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Observer-based adaptive fuzzy backstepping dynamic surface control design and stability analysis for MIMO stochastic nonlinear systems

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Abstract In this paper, an adaptive fuzzy backstepping output feedback dynamic surface control (DSC) approach is developed for a class of multiinput and multioutput (MIMO) stochastic nonlinear systems with immeasurable states. Fuzzy logic systems are firstly utilized to approximate the unknown nonlinear functions, and then a fuzzy state observer is designed to estimate the immeasurable states. By combining adaptive backstepping technique and dynamic surface control (DSC) technique, an adaptive fuzzy output feedback backstepping DSC approach is developed. The proposed control method not only overcomes the problem of "explosion of complexity" inherent in the backstepping design methods, but also the problem of the immeasurable states. It is proved that all the signals of the closed-loop adaptive control stochastic system are semiglobally uniformly ultimately bounded (SUUB) in probability, and the observer errors and the output of the system converge to a small neighborhood of the origin. Simulation results are provided to show the effectiveness of the proposed approach.

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1 Introduction

In the past decades, many approximation-based adaptive backstepping control approaches have been developed to deal with uncertain nonlinear strict-feedback systems via fuzzy-logic-systems (FLSs) or neural-networks (NNs) approximators; see, for example, [1–16] and references herein. Adaptive fuzzy or Neural network backstepping control approaches in [1–10] are for single-input and single-output (SISO) nonlinear systems, and in [11, 12] are for multiple-input and multiple-output (MIMO) nonlinear systems, while those in [13–16] are for SISO/MIMO nonlinear systems with immeasurable states. Adaptive fuzzy or neural network backstepping control approaches can provide a systematic methodology of solving tracking or regulation control problems for a larger of unknown nonlinear systems, where FLSs or NNs are used to approximate unknown nonlinear functions, and the backstepping design technique is applied to construct adaptive controllers and the adaptation adjusted laws of the parameters. Two of the main features of these adaptive approaches are (i) they can be used to deal with those nonlinear systems without satisfying the matching conditions, and (ii) they do not require the unknown nonlinear functions being linearly parameterized. Therefore, the approximator-based adaptive fuzzy or neural network backstepping control becomes one of the most popular design approaches to a large class of uncertain nonlinear systems.

Despite that many developments have been achieved for the adaptive backstepping control of uncertain nonlinear strict-feedback systems using FLSs or NNs, the mentioned above adaptive control approaches are only applied to the deterministic nonlinear strict-feedback systems without stochastic disturbances. It is well known that stochastic disturbances often exist in many practical systems. Their existence is a source of instability of the control systems, thus, the investigations on stochastic systems modeling and control have received considerable attention in recent years [17]. Authors in [18] first proposed an adaptive backstepping control design approach for strict-feedback stochastic systems by a risk-sensitive cost criterion. Authors in [19] solved the output feedback stabilization problem of strict-feedback stochastic nonlinear systems by using the quartic Lyapunov function, while authors in [20] and [21] developed backstepping control design approaches for nonlinear stochastic systems with Markovian switching. Meanwhile, by using the linear reduced-order state observer, several different output-feedback controllers are developed in [22–24] for strict-feedback nonlinear stochastic systems with unmeasured states. However, these schemes are only suitable for those nonlinear stochastic systems with nonlinear dynamics models known exactly or with the unknown parameters appearing linearly with respect to known nonlinear functions.

To handle the above the problems, authors in [25] and [26] first developed adaptive output feedback control approaches for a class of uncertain nonlinear stochastic systems by using neural networks and the stability proofs of the control systems are given on the stochastic stability theory [27]. Afterward, authors in [28] extended the results of [25] and [26] to a class of uncertain large-scale nonlinear stochastic systems and developed adaptive NN decentralized output feedback control schemes. The adaptive NN backstepping control approaches in [25, 26], and [28] can control a class of nonlinear stochastic systems with immeasurable states, however, the nonlinear uncertainties in the nonlinear stochastic systems are only the functions of the system output, not related with the other states variables. Moreover, the mentioned above approaches are only limited to those SISO or large-scale nonlinear stochastic systems. To our best knowledge, to date, there are few results on MIMO stochastic nonlinear systems with immeasurable states.

Motivated by the above observations, in this paper, an observer-based adaptive fuzzy backstepping output feedback DSC approach is proposed for a class of MIMO stochastic nonlinear strict-feedback systems. In the design, the FLSs are first used to approximate the unknown functions, and a nonlinear fuzzy state observer is designed to estimate the unmeasured states. Combining the adaptive backstepping design along with the DSC technique, an observer-based adaptive fuzzy backstepping control approach is developed. It is proved that this control approach can guarantee that all the signals of the closed-loop system are semiglobally uniformly ultimately bounded (SUUB) in probability, and the observer errors and the output of the system converge to a small neighborhood of the origin by appropriate choice of the design parameters. Compared with the existing results, the main advantages of the proposed control schemes are as follows: (i) by designing a fuzzy nonlinear state observer, the proposed adaptive control method does not require that all the states of the system are measured directly. Meanwhile, the designed state observer can achieve the better estimation results for the unmeasured states than the linear reduced-order state observer in [25, 26, 28]. (ii) DSC technique is incorporated in adaptive fuzzy backstepping control design, thus the proposed adaptive control method can overcome the problem of "explosion of complexity" inherent in the methods of [25, 26, 28].

2 Problem formulation and some preliminaries

2.1 Problem formulation

Consider the following MIMO uncertain strict-feedback stochastic nonlinear system

$$
\begin{aligned}
dx_{j,1} &= \big(x_{j,2} + f_{j,1}(x_{j,1})\big)dt + \phi_{j,1}(x_{j,1})^T dw \\
dx_{j,2} &= \big(x_{j,3} + f_{j,2}(\underline{x}_{j,2})\big)dt + \phi_{j,2}(\underline{x}_{j,2})^T dw \\
&\vdots \\
dx_{j,m_j-1} &= \big(x_{j,m_j} + f_{j,m_j-1}(\underline{x}_{j,m_j-1})\big)dt
\end{aligned}
\tag{1}
$$

$$+\phi_{j,m_j-1}(\underline{x}_{j,m_j-1})^T dw$$

$$dx_{j,m_j} = \big(u_j + f_{j,m_j}(X, \underline{u}_{j-1})\big)dt + \phi_{j,m_j}(\underline{x}_{j,m_j})^T dw$$

$$y_j = x_{j,1}, \quad j = 1, 2, \dots, n$$

where $\underline{x}_{j,i_j} = (x_{j,1}, \dots, x_{j,i_j})^T \in R^{i_j}$, $i_j = 1, 2, \dots, m_j$ is the state vector for the first i_j differential equations of the jth subsystem, u_j and y_j are the input and output of the first j subsystems. $f_{j,i_j}(\cdot)$ is an unknown smooth nonlinear function. $X = (x_1^T, \dots, x_n^T)^T$ with $x_j = (x_{j,1}, \dots, x_{j,m_j})^T$. w is an independent r-dimensional standard Wiener process. In this paper, it is assumed that the only output variable $y_j = x_{j,1}$ is available for measurement.

