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Robust stability of nonlinear model-based networked control systems with time-varying transmission times

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Abstract Robust stability of a class of model-based networked control systems (MB-NCSs, for short) with nonlinear perturbation is analyzed. Based on the model-based networked control algorithm, a linear model of the plant is used to estimate the plant state behavior between transmission times. The case that the nonlinear plant and the linear plant model are connected via a network channel with transmission times that are varying within a time interval is of particular interest. Sufficient condition on stability of MB-NCSs with nonlinear perturbation is given. One advantage of the proposed method is that the maximum transmission interval and the robustness bound on nonlinear perturbation can be computed. Finally, numerical simulation is worked out to show our main result.

Keywords Nonlinear perturbation · Exponentially stable · Ultimately bounded · Time-varying

1 Introduction

Recently, there has been much interest in networked control systems (NCSs, for short, [1, 2]), that is, control systems with a feedback loop closed through a communication network. It is clear that the reduction of bandwidth needed by the communication network in a NCS is a major concern. To overcome the bandwidth constraint, several approaches have been proposed. Some of these include the study of NCS with transport delays, and under noise disturbances, quantization effects and algorithms, and scheduling algorithms [1, 2].

A particular class of NCSs is model-based networked control systems (MB-NCSs), introduced in [3]. Model-based networked control systems are used widely, which provide a plant model in the actuator side of the plant to approximate the plant behavior during time periods when sensor data are not available, such that the amount of the bandwidth necessary for feedback control to maintain certain stability and performance is minimized. At each transmission time, the plant model is updated with the measured plant state. Stability results for linear systems with different variations of transmissions times are shown in [3–6]. Different aspects of NCS such as time delay, transmission policies, multirate sampling, quantization, etc. are further pursued based on the idea of model-based NCS in [7–12].

A constant updating period is discussed for systems under MB-NCS control scheme in [3–5, 7–12]; few works are done for systems with time-varying update times. Although stability of linear MB-NCSs with time-varying update times is studied in [6], their approach does not work in the case with nonlinear MB-NCSs. The reason is that its main results are based on the representation of its response (*related to ini-*

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tial condition), while the one for nonlinear MB-NCSs cannot be shown directly. In addition, as we all know that stability of systems with nonlinearity is also very popular; see [13, 14]. Motivated by above, stability of nonlinear MB-NCS with time-varying update times is pursued here.

Nonlinear systems of interest are those in the paper, which are composed by linear part plus nonlinear perturbation. Here, the transmission times are assumed to be varying within a time interval $[h_{\min}, h_{\max}]$ and a linear system is chosen as the compensated plant model at the controller/actuator side to approximate the plant behavior during time periods when sensor data are not available.

Since the approach for stability of linear MB-NCSs in [6] depends on the accurate representation of its response, when we investigate stability of nonlinear MB-NCSs with time-varying update times, the main difficulty is that a new approach is needed. To overcome this difficulty, the Lyapunov function is adopted here. Firstly, stability of a sequence of the whole closed-loop systems is discussed based on Lyapunov function. Then stability of the whole closed systems will be studied. Since a different method is adopted to deal with the nonlinear case; a new test matrix and a sufficient condition for stability are given in the paper. If supposedly the condition is satisfied, it is claimed that there exists a robust bound on nonlinear perturbation such that MB networked control systems with nonlinear perturbation within the range is stable or uniformly ultimately bounded.

On the other hand, since MB networked control systems with time-varying update times are time-varying systems; sufficient condition for its stability is required to hold uniformly for all update times within an interval $[h_{\min}, h_{\max}]$. Therefore, another difficulty is how to derive such an update time interval. An algorithm for determining the interval is proposed in the paper based on a new test matrix and sufficient condition for stability, which is shown in the numerical example. It is worth pointing out that a better approximation of the actual plant, a larger stable interval $[h_{\min}, h_{\max}]$ will be obtained.

In a word, stability of nonlinear MB-NCS with time-varying update times is discussed here. The case that the nonlinear plant and the linear plant model are connected via a network channel with transmission times that are varying within a time interval is of particular interest. Sufficient condition for stability of

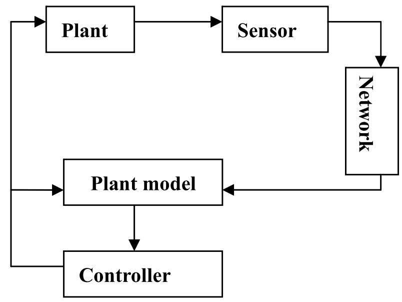


Fig. 1 State feedback of model-based networked systems

MB-NCSs with nonlinear perturbation is proposed. If supposedly the condition is satisfied, it is claimed that there exists a robust bound on nonlinear perturbation such that MB networked control systems with nonlinear perturbation within the range is stable or uniformly ultimately bounded. An algorithm for determining the interval is proposed. Meanwhile, stability robustness bound can be obtained via Lyapunov function, which is illustrated in the numerical simulation.

The rest of the paper is organized as follows. In Sect. 2, the problem formulation for nonlinear MB-NCS under time-varying updating times is given. In Sect. 3, stability results are presented. Next, an algorithm to derive a stable interval is proposed. Finally, two numerical examples are shown to verify our results.

2 Problem formulation

The control system is shown in Fig. 1. The actual plant labeled by "Plant" in Fig. 1 is described by

$$\dot{x} = Ax + Bu + f(x, u) \tag{1}$$

where $x \in \mathbf{R}^n$, $u \in \mathbf{R}^m$ are the state and input of the plant (1), respectively.

Nonlinear perturbation f satisfies

$$\|f\| \le \alpha\|x\| + \beta\|u\| + \delta \tag{2}$$

where α, β, δ are nonnegative real numbers.

The plant model is chosen as follows:

$$\dot{\widehat{x}} = \widehat{A}\,\widehat{x} + \widehat{B}\,u \tag{3}$$

where $\widehat{x} \in \mathbf{R}^n$ is the state of the plant model.

We chose the controller as $u = K\widehat{x}$ where matrix K will be designed later.

As shown in Fig. 1, the plant model (3) will receive the state $x(t_k)$ of the plant (1) at times t_k, periodically, where $t_{k+1} - t_k = h(k)$ $(k = 0, 1, 2, \ldots)$ is a time-varying period. For simplicity, no transmission delay is considered here, that is, the plant model (3) will receive the state $x(t_k)$ at the instance when it is transmitted. Moreover, the state $\widehat{x}(t_k)$ of the plant model (3) will be updated by $x(t_k)$ as soon as it receives $x(t_k)$.

