [20, 22], an equation of fractional Fokker–Planck type [20, 21]. This is an example of how macroscopic objects start to behave classically (decoherence), since the diffusion coefficients are roughly proportional to the size of these objects. Thus an object will evolve according to classical dynamics if it has a strong interaction with its environment.

4 Examples of the quantum Markovian fractional equation

Let us consider some special cases of quantum Markovian fractional equation in the phase-space representation.

(1) In the case of a free particle, i.e., $U(q) = 0$, (32) takes the form:

$$\frac{\partial \rho_W}{\partial t} = -\frac{p}{m} \frac{\partial \rho_W}{\partial q} + N_\alpha(\rho_W). \quad (35)$$

(2) For the linear potential $U = \gamma q$ (for example, $\gamma = mg$ for the free fall or $\gamma = eE$ for the motion in a uniform electric field), we have

$$\frac{\partial \rho_W}{\partial t} = -\frac{p}{m} \frac{\partial \rho_W}{\partial q} + \gamma \frac{\partial \rho_W}{\partial p} + N_\alpha(\rho_W). \quad (36)$$

(3) For harmonic oscillator $U = m\omega^2 q^2/2$, and (32) takes the form:

$$\frac{\partial \rho_W}{\partial t} = -\frac{p}{m} \frac{\partial \rho_W}{\partial q} + m\omega^2 q \frac{\partial \rho_W}{\partial p} + N_\alpha(\rho_W). \quad (37)$$

Since the coefficients are linear in the variables q and p and the diffusion coefficients are constant with respect to q and p, (35)–(37) describe a fractional analog of Ornstein–Uhlenbeck process. Equations (35)–(37) are exactly fractional equations of the Fokker–Planck type.

Note that not every function $\rho(0, q, p)$ on the phase space is the Weyl symbol of a density operator. Hence, the quantum mechanics appears now in the restrictions imposed on the initial condition $\rho_W(0, q, p)$ for (32). The most frequently used choice for $\rho_W(0, q, p)$ is a Gaussian function and (35)–(37) preserve this Gaussian type, i.e., $\rho_W(t, q, p)$ is always a Gaussian function in time, so that the differences between quantum and classical mechanics are completely lost in this representation of the master equation.

(4) It is easy to see that the potential $U(q)$ of the finite polynomial form gives the sum with finite number of the terms. As an example, we can consider an anharmonic oscillator with the potential

$$U(q) = m\omega^2 q^2/2 + cq^4. \quad (38)$$

In this case, (32) becomes

$$\frac{\partial \rho}{\partial t} = -\frac{p}{m}\frac{\partial \rho_W}{\partial q} + \left(m\omega^2 q + 4cq^3\right)\frac{\partial \rho_W}{\partial p} - c\hbar^2 q \frac{\partial^3 \rho_W}{\partial p^3} + N_\alpha(\rho_W). \quad (39)$$

Equation (39) has one term with third derivative, associated with the nonlinear potential (38). In fact, the first three terms on the right-hand side of (39) give the usual Wigner equation of an isolated anharmonic oscillator. The third derivative term is of order $\hbar^2$ and is the quantum correction. In the classical limit, when this term is neglected, the Wigner equation becomes the fractional Fokker–Planck equation.

(5) If we use the periodic potential $U(q) = U_0 \cos(kq)$, then

$$\frac{\partial^{2n+1} U}{\partial q^{2n+1}} = (-1)^n k^{2n} \frac{\partial U}{\partial q}, \quad (40)$$

and we obtain the equation

$$\frac{\partial \rho_W}{\partial t} = -\frac{p}{m}\frac{\partial \rho_W}{\partial q} + \frac{1}{\hbar k}\frac{\partial U}{\partial q}\big(\rho_W(t,q,p+\hbar k/2) - \rho_W(t,q,p-\hbar k/2)\big) + N_\alpha(\rho_W). \quad (41)$$

Equation (41) takes a simpler form when $\hbar k$ is large compared to the momentum spread Δp of the particle being considered, i.e., when the spatial extension of the wave packet representing the particle is large compared to the spatial period of the potential. For the condition $\hbar k \gg \Delta p$, we have $(\rho_W(t,q,p+\hbar k/2) - \rho_W(t,q,p-\hbar k/2))/\hbar k \approx 0$ for all p. As a result the term $N_\alpha(\rho_W)$ with fractional Laplacian gives an important contribution to the Wigner distribution function. Equation (41) is then reduced to (35) for a free particle moving in an environment.