Assumption 1 $\phi_{j,i_j}(\underline{x}_{j,i_j}) = g_{j,i_j}(y_j)$, *where* $g_{j,i_j}(y_j)$ *is a smooth function satisfying locally Lipschitz condition.*

Write (1) in the state space form

$$d\underline{x}_{m_j} = \left(A_j \underline{x}_{m_j} + K_j y_j + \sum_{k=1}^{m_j} B_{j,k}\big(f_{j,k}(X_{j,k})\big) + b_j u_j\right)dt + G_j(y_j)^T dw \qquad (2)$$

$$y_j = C_j^T \underline{x}_{n_i}, \quad j = 1, 2, \dots, n; i_j = 1, 2, \dots, m_j - 1$$

where

$$A_j = \begin{bmatrix} -k_{j,1} & & & \\ \vdots & & I & \\ -k_{j,m_j} & 0 & \cdots & 0 \end{bmatrix}_{m_j \times m_j},$$

$$K_j = \begin{bmatrix} k_{j,1} \\ \vdots \\ k_{j,m_j} \end{bmatrix},$$

$$B_{j,k}^T = [\underbrace{0 \cdots 1}_{k} \cdots 0]_{1\times m_j}, \quad b_j^T = [0 \cdots 0 \cdots 1]_{1\times m_j},$$

$$G_j(y_j) = \big[\, g_{j,1}(y_j) \quad \cdots \quad g_{j,m_j}(y_j) \,\big],$$

$$C_j^T = [1 \cdots 0 \cdots 0]_{1\times m_j}, u = [u_1, u_2, \dots, u_n]^T$$

Choose vector K_j such that matrix A_j is a strict Hurwitz, therefore, given $Q_j = Q_j^T > 0$, there exists a positive definite matrix $P_j = P_j^T$ such that

$$A_j^T P_j + P_j A_j = -Q_j \qquad (3)$$

Control objective: Using fuzzy logic systems to determine an output feedback controller and parameters adaptive laws such that all the signals involved in the closed-loop system are SUUB in probability and the observer errors and the output of the system are as small as the desired.

2.2 Stochastic system and stability

To establish stochastic stability as preliminary, we consider the following stochastic nonlinear system:

$$d\chi(t) = f\big(\chi(t)\big)dt + g\big(\chi(t)\big)d\omega(t) \qquad (4)$$

where $\chi \in R^n$ is the state, ω is an r-dimensional independent standard Wiener process, and $f(\cdot): R^n \to R^n$ and $g(\cdot): R^n \to R^{n\times r}$ are locally Lipschitz and satisfy $f(0) = 0$, $g(0) = 0$.

Define a differential operator ℓ for twice continuously differentiable function $V(\chi)$ as follows:

$$\ell V(\chi) = \frac{\partial V}{\partial \chi} f(\chi) + \frac{1}{2} Tr\left\{ g^T(\chi) \frac{\partial^2 V}{\partial \chi^2} g(\chi) \right\} \qquad (5)$$

Recall two stability notions for nonlinear stochastic system (4).

Definition 1 [27] Consider system (4) with $f(0) = 0$ and $g(0) = 0$. The solution $\chi(t) = 0$ is said to be asymptotically stable in the large if for any $\varepsilon > 0$,

$$\lim_{\chi(0)\to 0} P\left\{ \sup_{t\ge 0} \|\chi(t)\| \ge \varepsilon \right\} = 0$$

And for any initial condition $\chi(0)$,

$$P\left\{ \lim_{t\to\infty} \chi(t) = 0 \right\} = 1$$

Definition 2 [27] The solution process $\{\chi(t), t \ge 0\}$ of stochastic differential system (4) is said to be bounded in probability, if

$$\lim_{c\to\infty} \sup_{0\le t\le\infty} P\big\{ \|\chi(t)\| \ge c \big\} = 0$$

Lemma 1 *Consider the stochastic nonlinear system (4). If there exists a positive definite, radially unbounded, twice continuously differentiable Lyapunov*

$V: R^n \to R$, and constants $\rho > 0$ and $\mu \geq 0$, such that

$$\ell V(\chi) \leq -\rho V(\chi) + \mu \tag{6}$$

then the following conclusions are true:

(1) *the system has a unique solution almost surely*;
(2) *the system is bounded in probability*;
(3) *in addition, if $f(0) = 0$ and $g(0) = 0$ and $\mu = 0$. Then the system is asymptotically stable in the large.*

Lemma 2 (Young's inequality) *For any vectors $x, y \in R^n$, there is inequality, $x^T y \leq \frac{a^p}{p}\|x\|^p + \frac{1}{qa^q}\|y\|^q$, where $a > 0$, $p > 1, q > 1$, and $(p-1)(q-1) = 1$.*

2.3 Fuzzy logic systems

A FLS consists of four parts: the knowledge base, the fuzzifier, the fuzzy inference engine, and the defuzzifier. The knowledge base is composed of a collection of fuzzy. If-then rules of the following form:

$$R^l: \text{If } x_1 \text{ is } F_1^l \text{ and } x_2 \text{ is } F_2^l \text{ and } \dots \text{ and } x_n \text{ is } F_n^l, \quad \text{then } y \text{ is } G^l, l = 1, 2, \dots, N \tag{7}$$

where $x = (x_1, x_2, \dots, x_n)^T$ and y are FLS input and output, respectively, $\mu_{F_i^l}(x_i)$ and $\mu_{G^l}(y)$ are the membership function of fuzzy sets F_i^l and G^l, N is the number of inference rules.

Through singleton fuzzifier, center average defuzzification and product inference [29], the FLS can be expressed as

$$y(x) = \frac{\sum_{l=1}^N \bar{y}_l \prod_{i=1}^n \mu_{F_i^l}(x_i)}{\sum_{l=1}^N [\prod_{i=1}^n \mu_{F_i^l}(x_i)]} \tag{8}$$

where $\bar{y}_l = \max_{y \in R} \mu_{G^l}(y)$.