Remark 1 The plant model (3) is an approximating model of the actual plant (1) and is connected with the actual plant (1) by networks with a time-varying period $h(k) \in [h_{\min}, h_{\max}]$. Our main goal is to study whether the plant (1) can be stabilized by the controller based on the state feedback of its approximating model (3).

Assume that the property of time-varying update times $h(k)$ is unknown, but $h(k)$ is contained within some interval $[h_{\min}, h_{\max}]$.

Let $e = x - \widehat{x}$ and $z = (x^T, e^T)^T$. Then, based on the model-based control architecture, the dynamic of the system (1)–(3) shown in Fig. 1 with time-varying update times $h(k)$ are given by

$$\dot{z} = \Lambda z + F, \quad z(t_k) = \begin{pmatrix} x(t_k) \\ e(t_k) \end{pmatrix} = \begin{pmatrix} x(t_k^-) \\ 0 \end{pmatrix}, \qquad (4)$$
$$t_k - t_{k-1} = h(k), k = 0, 1, 2, \ldots$$

where

$$\Lambda = \begin{bmatrix} A + BK & -BK \\ \Delta A + \Delta BK & \widehat{A} - \Delta BK \end{bmatrix}, \quad F = \begin{bmatrix} f \\ f \end{bmatrix},$$
$$\Delta A = A - \widehat{A}, \Delta B = B - \widehat{B}, \quad z(t_k^-) = \lim_{t \to t_k^-} z(t)$$

It is easy to get

$$\|F\| \le \alpha_0\|x\| + \beta_0\|e\| + \delta_0 \le \sqrt{\alpha_0^2 + \beta_0^2}\|z\| + \delta_0 \quad (5)$$

where $\alpha_0 = \sqrt{2}(\alpha + \beta\|K\|), \beta_0 = \sqrt{2}\beta\|K\|$, and $\delta_0 = \sqrt{2}\delta$.

Here, the vector norm $\|.\|$ is chosen as 2-norm.

3 Main results

Define

$$M_0(h) = \exp(\Lambda h) \begin{bmatrix} I & O \\ O & O \end{bmatrix} \qquad (6)$$

where $h \in [h_{\min}, h_{\max}]$.

Remark 2 The previous matrix $M_0(h)$ is called "**the test matrix**," which is very important for stability of dynamic (4).

The following are the main results of the paper.

Theorem 1 *Suppose that there exist positive-definite matrices $X > 0$ and $\widehat{Q} > 0$ such that*

$$X - M_0(h)^T X M_0(h) - \widehat{Q} \ge 0 \qquad (7)$$

holds for all $h \in [h_{\min}, h_{\max}]$.

Then there exists $d^* > 0$ such that for all $\sqrt{\alpha_0^2 + \beta_0^2} \le d^*$ the following holds:

(1) When $\delta = 0$, the system (4) is exponentially stable around the solution $z(t) = (O^T, O^T)^T$ with time-varying update times $h(k) \in [h_{\min}, h_{\max}]$.
(2) When $\delta \ne 0$, the system (4) is ultimately bounded with time-varying update times $h(k) \in [h_{\min}, h_{\max}]$.

Remark 3 Theorem 1 tells us that exponentially stability or ultimately bounded hold for small perturbation if there are positive-definite matrices X and $\widehat{Q}$ such that inequality (7) holds for all time-varying update times $h \in [h_{\min}, h_{\max}]$. Here, (7) is called as "**the Lyapunov inequality**," which is a sufficient condition for stability of (4) with time-varying update times $h \in [h_{\min}, h_{\max}]$.

Remark 4 In order to guarantee stability of system (4), the Lyapunov inequality (7) is requested to hold for all update times $h \in [h_{\min}, h_{\max}]$ uniformly. The transfer interval $[h_{\min}, h_{\max}]$ is called "stable interval" for system (4). Next, we will continue to find such a stable interval for system (4) and the corresponding Lyapunov inequality (7).

Remark 5 Here, two positive matrices $X > 0$ and $\widehat{Q} > 0$ independent of time-varying update times $h(k) \in [h_{\min}, h_{\max}]$ are required to ensure stability of the whole system (4). It is worth pointing out that only one positive matrix is required for linear systems in [6]. Since the condition (7) is improved, a better result can be obtained, that is, the system (4) is uniformly exponentially stable and supposedly (7) holds.

To get Theorem 1, we need the following lemma.

Lemma 1 *There exist positive constants* c *and* γ *for the solution of* (4) *with time-varying update times* $h(k) \in [h_{\min}, h_{\max}]$ *such that*

$$\|z(t)\| \le c\|z(t_k^-)\| + \gamma \tag{8}$$

for $t \in [t_k, t_{k+1})$, *where* $z(t_k^-) = \lim_{t \to t_k^-} z(t)$.

Proof For $t \in [t_k, t_{k+1})$, the response of dynamic (4) has the following response:

$$\begin{aligned} z(t) &= \exp\big(\Lambda(t-t_k)\big)\begin{bmatrix} I & O \\ O & O \end{bmatrix} z(t_k^-) \\ &\quad + \int_{t_k}^{t} \exp\big(\Lambda(t-s)\big)F(s)\,\mathrm{d}s \\ &= M_0(t-t_k)z(t_k^-) + \int_{t_k}^{t} \exp\big(\Lambda(t-s)\big)F(s)\,\mathrm{d}s \end{aligned} \tag{9}$$

Thus,

$$\begin{aligned} \|z(t)\| &\le \|M_0(t-t_k)\|\|z(t_k^-)\| \\ &\quad + \int_{t_k}^{t} \|\exp\big(\Lambda(t-s)\big)\|\|F(s)\|\,\mathrm{d}s \end{aligned}$$

Let

$$\Gamma = \max_{t\in[0,h_{\max}]}\|M_0(t)\| \quad \text{and} \quad \eta = \max_{t\in[0,h_{\max}]}\|e^{\Lambda t}\|$$

Combine the previous inequality with (5), one has that

$$\begin{aligned} \|z(t)\| &\le \big(\Gamma\|z(t_k^-)\| + \eta\delta_0 h_{\max}\big) \\ &\quad + \eta\sqrt{\alpha_0^2+\beta_0^2}\int_{t_k}^{t}\|z(s)\|\,\mathrm{d}s \end{aligned} \tag{10}$$

By the Grownwall–Bell inequality, we know that

$$\|z(t)\| \le c\|z(t_k^-)\| + \gamma \tag{11}$$

where $c = \Gamma e^{\eta h_{\max}\sqrt{\alpha_0^2+\beta_0^2}}$, $\gamma = \eta\delta_0 h_{\max} e^{\eta h_{\max}\sqrt{\alpha_0^2+\beta_0^2}}$. Therefore, (8) holds. □

Remark 6 Lemma 1 tells us a relation between the state of the system (4) and its sequence at instants $\{t_k^-\}$. Here, c and γ are independent of the time-varying update times $h(k)$.