Using the cases (1)–(3), we see that for Hamiltonians at most quadratic in q and p, that the equation of motion of the Wigner distribution *contains only the classical part* and the contributions from the non-Hamiltonian properties of the system and obeys classical fractional Fokker–Planck equations of motion (35)–(37). In general, the potential U has terms of order higher than q^2 and one has to deal with a partial differential equation of order higher than two or generally of infinite order. When the potential deviates only slightly from the harmonic potential, one can take the classical limit $\hbar \to 0$ in (32) as the lowest-order approximation to the quantum dynamics and construct higher-order approximations that contain quantum corrections to the classical behavior using the perturbation technique.

(6) Equation (32) can be written in the form

$$\frac{\partial \rho_W}{\partial t} = -\frac{p}{m}\frac{\partial \rho_W}{\partial q} + \frac{\partial U_{\text{eff}}}{\partial q}\frac{\partial \rho_W}{\partial p} + N_\alpha(\rho_W), \quad (42)$$

where the effective potential is defined as

$$\frac{\partial U_{\text{eff}}}{\partial q}\frac{\partial \rho_W}{\partial p} = \frac{\partial U}{\partial q}\frac{\partial \rho_W}{\partial p} + \sum_{n=1}^{\infty}(-1)^n \frac{(\hbar)^{2n}}{2^{2n}(2n+1)!}\frac{\partial^{2n+1}U(q)}{\partial q^{2n+1}}\frac{\partial^{2n+1}\rho_W}{\partial p^{2n+1}}. \quad (43)$$

Then the phase-space points move under the influence of the effective potential U_{eff}. We note that (43) indicates that only when at least an approximate solution for ρ is known, the effective potential can be determined. If U does not deviate much from the harmonic potential, the zeroth-order approximation for ρ_W can be taken as that resulting from classical propagation. The main limitation of the effective potential method is that it can be applied only to systems whose behavior is not much different from that of the harmonic oscillator or the free particle.

Phase-space representation of fractional quantum dynamics provides a natural framework to study the consequences of the chaotic dynamics and its interplay with decoherence. Equation (32) can be applied in order to investigate the process of decoherence for quantum chaos. Since decoherence induces a transition from quantum to classical mechanics, it can be used to find the connection between the classical and quantum chaotic systems. In this case $U(q)$ is the potential of a classically chaotic system, coupled to the external environment.

(7) Note that the power series involving third and higher derivative terms sometimes may be neglected. The anomalous diffusion terms may smooth out the Wigner function, suppressing contributions from the higher-order terms. When these terms can be neglected, the Wigner function evolution equation (32) then becomes

$$\frac{\partial \rho_W}{\partial t} = -\frac{p}{m}\frac{\partial \rho_W}{\partial q} + \frac{\partial U}{\partial q}\frac{\partial \rho_W}{\partial p} + N_\alpha(\rho_W). \quad (44)$$

In the case of a thermal bath and if

$$\lambda_Q(\alpha) = 0, \qquad \lambda_P(\alpha) = 2m\gamma k_B T,$$

equation (44) becomes the Kramers equation

$$\frac{\partial \rho_W}{\partial t} = -\frac{p}{m}\frac{\partial \rho_W}{\partial q} + \frac{\partial U}{\partial q}\frac{\partial \rho_W}{\partial p} + 2m\gamma k_B T D^{\alpha}_{p}\rho_W. \quad (45)$$

Here we have fractional Laplacian with respect of momentum only.

The phase-space formulation of quantum dynamics is an alternative to the standard operator formulation of equation for quantum states. The main difficulty with the phase-space formulation is that the time development of the phase-space Wigner distribution is given in terms of an infinite-order partial differential equation (see (32)). This difficulty is a result of the fact that superoperator Λ is not a Weyl ordered superoperator. If Λ is a Weyl ordered superoperator [10], then we obtain a finite-order differential equation for Wigner distribution $\rho_W(q, p)$. The Weyl ordered superoperator Λ gives an operator equation with Weyl ordered operators [10]. As a result, the correspondent equation for the Wigner distribution function is finite order.

It is interesting to find a stationary solution of these equations by analogy with the stationary states of open quantum systems [26]. The positive solutions of fractional equations [27] play an important role in quantum theory.

Acknowledgements The author expresses his gratitude to the guest editors Professor Juan J. Trujillo, Professor Dumitru Baleanu, Professor Donal O'Regan for kind invitation to contribute to a special issue titled "Nonlinear Fractional Differential equations and their applications", in the journal *Nonlinear Dynamics*, which celebrates Professor Ravi P. Agarwal's 65th birthday.

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