Define the fuzzy basis functions as

$$\varphi_l = \frac{\prod_{i=1}^n \mu_{F_i^l}(x_i)}{\sum_{l=1}^N [\prod_{i=1}^n \mu_{F_i^l}(x_i)]} \tag{9}$$

Denoting $\theta^T = [\bar{y}_1, \bar{y}_2, \dots, \bar{y}_N] = [\theta_1, \theta_2, \dots, \theta_N]$ and $\varphi(x) = [\varphi_1(x), \varphi_2(x), \dots, \varphi_N(x)]^T$, then fuzzy logic system (8) can be rewritten as

$$y(x) = \theta^T \varphi(x) \tag{10}$$

Lemma 3 [29] *Let $f(x)$ be a continuous function defined on a compact set Ω. Then for any constant $\varepsilon > 0$, there exists a fuzzy logic system (10) such as*

$$\sup_{x \in \Omega} \left| f(x) - \theta^T \varphi(x) \right| \leq \varepsilon \tag{11}$$

By Lemma 3, we can assume that the nonlinear functions in (1) can be approximated by the following fuzzy logic systems as

$$\hat{f}_{j,i_j}(X_{j,i_j}|\theta_{j,i_j}) = \theta_{j,i_j}^T \varphi_{j,i_j}(X_{j,i_j}), \hat{f}_{j,i_j}(\hat{X}_{j,i_j}|\theta_{j,i_j}) = \theta_{j,i_j}^T \varphi_{j,i_j}(\hat{X}_{j,i_j}) \tag{12}$$

where $1 \leq j \leq n, i_j = 1, 2, \dots, m_j$. $\hat{X}_{j,i_j}$ is the estimation of state vector X_{j,i_j}.

The optimal parameter vector θ_{j,i_j}^* is defined as

$$\theta_{j,i_j}^* = \arg\min_{\theta_{j,i_j} \in \Omega_{j,i_j}} \Big[\underbrace{\sup}_{X_{j,i_j} \in U_{j,i_j}, \hat{X}_{j,i_j} \in \hat{U}_{j,i_j}} \left| \hat{f}_{j,i_j}(\hat{X}_{j,i_j}|\theta_{j,i_j}) - f_{j,i_j}(X_{j,i_j}) \right| \Big] \tag{13}$$

where Ω_{j,i_j}, U_{j,i_j}, and $\hat{U}_{j,i_j}$ are compact regions for θ_{j,i_j}, X_{j,i_j}, and $\hat{X}_{j,i_j}$, respectively. The fuzzy minimum approximation errors ε_{j,i_j} and approximation errors δ_{j,i_j} are defined as

$$\varepsilon_{j,i_j} = f_{j,i_j}(X_{j,i_j}) - \hat{f}_{j,i_j}\big(\hat{X}_{j,i_j}|\theta_{j,i_j}^*\big), \delta_{j,i_j} = f_{j,i_j}(X_{j,i_j}) - \hat{f}_{j,i_j}(\hat{X}_{j,i_j}|\theta_{j,i_j}) \tag{14}$$

Assumption 2 [9, 15, 30] *There are unknown positive constants ε_{j,i_j}^* and δ_{j,i_j}^* such that $|\varepsilon_{j,i_j}| \leq \varepsilon_{j,i_j}^*$ and $|\delta_{j,i_j}| \leq \delta_{j,i_j}^*$.*

Denote $\omega_{j,i_j} = \varepsilon_{j,i_j} - \delta_{j,i_j}$, by Assumption 2, one has $|\omega_{j,i_j}| \leq \varepsilon_{j,i_j}^* + \delta_{j,i_j}^* = \omega_{j,i_j}^*$, where ω_{j,i_j}^* is also an unknown constant. ε_{j,i_j}^* and ω_{j,i_j}^* can be estimated by the parameters adaptation laws to be designed in the next section.

3 Nonlinear fuzzy adaptive observer design

Note that the states $(x_{j,2}, \dots, x_{j,m_j})^T$ in system (1) are not available for measurement, thus a state observer should be designed to estimate the unmeasured states. A fuzzy adaptive observer is designed for (1) as

$$\begin{aligned}
\dot{\hat{x}}_{j,1} &= \hat{x}_{j,2} + \hat{f}_{j,1}(\hat{X}_{j,1}|\theta_{j,1}) - k_{j,1}(\hat{x}_{j,1} - y_j) \\
\dot{\hat{x}}_{j,i_j} &= \hat{x}_{j,i_j+1} + \hat{f}_{j,i_j}(\hat{X}_{j,i_j}|\theta_{j,i_j}) - k_{j,2}(\hat{x}_{j,1} - y_j) \\
& \quad j = 1,2\ldots n; \quad i_j = 2,\ldots,m_j - 1 \\
\dot{\hat{x}}_{j,m_j} &= u_j + \hat{f}_{j,m_j}(\hat{X}_{j,m_j}|\theta_{j,m_j}) - k_{j,m_j}(\hat{x}_{j,1} - y_j)
\end{aligned} \tag{15}$$

Rewrite (15) in state space form

$$\begin{aligned}
\dot{\underline{\hat{x}}}_{m_j} &= A_j \underline{\hat{x}}_{m_j} + K_j y_j \\
&\quad + \sum_{k=1}^{m_j} B_{j,k} \hat{f}_{j,k}(\hat{X}_{j,k}|\theta_{j,k}) + b_j u_j \\
y_j &= C_j^T \underline{\hat{x}}_{m_j}, \quad j = 1,2,\ldots,n
\end{aligned} \tag{16}$$

Let $e_j = \underline{x}_{m_j} - \underline{\hat{x}}_{m_j}$ be state estimation error vector. From (1) and (16), one has a composite error dynamic equation

$$\begin{aligned}
de_j &= \left(A_j e_j + \sum_{k=1}^{m_j} B_{j,k}\big(f_{j,k}(X_{j,k}) - \hat{f}_{j,k}(\hat{X}_{j,k}|\theta_{j,k})\big) \right) dt + G_j(y_j)^T dw \\
&= (A_j e_j + \delta_j)dt + G_j(y_j)^T dw
\end{aligned} \tag{17}$$

where $\delta_j = (\delta_{j,1}, \delta_{j,2}, \ldots, \delta_{j,m_j})^T$.