Next, we will prove Theorem 1.

Proof of Theorem 1

By (9), we have that

$$z(t_{k+1}^-) = M_0\big(h(k)\big)z(t_k^-) + \Omega_k \tag{12}$$

where

$$M_0(h(k)) = \exp(\Lambda h(k))\begin{bmatrix} I & O \\ O & O \end{bmatrix},$$

$$\Omega_k = \int_{t_k}^{t_{k+1}} \exp\big(\Lambda(t-s)\big)F(s)\,\mathrm{d}s$$

Construct a Lyapunov function $V(z(t_k^-)) = (z(t_k^-))^T X z(t_k^-)$ and the difference ΔV along (12) is

$$\begin{aligned} \Delta V &= \big(z(t_k^-)\big)^T\big(M_0\big(h(k)\big)^T X M_0\big(h(k)\big) - X\big)z(t_k^-) \\ &\quad + 2\Omega_k^T X M_0\big(h(k)\big)z(t_k^-) + \Omega_k^T X \Omega_k \end{aligned}$$

By (7), we have that

$$M_0(h)^T X M_0(h) - X \le -\widehat{Q} < 0, \quad \text{for all } h \in [h_{\min}, h_{\max}]$$

Therefore, one has

$$\begin{aligned} \Delta V &\le -\big(z(t_k^-)\big)^T \widehat{Q} z(t_k^-) \\ &\quad + 2\Omega_k^T X M_0\big(h(k)\big)z(t_k^-) + \Omega_k^T X \Omega_k \end{aligned} \tag{13}$$

By (5), we have

$$\begin{aligned} \|\Omega_k\| &\le \eta\int_{t_k}^{t_{k+1}}\|F(s)\|\,\mathrm{d}s \\ &\le \eta\int_{t_k}^{t_{k+1}}\big(d_0\|z(s)\| + \delta_0\big)\,\mathrm{d}s \end{aligned} \tag{14}$$

where $d_0 = \sqrt{\alpha_0^2+\beta_0^2}$. Due to Lemma 1, one can get

$$\begin{aligned} \|\Omega_k\| &\le \eta\int_{t_k}^{t_{k+1}}\big(d_0\big[c\|z(t_k^-)\| + \gamma\big] + \delta_0\big)\,\mathrm{d}s \\ &\le N_1\|z(t_k^-)\| + N_2 \end{aligned} \tag{15}$$

where $N_1 = \eta c h_{\max} d_0$, $N_2 = \eta h_{\max}(\delta_0 + \gamma d_0)$.

After the calculation, we have

$$\Delta V \le -\rho\|z(t_k^-)\|^2 + \kappa\|z(t_k^-)\| + \omega \tag{16}$$

where $\lambda_{\max}(X)$ and $\lambda_{\min}(X)$ are the maximum and minimum eigenvalue of X,

$$\begin{aligned} &\rho = \lambda_{\min}(\widehat{Q}) - \lambda_{\max}(X)(2\Gamma + N_1)N_1, \\ &\kappa = 2(\Gamma + N_1)\lambda_{\max}(X)N_2, \quad \omega = \lambda_{\max}(X)N_2^2 \end{aligned} \tag{17}$$

Note that $N_1 \to 0$ as $d_0 \to 0$ and $\rho \to \lambda_{\min}(\widehat{Q}) > 0$ as $N_1 \to 0$.

By continuous property, we know that there exists $d^* > 0$ such that

$$\rho > 0 \quad \text{holds for all } d_0 = \sqrt{\alpha_0^2+\beta_0^2} \le d^*$$

(1) for the special case with $\delta = 0$.

When $\delta = 0$, $N_2 = 0$, $\kappa = 0$, and $\omega = 0$ hold. Therefore, (16) turns to be

$$\Delta V \le -\rho \left\| z\left(t_k^-\right) \right\|^2$$

Moreover,

$$\Delta V \le -\rho \left\| z\left(t_k^-\right) \right\|^2 \le 0 \quad \text{holds for all}$$

$$d_0 = \sqrt{\alpha_0^2 + \beta_0^2} \le d^*$$

Therefore, $\{z(t_k^-)\}$ is exponentially stable.

On the other hand, the inequality (8) in Lemma 1 with $\delta = 0$ turns out to be

$$\left\| z(t) \right\| \le c \left\| z\left(t_k^-\right) \right\|$$

Hence, when $\delta = 0$, the system described by (4) is exponentially stable around the solution $z(t) = (O^T, O^T)^T$ for all $d_0 = \sqrt{\alpha_0^2 + \beta_0^2} \le d^*$ with time-varying update times $h(k) \in [h_{\min}, h_{\max}]$.

(2) If $\delta \ne 0$, (16) can be written as

$$\Delta V \le -(1-\theta)\rho \left\| z\left(t_k^-\right) \right\|^2 + \kappa \left\| z\left(t_k^-\right) \right\| - \theta\rho \left\| z\left(t_k^-\right) \right\|^2 + \omega \quad 0 < \theta < 1 \tag{18}$$

Then

$$\Delta V < 0 \quad \text{if } \left\| z(t_k^-) \right\| > \max\{\frac{\kappa}{(1-\theta)\rho} \sqrt{\omega/\theta\rho}\}$$

$$\text{for all } d_0 = \sqrt{\alpha_0^2 + \beta_0^2} \le d^*$$

Hence, $\{z(t_k^-)\}$ is ultimately bounded.

Combined with Lemma 1, we can get the system described by (4) with time-varying update times $h(k) \in [h_{\min}, h_{\max}]$ is ultimately bounded if $\delta \ne 0$ for all $d_0 = \sqrt{\alpha_0^2 + \beta_0^2} \le d^*$. □

Remark 7 Nonlinear bound d_0 can be derived from (16), that is, to ensure $\rho > 0$. From the representation of ρ in (17), we know that different choice of positive matrices X and $\widehat{Q}$ will affect the size of robustness bound d^*.

Two corollaries concerning nonlinear MB-NCSs with constant update time and linear MB-NCSs with time-varying update times can be obtained from Theorem 1 directly.

When the update times $h(k)$ are equal, then the following corollary is deduced from Theorem 1 directly for constant update time.

Corollary 1 *If the eigenvalues of $M_0(h)$ are strictly inside the unit circle, then there is $d^* > 0$ such that for all $\sqrt{\alpha_0^2 + \beta_0^2} \le d^*$ the following holds:*

(1) *When $\delta = 0$, the system (4) is exponentially stable around the solution $z(t) = (O^T, O^T)^T$ with constant update time $h(k) = h$.*

(2) *When $\delta \ne 0$, the system (4) is ultimately bounded with constant update time $h(k) = h$.*

Proof Since M_0 is Schur stable, there exist positive matrices $P > 0$ and $Q > 0$ such that $M_0^T P M_0 - P = -Q$. Therefore, (7) is satisfied. The rest is omitted.

When nonlinear perturbation disappears, that is, $f \equiv 0$, then the following corollary can be deduced from Theorem 1 directly for linear MB networked control systems. □

Corollary 2 *When $f \equiv 0$, the dynamic system (4) is exponentially stable around the solution $z(t) = (O^T, O^T)^T$ with time-varying update times $h(k) \in [h_{\min}, h_{\max}]$, if there exist positive-definite matrices $X > 0$ and $\widehat{Q} > 0$ such that*

$$X - M_0(h)^T X M_0(h) - \widehat{Q} \ge 0$$

holds for all $h \in [h_{\min}, h_{\max}]$.

4 Some comments concerning stable intervals

Theorem 1 can be used to derive an interval $[h_{\min}, h_{\max}]$ for h for which stability is guaranteed. In fact, the Lyapunov inequality (7) is required to hold for all update times h within the interval $[h_{\min}, h_{\max}]$. Moreover, it is clear that the range $[h_{\min}, h_{\max}]$ will vary with the choice of positive matrices X and $\widehat{Q}$. In the following, we will pursue how to find such an interval $h \in [h_{\min}, h_{\max}]$ and corresponding positive matrices X and $\widehat{Q}$.

An approach for linear MB-NCSs is given in [6], which is based on the observation that the stable interval obtained this way will always be contained in the set of constant update times for which the system is stable. That is, an update time contained in the stable interval $[h_{\min}, h_{\max}]$ will always be a stable constant update time.

With that in mind, the way to obtain the values for $h_{\min}$ and $h_{\max}$ for nonlinear MB-NCSs with time-varying update times $h(k) \in [h_{\min}, h_{\max}]$ can be described as follows:

Step 1. Find a constant update time h_0 which is stable. This stable constant update time h_0 can be found based on Corollary 1. For such an update time h_0, find the corresponding positive matrices X and $\widehat{Q}$ such that the Lyapunov inequality (7) holds for this constant update time h_0.

Step 2. Use the previous value of X and $\widehat{Q}$. Let the update time h vary near h_0. Next, consider the positiveness of the expression

$$X - M_0(h)^T X M_0(h) - \widehat{Q}$$

which will vary with the update time h. Then find the largest interval $[h_{\min}, h_{\max}]$ such that (7) holds for all $h \in [h_{\min}, h_{\max}]$.

Step 3. This can be repeated for all the values of h known to be stable to find the widest interval $[h_{\min}, h_{\max}]$.

5 Example

We take the same system with [6]: the double integrator with nonlinear perturbation in the following example to illustrate our proposed approach.

Example 1 The actual plant is now chosen as

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u + f \tag{19}$$

where nonlinear perturbation f satisfies (2).

The plant model is chosen as

$$\begin{pmatrix} \dot{\hat{x}}_1 \\ \dot{\hat{x}}_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \hat{x}_1 \\ \hat{x}_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \end{pmatrix} u \tag{20}$$

which behaves as a zero-order hold device as the one in [6].

The controller is taken as $u = K\widehat{x}$ with $K = (-1, -2)$.

After calculation, we can get that

$$\Lambda = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & -2 & 1 & 2 \\ 0 & 1 & 0 & 0 \\ -1 & -2 & 1 & 2 \end{bmatrix}$$

and

$$M_0(h) = \exp(\Lambda h) \begin{bmatrix} I_2 & O \\ O & O \end{bmatrix} = \begin{bmatrix} 1-0.5h^2 & h-h^2 & 0 & 0 \\ -h & 1-2h & 0 & 0 \\ -0.5h^2 & h-h^2 & 0 & 0 \\ -h & -2h & 0 & 0 \end{bmatrix}$$

Next, we will find a stable interval for nonlinear system (19)–(20) based on Theorem 1.

Step 1. First, find one constant update time such that inequality (7) holds at this point.

Fix the update time $h_0 = 0.5$, and after calculation, we know that

$$M_0(0.5) = \begin{bmatrix} W & O \\ Y & O \end{bmatrix}, \quad W = \begin{bmatrix} 0.875 & 0.25 \\ -0.5 & 0 \end{bmatrix},$$

$$Y = \begin{bmatrix} -0.125 & 0.25 \\ -0.5 & -1 \end{bmatrix}$$

It is easy to check that the matrix $M_0(0.5)$ is Schur stable. Hence, by Corollary 1, we know that the update time $h_0 = 0.5$ is a stable constant update time.

Correspondingly, we can choose the positive-definite matrices X and $\widehat{Q}$ by the Matlab tool box as follows:

$$X = \begin{bmatrix} 7.7529 & 2.3953 & -0.0001 & 0.0003 \\ & 4.8766 & 0.0001 & 0.0001 \\ & * & 2.1276 & -0.0006 \\ & * & * & 2.1289 \end{bmatrix}, \tag{21}$$
$$\widehat{Q} = 1.5I_4$$

such that

$$X - M_0^T(0.5) X M_0(0.5) - 1.5I_4 > 0 \tag{22}$$

holds, that is, (7) with the previous positive matrix X and $\widehat{Q} = 1.5I_4$ holds for $h_0 = 0.5$. Moreover,

$$\lambda_{\max}(X) = 9.1086, \quad \lambda_{\min}(\widehat{Q}) = 1.5 \tag{23}$$

Step 2. Expand near the fixed point $h_0 = 0.5$.