Consider the following Lyapunov candidate $V_{j,0}$ as

$$V_{j,0} = \frac{1}{2}\big(e_j^T P_j e_j\big)^2 \tag{18}$$

Using (6) and (17), one has

$$\begin{aligned}
\ell V_{j,0} &= e_j^T P_j e_j \big\{ e_j^T \big(A_j^T P_j + P_j A_j\big) e_j + 2 e_j^T P_j \delta_j \big\} \\
&\quad + 2Tr\big\{ G_j(y_j)^T \big(2P_j e_j e_j^T P_j + e_j^T P_j e_j P_j\big) G_j(y_j) \big\} \\
&\le -\lambda_j \|e_j\|^4 + 2 e_j^T P_j e_j e_j^T P_j \delta_j \\
&\quad + 2Tr\big\{ G_j(y_j)^T \big(2P_j e_j e_j^T P_j + e_j^T P_j e_j P_j\big) G_j(y_j) \big\}
\end{aligned} \tag{19}$$

where $\lambda_j = \lambda_{\min}(P_j) \cdot \lambda_{\min}(Q_j), \lambda_{\min}(P_j)$, and $\lambda_{\min}(Q_j)$ are the smallest eigenvalues of the matrices P_j and Q_j, respectively.

Choosing an appropriate constant $\eta_{j,0} > 0$ such that

$$p_{j,0} = \lambda_j - \frac{3}{2}\eta_{j,0}^{\frac{4}{3}} \|P_j\|^{\frac{8}{3}} - 3 m_j \sqrt{m_j}\, \eta_{j,0}^2 \|P_j\|^4 > 0$$

By using the well-known mean value theorem in [26], $g_{j,i_j}(y_j)$ can be expressed as $g_{j,i_j}(y_j) = y_j \psi_{j,i_j}(y_j)$, thus

$$G_j(y_j) = y_j \big[\psi_{j,1}(y_j) \cdots \psi_{j,m_j}(y_j)\big] = y_j \psi_j(y_j) \tag{20}$$

By Lemma 2, one can obtain the following inequalities:

$$\begin{aligned}
& 2 e_j^T P_j e_j e_j^T P_j \delta_j \\
&\le 2\|e_j\|^3 \|P_j\|^2 \|\delta_j\| \le \frac{3}{2}\eta_{j,0}^{\frac{4}{3}} \|P_j\|^{\frac{8}{3}} \|e_j\|^4 + \frac{1}{2\eta_{j,0}^4} \|\delta_j^*\|^4 \\
& \times 2Tr\big\{ G_j(y_j)^T \big(2P_j e_j e_j^T P_j + e_j^T P_j e_j P_j\big) G_j(y_j) \big\} \\
&\le 2 m_j \big| G_j(y_j)^T \big(2P_j e_j e_j^T P_j + e_j^T P_j e_j P_j\big) \times G_j(y_j) \big|_\infty \\
&\le 2 m_j \sqrt{m_j} \big| G_j(y_j)^T \big(2P_j e_j e_j^T P_j + e_j^T P_j e_j P_j\big) G_j(y_j) \big| \\
&\le 6 m_j \sqrt{m_j}\, y_j^2 \big\|\psi_j(y_j)\big\|^2 \|P_j\|^2 \|e_j\|^2 \\
&\le \frac{3 m_j \sqrt{m_j}}{\eta_{j,0}^2} y_j^4 \big\|\psi_j(y_j)\big\|^4 + 3 m_j \sqrt{m_j}\, \eta_{j,0}^2 \|P_j\|^4 \|e_j\|^4
\end{aligned} \tag{21}$$

where $\delta_j^* = (\delta_{j,1}^*, \delta_{j,2}^*, \ldots, \delta_{j,m_j}^*)^T$.

Substituting (20) and (21) into (19) results in

$$\ell V_{j,0} \le -p_{j,0}\|e_j\|^4 + \varXi_j + \frac{3 m_j \sqrt{m_j}}{\eta_{j,0}^2} y_j^4 \big\|\psi_j(y_j)\big\|^4 \tag{22}$$

where $\varXi_j = \frac{1}{2\eta_{j,0}^4}\|\delta_j^*\|^4$.

Remark Note that if the stochastic disturbance $dw = 0$ (the third term is zero) and fuzzy logic systems

$\hat{f}_{j,i_j}(\hat{X}_{j,i_j}|\theta_{j,i_j})$ can well approximate $f_{j,i_j}(X_{j,i_j})$ in system (1), then Ξ_j will be small, and by (22), it is concluded that the design state observer (15) is asymptotically stable. It should be pointed that the linear reduced-order state observer used in [22–26], even if $dw = 0$, it cannot be concluded that the state observer is asymptotically stable.

4 Control design and stability analysis

In this section, a fuzzy controller and parameter adaptive laws are to be developed by using the backstepping design and DSC technique so that all the signals in the closed-loop systems are SUUB, the observer errors and the system outputs are as small as the desired.

The m_j-steps adaptive fuzzy output-feedback backstepping design is based on the following changes of coordinates:

$$\chi_{j,1} = y_j, \tag{23}$$

$$\chi_{j,i_j} = \hat{x}_{j,i_j} - z_{j,i_j}, \tag{24}$$

$$\xi_{j,i_j} = z_{j,i_j} - \alpha_{j,i_j-1} \tag{25}$$

where χ_{j,i_j} is called the error surface, z_{j,i_j} ($j = 1,\dots,n;\ i_j = 2,\dots,m_j$) is called the output error of the first-order filter.

Step $j.1$ ($j = 1,2,\dots,n$) Using (1), (15), (24), and (25), one has

$$d\chi_{j,1} = \big(\chi_{j,2} + \alpha_{j,1} + \xi_{j,2} + e_{j,2} + \theta_{j,1}^T\varphi_{j,1}(\hat{X}_{j,1}) + \tilde{\theta}_{j,1}^T\varphi_{j,1}(\hat{X}_{j,1}) + \varepsilon_{j,1}\big)dt + g_{j,1}(y_j)^T dw \tag{26}$$

Consider the following Lyapunov function candidate:

$$V_{j,1} = V_{j,0} + \frac{1}{4}\chi_{j,1}^4 + \frac{1}{2\gamma_{j,1}}\tilde{\theta}_{j,1}^T\tilde{\theta}_{j,1} + \frac{1}{2\bar{\gamma}_{j,1}}\tilde{\varepsilon}_{j,1}^2 \tag{27}$$

where $\gamma_{j,1} > 0$ and $\bar{\gamma}_{j,1} > 0$ are design parameters. $\tilde{\theta}_{j,1} = \theta_{j,1}^* - \theta_{j,1}$ and $\tilde{\varepsilon}_{j,1} = \varepsilon_{j,1}^* - \hat{\varepsilon}_{j,1}$ are the parameter errors. $\theta_{j,1}$ and $\hat{\varepsilon}_{j,1}$ are the estimates of $\theta_{j,1}^*$ and $\varepsilon_{j,1}^*$, respectively.

From (22) and (26), one has

$$\begin{aligned}
\ell V_{j,1} &= \ell V_{j,0} + \ell\left(\frac{1}{4}\chi_{j,1}^4\right) + \frac{1}{\gamma_{i,1}}\tilde{\theta}_{j,1}^T\dot{\tilde{\theta}}_{j,1} + \frac{1}{\bar{\gamma}_{j,1}}\tilde{\varepsilon}_{j,1}\dot{\tilde{\varepsilon}}_{j,1} \\
&\le \ell V_{j,0} + \chi_{j,1}^3\big(\chi_{j,2} + \alpha_{j,1} + \xi_{j,2} + e_{j,2} + \theta_{j,1}^T\varphi_{j,1}(\hat{X}_{j,1}) + \tilde{\theta}_{j,1}^T\varphi_{j,1}(\hat{X}_{j,1}) + \varepsilon_{j,1}\big) \\
&\quad + \frac{3}{2}\chi_{j,1}^2 g_{j,1}(y_j)^T g_{j,1}(y_j) - \frac{1}{\gamma_{j,1}}\tilde{\theta}_{j,1}^T\dot{\theta}_{j,1} - \frac{1}{\bar{\gamma}_{j,1}}\tilde{\varepsilon}_{j,1}\dot{\hat{\varepsilon}}_{j,1} \\
&\le -p_{j,0}\|e_j\|^4 + \Xi_j + \frac{3m_j\sqrt{m_j}}{\eta_{j,0}^2}y_j^4\|\psi_j(y_j)\|^4 + \chi_{j,1}^3\big(\chi_{j,2} + \alpha_{j,1} + \xi_{j,2} + e_{j,2} + \theta_{j,1}^T\varphi_{j,1}(\hat{X}_{j,1})\big) + |\chi_{j,1}^3|\varepsilon_{j,1}^* \\
&\quad + \frac{3}{2}\chi_{j,1}^2 g_{j,1}(y_j)^T g_{j,1}(y_j) + \tilde{\theta}_{j,1}^T\left(\varphi_{j,1}(\hat{X}_{j,1})\chi_{j,1}^3 - \frac{1}{\gamma_{j,1}}\dot{\theta}_{j,1}\right) - \frac{1}{\bar{\gamma}_{j,1}}\tilde{\varepsilon}_{j,1}\dot{\hat{\varepsilon}}_{j,1}
\end{aligned} \tag{28}$$

By Lemma 2, the following inequalities can be obtained:

$$\chi_{j,1}^3 e_{j,2} \le \frac{3}{4}\eta_{j,1}^{\frac{4}{3}}\chi_{j,1}^4 + \frac{1}{4\eta_{j,1}^4}\|e_j\|^4, \tag{29}$$

$$\frac{3}{2}\chi_{j,1}^2 g_{j,1}(y_j)^T g_{j,1}(y_j) = \frac{3}{2}\chi_{j,1}^4\psi_{j,1}(y_j)^T\psi_{j,1}(y_j) \tag{30}$$

where $\eta_{j,1} > 0$ is a design parameter. Substituting (29)–(30) into (28), one has

$$\begin{aligned}
\ell V_{j,1} \le{}& -\left(p_{j,0} - \frac{1}{4\eta_{j,1}^4}\right)\|e_j\|^4 + \Xi_j + \chi_{j,1}^3\Bigg(\chi_{j,1} + \xi_{j,2} + \alpha_{j,1} + \frac{3}{4}\eta_{j,1}^{\frac{4}{3}}\chi_{j,1} + \theta_{j,1}^T\varphi_{j,1}(\hat{X}_{j,1}) \\
&+ \frac{3m_j\sqrt{m_j}}{\eta_{j,0}^2}\chi_{j,1}\|\psi_j(y_j)\|^4 + \frac{3}{2}\chi_{j,1}\psi_{j,1}(y_j)^T\psi_{j,1}(y_j)\Bigg) + |\chi_{j,1}^3|\varepsilon_{j,1}^*
\end{aligned}$$

$$-\frac{1}{\bar{\gamma}_{j,1}}\tilde{\varepsilon}_{j,1}\dot{\hat{\varepsilon}}_{j,1}+\tilde{\theta}_{j,1}^{T}\times\left(\varphi_{j,1}(\hat{X}_{j,1})\chi_{j,1}^{3}-\frac{1}{\gamma_{j,1}}\dot{\theta}_{j,1}\right) \quad (31)$$

Design the intermediate control function $\alpha_{j,1}$ and the adaptation functions $\theta_{j,1}$ and $\hat{\varepsilon}_{j,1}$ as

$$\begin{aligned}\alpha_{j,1}=&-c_{j,1}\chi_{i,1}-\frac{3}{4}\eta_{j,1}^{\frac{4}{3}}\chi_{j,1}-\theta_{j,1}^{T}\varphi_{j,1}(\hat{X}_{j,1})\\&-\hat{\varepsilon}_{j,1}\tanh\left(\chi_{j,1}^{3}/k\right)\\&-\frac{3m_j\sqrt{m_j}}{\eta_{j,0}^{2}}\chi_{j,1}\left\|\psi_j(y_j)\right\|^4\\&-\frac{3}{2}\chi_{j,1}\psi_{j,1}(y_j)^T\psi_{j,1}(y_j),\end{aligned} \quad (32)$$

$$\dot{\theta}_{j,1}=\gamma_{j,1}\varphi_{j,1}(\hat{X}_{j,1})\chi_{j,1}^{3}-\sigma_{j,1}\theta_{j,1} \quad (33)$$

$$\dot{\hat{\varepsilon}}_{j,1}=\bar{\gamma}_{j,1}\chi_{j,1}^{3}\tanh\left(\frac{\chi_{j,1}^{3}}{k}\right)-\bar{\sigma}_{j,1}\hat{\varepsilon}_{j,1} \quad (34)$$

where $\sigma_{j,1}>0$ and $\bar{\sigma}_{j,1}>0$ are design parameters, and $\theta_{j,1}(0)=\hat{\varepsilon}_{j,1}(0)=0$.