When matrix $M_0(h) = \exp(\Lambda h)\left[\begin{smallmatrix} I_2 & O \\ O & O \end{smallmatrix}\right]$ takes the place of $M_0(0.5)$ in the left equation of (22), the minimum eigenvalue magnitude for the matrix

$$X - M_0^T(h) X M_0(h) - 1.5I_4$$

is shown in Fig. 2.

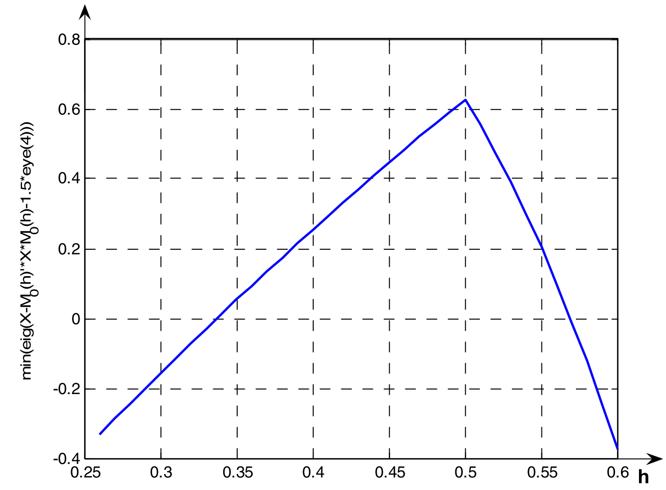


Fig. 2 Minimum eigenvalue for $X - M_0^T(h)XM_0(h) - 1.5I_4$

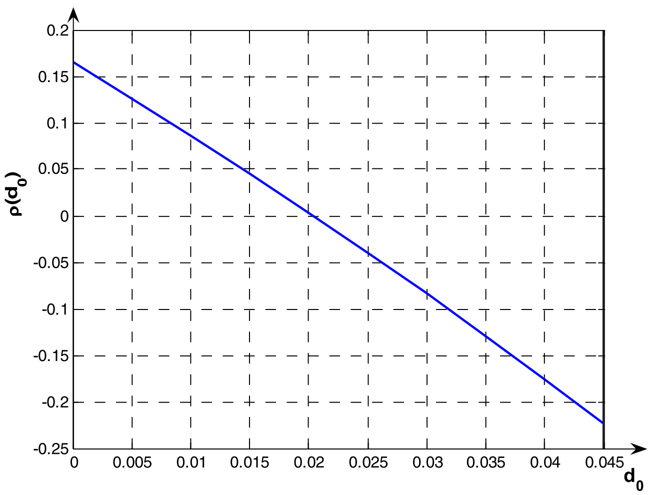


Fig. 3 The curve of function $\rho(d_0)$

From Fig. 2, we have that matrix

$$X - M_0^T(h)XM_0(h) - 1.5I_4 \geq 0$$

holds for the interval $h \in [h_{\min}, h_{\max}] = [0.34, 0.57]$.

Therefore, choose the stable interval as $h \in [h_{\min}, h_{\max}] = [0.34, 0.57]$. That is, (7) with previous positive matrix X defined in (21) and $\widehat{Q} = 1.5I_4$ holds for all $h \in [0.34, 0.57]$.

Therefore, by Theorem 1, there exists a stability robustness bound for nonlinear MB networked control system (19)–(20) with time-varying update times $h(k)$ within the interval [0.34, 0.57].

Next, we will compute the robust bound for system (19)–(20) with time-varying update time $h(k)$ within the interval [0.34, 0.57].

Step 3. Calculate the stable bound for nonlinear perturbation.

Here, $h_{\max} = 0.57$, $\Gamma = 1.5179$, $\eta = 2.9255$.

Since $N_1 = \eta c h_{\max} d_0$ and $c = \Gamma e^{\eta h_{\max}\sqrt{\alpha_0^2+\beta_0^2}}$, it is easy to get the relationship between ρ and nonlinear bound d_0 by (17):

$$\begin{aligned}\rho(d_0) &= \lambda_{\min}(1.5I_4) - \lambda_{\max}(X)N_1(2\Gamma + N_1)\\ &= 1.5 - 9.1086N_1(2\Gamma + N_1)\end{aligned}$$

Figure 3 plots the curve of the function $\rho(d_0)$, which shows that

$$\rho > 0 \quad \text{holds for all } d_0 \in \left[0, \quad d^*\right] = [0, 0.02]$$

Hence, the robustness bound $d^* = 0.02$.

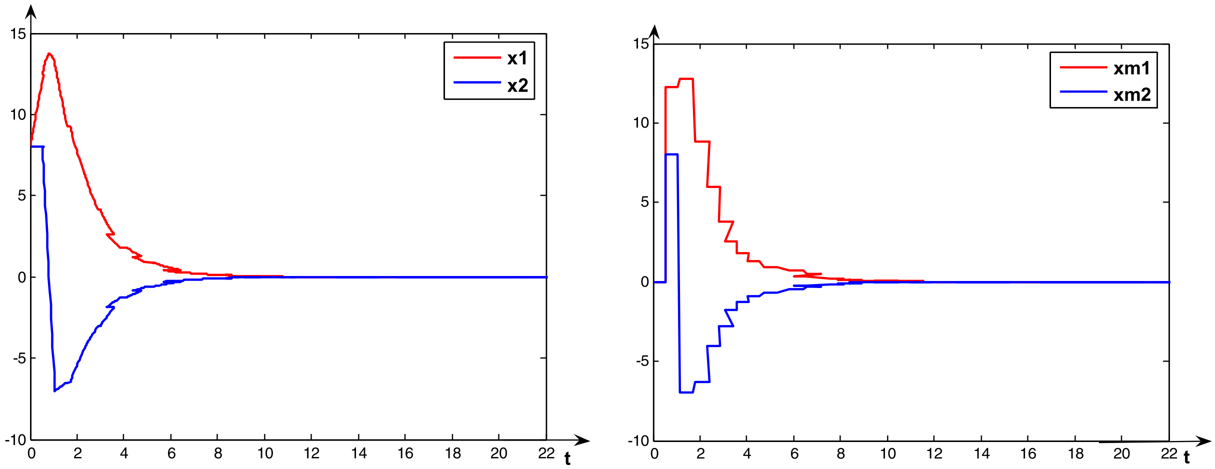
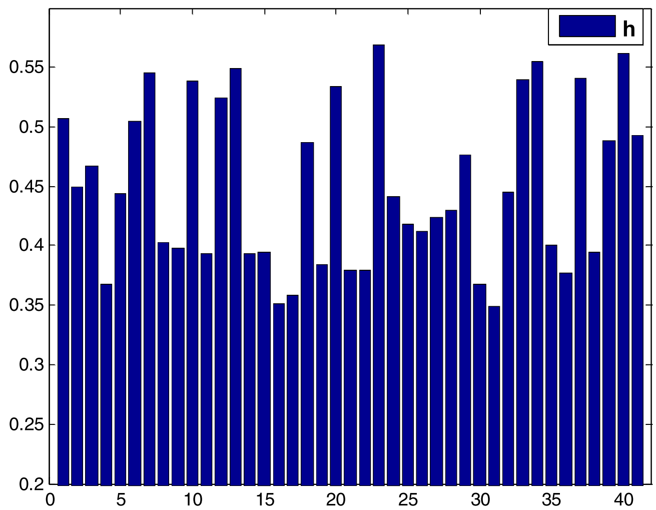