Substituting (32)–(34) into (31) and utilizing the inequalities

$$\left|\chi_{j,1}^{3}\right|-\chi_{j,1}^{3}\tanh\left(\chi_{j,1}^{3}/k_j\right)\le 0.2785k_j=k_j' \quad (\forall k_j>0).$$

(31) becomes

$$\begin{aligned}\ell V_{j,1}\le&-p_{j,1}\|e_j\|^4-c_{j,1}\chi_{j,1}^{4}+\varepsilon_{j,1}^{*}k_j'\\&+\Xi_j+\chi_{j,1}^{3}\chi_{j,2}\\&+\chi_{j,1}^{3}\xi_{j,2}+\frac{\sigma_{j,1}}{\gamma_{j,1}}\tilde{\theta}_{j,1}^{T}\theta_{j,1}+\frac{\bar{\sigma}_{j,1}}{\bar{\gamma}_{j,1}}\tilde{\varepsilon}_{j,1}\hat{\varepsilon}_{j,1}\end{aligned} \quad (35)$$

where $p_{j,1}=p_{j,0}-\frac{1}{4\eta_{j,1}^{4}}$.

Introduce a new state variable $z_{j,2}$ and let $\alpha_{j,1}$ pass through a first-order filter with the constant $\tau_{j,2}$ to obtain $z_{j,2}$

$$\tau_{j,2}\dot{z}_{j,2}+z_{j,2}=\alpha_{j,1},\quad z_{j,2}(0)=\alpha_{j,1}(0) \quad (36)$$

Step $j.i_j$ $(j=1,2,\ldots,n;\, i_j=2,\ldots m_j-1)$

From (24) and (25), the time derivative of χ_i is

$$\begin{aligned}\dot{\chi}_{j,i_j}=&\hat{x}_{j,i_j+1}-k_{j,i_j}\hat{x}_{j,1}+k_{j,i_j}y_j\\&+\theta_{j,i_j}^{T}\varphi_{j,i_j}(\hat{X}_{j,i_j})-\dot{z}_{j,i_j}\\=&\chi_{j,i_j+1}+\xi_{j,i_j+1}+\alpha_{j,i_j}-k_{j,i_j}\hat{x}_{j,1}+k_{j,i_j}y_j\\&+\theta_{j,i_j}^{T}\varphi_{j,i_j}(\hat{X}_{j,i_j})-\dot{z}_{j,i_j}\end{aligned} \quad (37)$$

To avoid repeatedly differentiating α_{j,i_j} in the traditional backstepping design, which leads to the so-called "explosion of complexity," we can incorporate the DSC technique proposed by [30–32] into the following backstepping design.

Introduce a new state variable $z_{i,j+1}$ and let α_{j,i_j} pass through a first-order filter with the constant τ_{j,i_j+1} to obtain z_{j,i_j+1}

$$\tau_{j,i_j+1}\dot{z}_{j,i_j+1}+z_{j,i_j+1}=\alpha_{j,i_j},\, z_{j,i_j+1}(0)=\alpha_{j,i_j}(0) \quad (38)$$

(37) can be rewritten as

$$\begin{aligned}\dot{\chi}_{j,i_j}=&\chi_{j,i_j+1}+\xi_{j,i_j+1}+\alpha_{j,i_j}-k_{j,i_j}\hat{x}_{j,1}+k_{j,i_j}y_j\\&+\theta_{j,i_j}^{T}\varphi_{j,i_j}(\hat{X}_{j,i_j})-\frac{1}{\tau_{j,i_j}}(-z_{j,i_j}+\alpha_{j,i_j-1})\end{aligned} \quad (39)$$

By the definition of $\xi_{j,i_j+1}=z_{j,i_j+1}-\alpha_{j,i_j}$, it yields $\dot{z}_{j,i_j+1}=-\frac{\xi_{j,i_j+1}}{\tau_{j,i_j+1}}$ and

$$\begin{aligned}d\xi_{j,i_j+1}=&\Bigg(-\frac{1}{\tau_{j,i_j+1}}\xi_{j,i_j+1}\\&+B_{j,i_j+1}(\underline{\chi}_{j,i_j},\hat{\varepsilon}_{j,1},\hat{\omega}_{j,2},\ldots,\hat{\omega}_{j,i_j},\\&\theta_{j,1},\ldots,\theta_{j,i_j},\underline{\xi}_{j,i_j+1})\Bigg)dt\\&+C_{j,i_j+1}(\underline{\chi}_{j,i_j},\hat{\varepsilon}_{j,1},\hat{\omega}_{j,2},\ldots,\hat{\omega}_{j,i_j},\\&\theta_{j,1},\ldots,\theta_{j,i_j},\underline{\xi}_{j,i_j+1})dw\end{aligned} \quad (40)$$

where $\underline{\chi}_{j,i_j}=[\chi_{j,1}\cdots\chi_{j,i_j}]^T,\underline{\xi}_{j,i_j+1}=[\xi_{j,2}\cdots\xi_{j,i_j+1}]^T$ and

$$\begin{aligned}B_{j,i_j+1}(\cdot)=&-\frac{\partial\alpha_{j,i_j}}{\partial y_j}\left(\hat{x}_{j,2}+e_{j,2}+f_{j,1}(y_j)\right)\\&-\sum_{k=1}^{i_j}\frac{\partial\alpha_{j,i_j}}{\partial\hat{X}_{j,k}}d\hat{X}_{j,k}\end{aligned}$$

$$-\sum_{k=1}^{i_j}\frac{\partial\alpha_{j,i_j}}{\partial\theta_{i,k}}\dot{\theta}_{i,k}-\frac{\partial\alpha_{j,i_j}}{\partial\hat{\varepsilon}_{j,1}}\dot{\hat{\varepsilon}}_{j,1}$$
$$-\sum_{k=2}^{i_j}\frac{\partial\alpha_{j,i_j}}{\partial\hat{\omega}_{j,k}}\dot{\hat{\omega}}_{j,k}-\sum_{k=2}^{i_j}\frac{\partial\alpha_{j,i_j}}{\partial z_{j,k}}\dot{z}_{j,k}$$
$$-\frac{1}{2}\frac{\partial^2\alpha_{j,i_j}}{\partial y_j^2}g_{j,1}(y_j)^T g_{j,1}(y_j), \tag{41}$$

$$C_{j,i_j+1}(\cdot)=-\frac{\partial\alpha_{j,i_j}}{\partial y_j}g_{j,1}(y_j) \tag{42}$$

Consider the following Lyapunov function candidate:

$$V_{j,i_j}=V_{j,i_j-1}+\frac{1}{4}\chi_{j,i_j}^4+\frac{1}{4}\xi_{j,i_j}^4+\frac{1}{2\gamma_{j,i_j}}\tilde{\theta}_{j,i_j}^T\tilde{\theta}_{j,i_j}+\frac{1}{2\bar{\gamma}_{j,i_j}}\tilde{\omega}_{j,i_j}^2 \tag{43}$$

where $\gamma_{j,i_j}>0$ and $\bar{\gamma}_{j,i_j}>0$ are design parameters. $\tilde{\theta}_{j,i_j}=\theta_{j,i_j}^*-\theta_{j,i_j}$ and $\tilde{\omega}_{j,i_j}=\omega_{j,i_j}^*-\hat{\omega}_{j,i_j}$ are the parameters errors. θ_{j,i_j} and $\hat{\omega}_{j,i_j}$ are the estimates of θ_{j,i_j}^* and ω_{j,i_j}^*, respectively.

From (39), (40), and (43), one has

$$\ell V_{j,i_j}\le -p_{j,1}\|e_j\|^4-\sum_{k=1}^{i_j-1}c_{j,k}\chi_{j,k}^4+\varepsilon_{j,1}^*k_j'$$
$$+\sum_{k=2}^{i_j-1}\omega_{j,k}^*k_j'+\Xi_j+\sum_{k=1}^{i_j-1}\frac{\sigma_{j,k}}{\gamma_{j,k}}\tilde{\theta}_{j,k}^T\theta_{j,k}$$
$$+\frac{\bar{\sigma}_{j,1}}{\bar{\gamma}_{j,1}}\tilde{\varepsilon}_{j,1}\hat{\varepsilon}_{j,1}+\sum_{k=2}^{i_j-1}\frac{\bar{\sigma}_{j,k}}{\bar{\gamma}_{j,k}}\tilde{\omega}_{j,k}\hat{\omega}_{j,k}$$
$$+\sum_{k=1}^{i_j}\chi_{j,k}^3(\chi_{j,k+1}+\xi_{j,k+1})+\chi_{j,i_j}^3$$
$$\times\Big(\alpha_{j,i_j}-k_{j,i_j}\hat{x}_{j,1}+k_{j,i_j}y_j$$
$$+\theta_{j,i_j}^T\varphi_{j,i_j}(\hat{X}_{j,i_j})$$
$$-\frac{1}{\tau_{j,i_j}}(-z_{j,i_j}+\alpha_{j,i_j-1})\Big)+|\chi_{j,i_j}^3|\omega_{j,i_j}^*$$
$$-\sum_{k=2}^{i_j}\left(\frac{\xi_{j,k}^4}{\tau_{j,k}}-\xi_{j,k}^3B_{j,k}(\cdot)\right)$$
$$+\frac{3}{2}\sum_{k=1}^{i_j-1}\xi_{k+1}^2 Tr\{C_{j,k+1}(\cdot)^T C_{j,k+1}(\cdot)\}$$
$$+\tilde{\theta}_{j,i_j}^T\left(\varphi_{j,i_j}(\hat{X}_{j,i_j})\chi_{j,i_j}^3-\frac{1}{\gamma_{j,i_j}}\dot{\theta}_{j,i_j}\right)$$
$$-\frac{1}{\bar{\gamma}_{j,i_j}}\tilde{\omega}_{j,i_j}\dot{\hat{\omega}}_{j,i_j} \tag{44}$$

Choose intermediate control function α_{j,i_j} and adaptation functions θ_{j,i_j} and $\hat{\omega}_{j,i_j}$ as:

$$\alpha_{j,i_j}=-c_{j,i_j}\chi_{j,i_j}+k_{j,i_j}\hat{x}_{j,1}-k_{j,i_j}y_j-\theta_{j,i_j}^T\varphi_{j,i_j}(\hat{X}_{j,i_j})-\frac{1}{\tau_{j,i_j}}(z_{j,i_j}-\alpha_{j,i_j-1})-\hat{\omega}_{j,i_j}\tanh\big(\chi_{j,i_j}^3/k\big), \tag{45}$$

$$\dot{\theta}_{j,i_j}=\gamma_{j,i_j}\varphi_{j,i_j}(\hat{X}_{j,i_j})\chi_{j,i_j}^3-\sigma_{j,i_j}\theta_{j,i_j}, \tag{46}$$

$$\dot{\hat{\omega}}_{j,i_j}=\bar{\gamma}_{j,i_j}\chi_{j,i_j}^3\tanh\left(\frac{\chi_{j,i_j}^3}{k}\right)-\bar{\sigma}_{j,i_j}\hat{\omega}_{j,i_j} \tag{47}$$

with $\theta_{j,i_j}(0)=0$, $\hat{\omega}_{j,i_j}(0)=0$.

Substituting (45)–(47) into (44) and utilizing the inequalities

$$|\chi_{j,i_j}^3|-\chi_{j,i_j}^3\tanh\big(\chi_{j,i_j}^3/k\big)\le 0.2785k_j=k_j'\quad(\forall k_j>0)$$

one can obtain

$$\ell V_{j,i_j}\le -p_{j,1}\|e_j\|^4-\sum_{k=1}^{i_j}c_{j,k}\chi_{j,k}^4+\varepsilon_{j,1}^*k_j'$$
$$+\sum_{k=2}^{i_j}\omega_{j,k}^*k_j'+\Xi_j+\sum_{k=1}^{i_j}\frac{\sigma_{j,k}}{\gamma_{j,k}}\tilde{\theta}_{j,k}^T\theta_{j,k}$$
$$+\frac{\bar{\sigma}_{j,1}}{\bar{\gamma}_{j,1}}\tilde{\varepsilon}_{j,1}\hat{\varepsilon}_{j,1}+\sum_{k=2}^{i_j}\frac{\bar{\sigma}_{j,k}}{\bar{\gamma}_{j,k}}\tilde{\omega}_{j,k}\hat{\omega}_{j,k}$$
$$+\sum_{k=1}^{i_j}\chi_{j,k}^3(\chi_{j,k+1}+\xi_{j,k+1})$$
$$-\sum_{k=1}^{i_j-1}\left(\frac{\xi_{j,k+1}^4}{\tau_{j,k+1}}-\xi_{j,k+1}^3B_{j,k+1}(\cdot)\right)$$

$$+\frac{3}{2}\sum_{k=1}^{i_j-1}\xi_{j,k+1}^2 Tr\{C_{j,k+1}(\cdot)^T C_{j,k+1}(\cdot)\} \quad (48)$$

Step $j.m_j$ $(j=1,2,\ldots,n)$ In the final step, the actual control input u_j appears. From (15) and (24), one has

$$d\chi_{j,m_j} = u_j + k_{j,m_j}\hat{x}_{j,1} - k_{j,m_j}y_j + \theta_{j,m_j}^T\varphi_{j,m_j}(\hat{X}_{j,m_j}) + \tilde{\theta}_{j,m_j}^T\varphi_{j,m_j}(\hat{X}_{j,m_j}) + \omega_{j,m_j} - \dot{z}_{j,m_j} \quad (49)$$