Fig. 4 The response of the MB networked control systems (19)–(20) (23)

Fig. 5 Time varying update times $h(k)$



Step 4. Numerical response

For the purpose of simulation, we choose the nonlinear perturbation f as

$$f = \begin{pmatrix} 0.1\alpha x_1 \sin(x_1) \\ 0 \end{pmatrix} \tag{24}$$

It is easy to get that $\|f\| \leq \alpha \|x\|$.

By previous analysis and (5), we can compute the nonlinear stable bound $\alpha^* = 0.1414$, that is, systems (19)–(20) with nonlinear perturbation (24) under the MB-networked control algorithm is exponentially stable if the nonlinear perturbation bound $\alpha \leq 0.1414$ for time-varying update times $h(k) \in [0.34, 0.57]$.

When the initial condition $x(0) = [8, 8]^T$ and $\alpha = 0.14$, the response of the controlled plant (19) and the plant model (20) with nonlinear perturbation (24) under MB-networked control algorithm for time-varying update time $h(k) \in [0.34, 0.57]$ are shown in the left and right side of Fig. 4, respectively, which verify the rightness of Theorem 1. Meanwhile, the corresponding time-varying update times $h(k)$ are shown in Fig. 5.

When the nonlinear perturbation f is chosen as

$$f = \begin{pmatrix} 0.1\alpha x_1 \sin(x_1) \\ 0 \end{pmatrix} + \begin{pmatrix} \frac{2x_1 x_2}{x_1^2 + x_2^2} \\ \frac{2x_1 x_2}{x_1^2 + x_2^2} \end{pmatrix} \tag{25}$$

It is easy to get that $\|f\| \leq \alpha \|x\| + 1$. By previous analysis and Theorem 1, we know that when the nonlinear stable bound $\alpha \leq 0.1414$, systems (19)–(20)

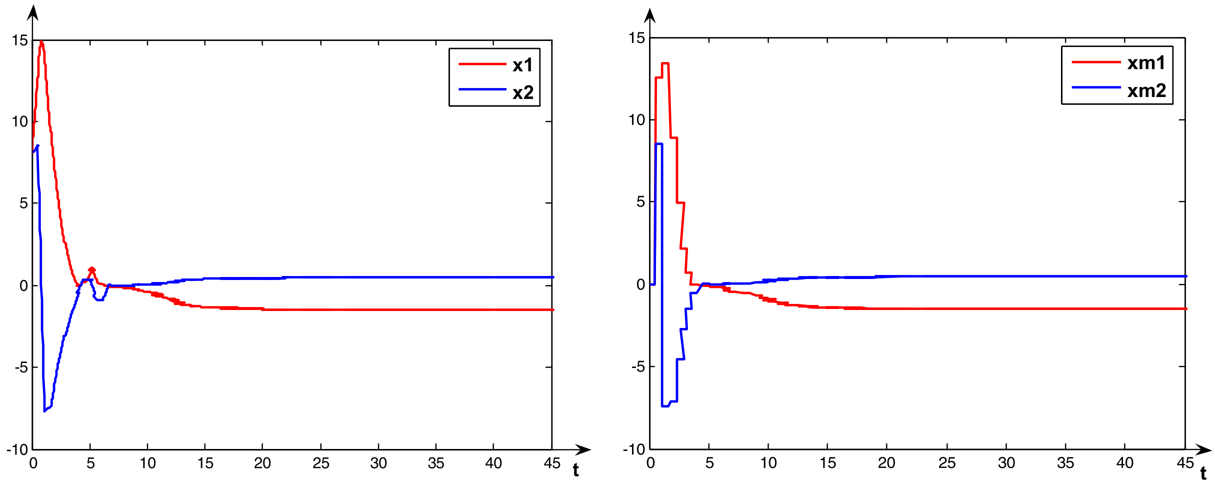
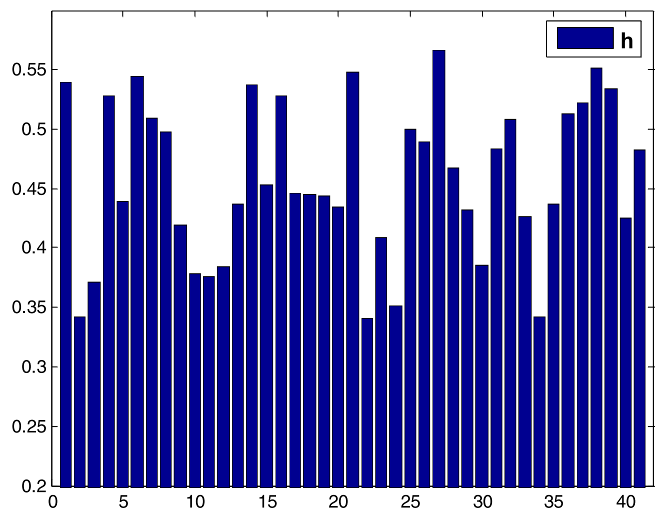


Fig. 6 The response of the MB networked control systems (19)–(20) (25)

Fig. 7 Time varying update times $h(k)$



with nonlinear perturbation (25) under MB-networked control algorithm is uniformly ultimately bounded for time-varying update time $h(k) \in [0.34, 0.57]$.

When the initial condition $x(0) = [8, 8]^T$ and $\alpha = 0.1414$, the state trajectories of the system (19)–(20) with nonlinear perturbation (25) under MB-networked control algorithm are shown in Fig. 6 for time-varying update time $h(k) \in [0.34, 0.57]$. Meanwhile, the corresponding time-varying update times $h(k)$ are shown in Fig. 7.

Remark 8 When the perturbation disappears, that is, $f = 0$, according to the Lyapunov equation represented in [6], the stable interval for the double integrator without nonlinear perturbation is $[h_{\min}, h_{\max}] = [0, 0.85]$, which is larger than ours result $h \in [0.34, 0.57]$. However, the Lyapunov equation represented in [6] cannot solve the nonlinear case.

In fact, two reasons lead our approach more conservative: one is that the existence of nonlinear perturbation; another is that a zero holder (20) is chosen as the plant model of (19), which is a crude approximation of the actual plant. The nearer approximation of the actual plant will lead to less conservative result. Next, we will take the nearer approximation of the plant (19) in the following example to enlarge the stable interval.

Example 2 The actual plant is also chosen as (19) with the nonlinear perturbation f satisfies (2).

Here, we choose a plant model that better resembles the plant as the one in [6]:

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0.0844 & 0.9353 \\ 0.0476 & -0.0189 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0.0871 \\ 1.0834 \end{pmatrix} u \tag{26}$$

The controller is also taken as $u = K\widehat{x}$ with $K = (-1, -2)$.

Remark 9 Compare the plant model (26) with the plant model (20) in Example 1, one can find that the plant model (26) is a better approximation of the actual plant (19).

After calculation, we can get that

$$\Lambda = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & -2 & 1 & 2 \\ 0.0027 & 0.2389 & -0.0027 & 0.7611 \\ 0.0358 & 0.1857 & -0.0358 & -0.1857 \end{bmatrix}$$

and the test matrix $M_0(h) = \exp(\Lambda h)\left[\begin{smallmatrix} I_2 & O \\ O & O \end{smallmatrix}\right]$ defined in (6) is a function of the update times h, for simplicity, it is omitted here.

Next, we will find a stable interval for nonlinear system (19) (26) based on Theorem 1.

Step 1. First, find one constant update time such that inequality (7) holds at this point.

Fix the update time $h_0 = 0.5$, and after the calculation, we know that

$$M_0(0.5) = \begin{bmatrix} W & O \\ Y & O \end{bmatrix}, \quad W = \begin{bmatrix} 0.9099 & 0.3113 \\ -0.3043 & 0.3440 \end{bmatrix},$$

$$Y = \begin{bmatrix} -0.0192 & 0.0873 \\ 0.0005 & 0.0571 \end{bmatrix}$$

and the matrix $M_0(0.5)$ is also Schur stable. Hence, by Corollary 1, we know that the update time $h_0 = 0.5$ is a stable constant update time.

Correspondingly, we can choose the positive-definite matrices X and $\widehat{Q}$ by the Matlab tool box as follows:

$$X = \begin{bmatrix} 4.0626 & 1.1923 & 0.0001 & 0 \\ * & 2.0746 & -0.0003 & -0.0002 \\ * & * & 1.1708 & -0.0001 \\ * & * & * & 1.1709 \end{bmatrix}, \tag{27}$$

$$\widehat{Q} = I_4$$

such that

$$X - M_0^T(0.5) X M_0(0.5) - I_4 > 0 \tag{28}$$

holds, that is, (7) with previous positive matrix X and $\widehat{Q} = I_4$ holds for $h_0 = 0.5$. Moreover,

$$\lambda_{\max}(X) = 4.6209, \quad \lambda_{\min}(\widehat{Q}) = 1 \tag{29}$$

Step 2. Expand near the fixed point $h_0 = 0.5$.

When matrix $M_0(h) = \exp(\Lambda h)\left[\begin{smallmatrix} I_2 & O \\ O & O \end{smallmatrix}\right]$ takes the place of $M_0(0.5)$ in the left equation of (28), the minimum eigenvalue magnitude for the matrix

$$X - M_0^T(h) X M_0(h) - I_4$$

is shown in Fig. 8.

From Fig. 8, we have that matrix

$$X - M_0^T(h) X M_0(h) - I_4 \geq 0$$

holds for the interval $h \in [h_{\min}, h_{\max}] = [0.5, 2.3]$.

Therefore, choose the stable interval as $h \in [h_{\min}, h_{\max}] = [0.5, 2.3]$. That is, (7) with the previous positive matrix X defined in (26) and $\widehat{Q} = I_4$ holds for all $h \in [0.5, 2.3]$.

Step 3. Calculate the stable bound for the nonlinear perturbation.

Here, $h_{\max} = 2.3$, $\Gamma = 0.9815$, $\eta = 3.5$. By (17), it is easy to get the relationship $\rho(d_0)$, for simplicity, it is omitted, whose figure is shown in Fig. 9.

From the curve of the function $\rho(d_0)$ in Fig. 9, one has that

$$\rho > 0 \quad \text{holds for all } d_0 \in \left[0, d^*\right] = [0, 0.015]$$

Hence, the robustness bound $d^* = 0.015$.

Step 4. Numerical response

For the purpose of simulation, we also choose the nonlinear perturbation f as (24).

By previous analysis and (5), we can get the nonlinear stable bound $\alpha^* = 0.1061$, that is, systems (19) (26) with nonlinear perturbation (24) under MB-networked control algorithm is exponentially stable if the nonlinear perturbation $\alpha \leq 0.1061$ for time-varying update times $h(k) \in [0.5, 2.3]$.

When the initial condition $x(0) = [8, 8]^T$ and $\alpha = 0.1$, the state trajectory of the system (19) (26) with nonlinear perturbation (24) under MB-networked control algorithm is shown in Fig. 10 for time-varying update time $h(k) \in [0.5, 2.3]$, which verify the rightness of Theorem 1. Meanwhile, the corresponding time-varying update times $h(k)$ are shown in Fig. 11.

Remark 10 When we choose a better approximation (26) of the actual plant (19) in Example 2, based on

Fig. 8 Minimum eigenvalue for $X - M_0^T(h)XM_0(h) - I_4$

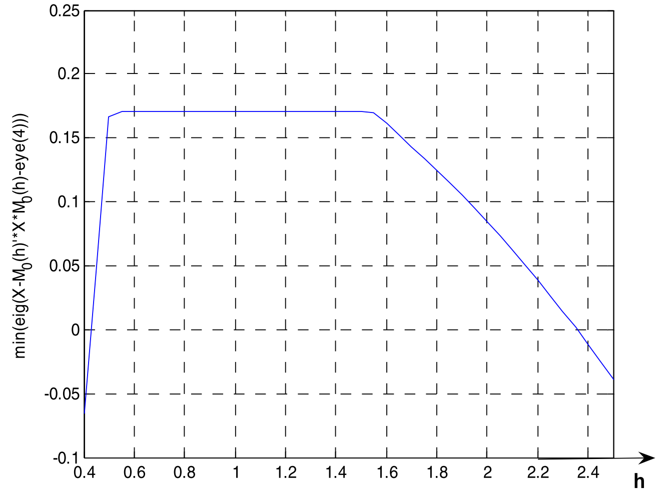
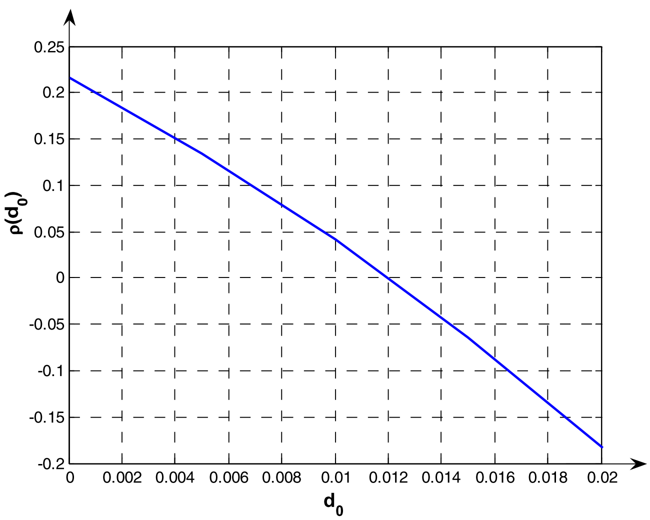


Fig. 9 The curve of function $\rho(d_0)$



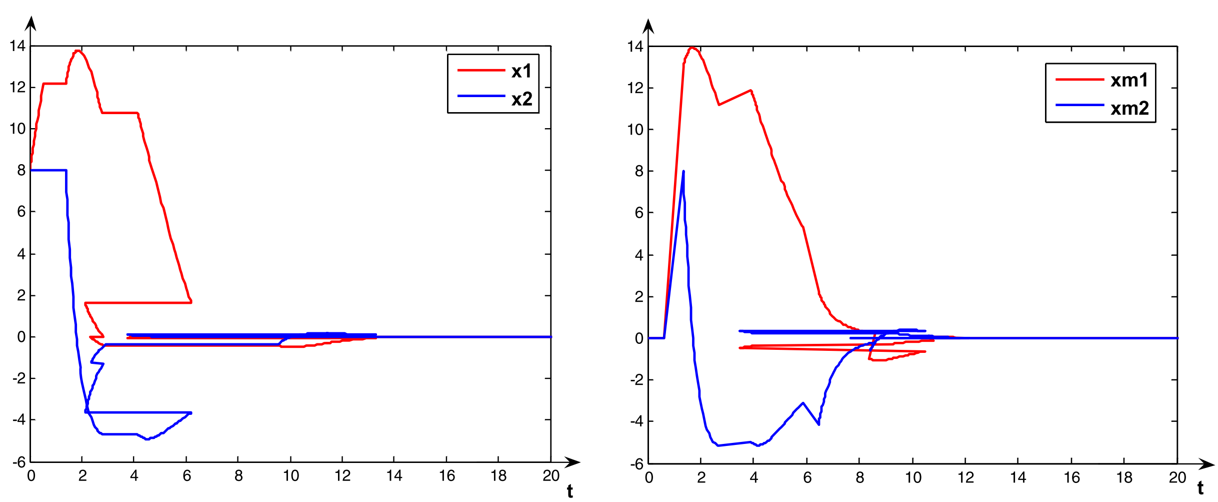


Fig. 10 The response of the MB networked control systems (19) (24) (26)

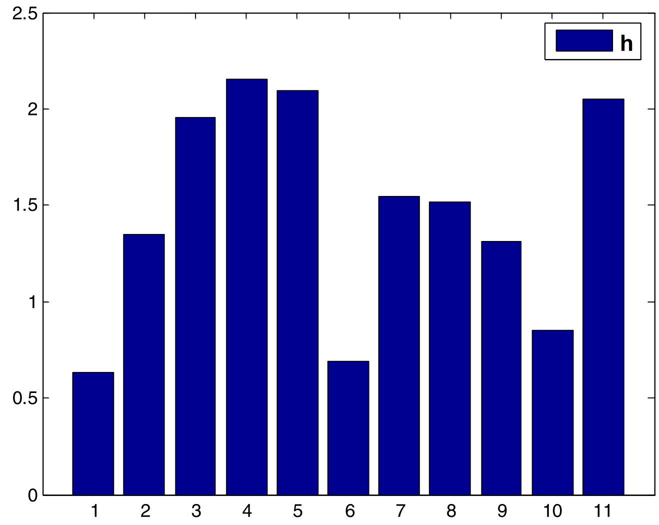


Fig. 11 Time varying update times $h(k)$

our method, we get the stable interval $[h_{\min}, h_{\max}] = [0.5, 2.3]$, which is larger than the one $[0.34, 0.57]$ in Example 1. Two examples shows that better approximation of the actual plant will lead to less conservative result.

Remark 11 The robustness bound for systems (19) (24) (26) with a stable interval $[0.5, 2.3]$ in Example 2 is 0.015, which is smaller than the one 0.02 for systems (19) (20) (24) with a stable interval $[0.34, 0.57]$ in Example 1. Two examples show that larger stable interval leads to smaller robustness bound even if a better approximation cannot avoid. How to balance between a larger stable interval and smaller robustness bound is something needed to be further considered.

6 Conclusions

The stability of nonlinear MB-NCSs with time-varying update times is discussed here. The nonlinear plant and the linear plant model are assumed to be connected via a network channel with transmission times which are varying within a time interval. Sufficient condition for stability of MB-NCSs with nonlinear perturbation is derived. It is claimed that there exists robust bound on nonlinear perturbation for stability of the systems. An algorithm to determine a stable interval $[h_{\min}, h_{\max}]$ for the update times during which the system is stable is given explicitly. Moreover, stability robustness bound can be obtained via the Lyapunov function. Numerical examples are given to illustrate the main result of the paper.

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