Choose the following Lyapunov function candidate:

$$V_{j,m_j} = V_{j,m_j-1} + \frac{1}{4}\chi_{j,m_j}^4 + \frac{1}{4}\xi_{j,m_j}^4 + \frac{1}{2\gamma_{j,m_j}}\tilde{\theta}_{j,m_j}^T\tilde{\theta}_{j,m_j} + \frac{1}{2\bar{\gamma}_{j,m_j}}\tilde{\omega}_{j,m_j}^2 \quad (50)$$

where $\tilde{\theta}_{j,m_j} = \theta_{j,m_j}^* - \theta_{j,m_j}$ and $\tilde{\omega}_{j,m_j} = \omega_{j,m_j}^* - \hat{\omega}_{j,m_j}$ are the parameter errors, θ_{j,m_j} and $\hat{\omega}_{j,m_j}$ are the estimates of $\theta_{j,m_j}^*, \omega_{j,m_j}^*$, respectively.

Design controller u_j and adaptation functions θ_{j,m_j} and $\hat{\omega}_{j,m_j}$ as

$$u_j = -c_{j,m_j}\chi_{j,m_j} + k_{j,m_j}(\hat{x}_{j,1} - y_j) - \theta_{j,m_j}^T\varphi_{j,m_j}(\hat{X}_{j,m_j}) - \frac{1}{\tau_{j,m_j}}(z_{j,m_j} - \alpha_{j,m_j-1}) - \hat{\omega}_{j,m_j}\tanh(\chi_{j,m_j}^3/k), \quad (51)$$

$$\dot{\theta}_{j,m_j} = \gamma_{j,m_j}\varphi_{j,m_j}(\hat{X}_{j,m_j})\chi_{j,m_j}^3 - \sigma_{j,m_j}\theta_{j,m_j}, \quad (52)$$

$$\dot{\hat{\omega}}_{j,m_j} = \bar{\gamma}_{j,m_j}\chi_{j,m_j}^3\tanh\left(\frac{\chi_{j,m_j}^3}{k}\right) - \bar{\sigma}_{j,m_j}\hat{\omega}_{j,m_j} \quad (53)$$

with $\theta_{j,m_j}(0) = \hat{\omega}_{j,m_j}(0) = 0$.

Similar to the derivations in step $j.i_j$, one has

$$\begin{aligned}\ell V_{j,m_j} \le& -p_{j,1}\|e_j\|^4 - \sum_{k=1}^{m_j}c_{j,k}\chi_{j,k}^4 + \varepsilon_{j,1}^* k_j' + \sum_{k=2}^{m_j}\omega_{j,k}^* k_j' + \Xi_j + \sum_{k=1}^{m_j}\frac{\sigma_{j,k}}{\gamma_{j,k}}\tilde{\theta}_{j,k}^T\theta_{j,k} \\ &+ \frac{\bar{\sigma}_{j,1}}{\bar{\gamma}_{j,1}}\tilde{\varepsilon}_{j,1}\hat{\varepsilon}_{j,1} + \sum_{k=2}^{m_j}\frac{\bar{\sigma}_{j,k}}{\bar{\gamma}_{j,k}}\tilde{\omega}_{j,k}\hat{\omega}_{j,k} + \sum_{k=1}^{m_j-1}\chi_{j,k}^3(\chi_{j,k+1} + \xi_{j,k+1}) \\ &- \sum_{k=1}^{m_j-1}\left(\frac{\xi_{j,k+1}^4}{\tau_{j,k+1}} - \xi_{j,k+1}^3 B_{j,k+1}(\cdot)\right) + \frac{3}{2}\sum_{k=1}^{m_j-1}\xi_{j,k+1}^2 Tr\{C_{j,k+1}(\cdot)^T C_{j,k+1}(\cdot)\}\end{aligned} \quad (54)$$

Applying Young's inequality, one has

$$\frac{\sigma_{j,k}}{\gamma_{j,k}}\tilde{\theta}_{j,k}^T\theta_{j,k} = \frac{\sigma_{j,k}}{\gamma_{j,k}}\tilde{\theta}_{j,k}^T(\theta_{j,k}^* - \tilde{\theta}_{j,k}) \le -\frac{\sigma_{j,k}\|\tilde{\theta}_{j,k}\|^2}{2\gamma_{j,k}} + \frac{\sigma_{j,k}\|\theta_{j,k}^*\|^2}{2\gamma_{j,k}}, \quad (55)$$

$$\frac{\bar{\sigma}_{j,1}}{\bar{\gamma}_{j,1}}\tilde{\varepsilon}_{j,1}\hat{\varepsilon}_{j,1} \le -\frac{\bar{\sigma}_{j,1}\tilde{\varepsilon}_{j,1}^2}{2\bar{\gamma}_{j,1}} + \frac{\bar{\sigma}_{j,1}\varepsilon_{j,1}^{*2}}{2\bar{\gamma}_{j,1}}, \quad (56)$$

$$\frac{\bar{\sigma}_{j,k}}{\bar{\gamma}_{j,k}}\tilde{\omega}_{j,k}\hat{\omega}_{j,k} \le -\frac{\bar{\sigma}_{j,k}\tilde{\omega}_{j,k}^2}{2\bar{\gamma}_{j,k}} + \frac{\bar{\sigma}_{j,k}\omega_{j,k}^{*2}}{2\bar{\gamma}_{j,k}}, \quad (57)$$

$$\begin{aligned}\sum_{k=1}^{m_j-1}\chi_{j,k}^3\chi_{j,k+1} &\le \frac{3}{4}\sum_{k=1}^{m_j-1}\upsilon_{j,k}^{\frac{4}{3}}\chi_{j,k}^4 + \frac{1}{4}\sum_{k=1}^{m_j-1}\frac{1}{\upsilon_{j,k}^4}\chi_{j,k+1}^4 \\ &\le \frac{3}{4}\sum_{k=1}^{m_j-1}\upsilon_{j,k}^{\frac{4}{3}}\chi_{j,k}^4 + \frac{1}{4}\sum_{k=2}^{m_j}\frac{1}{\upsilon_{j,k-1}^4}\chi_{j,k}^4,\end{aligned} \quad (58)$$

$$\sum_{k=1}^{m_j-1}\chi_{j,k}^3\xi_{j,k+1} \le \frac{3}{4}\sum_{k=1}^{m_j-1}\rho_{j,k}^{\frac{4}{3}}\chi_{j,k}^4 + \frac{1}{4}\sum_{k=1}^{m_j-1}\frac{1}{\rho_{j,k}^4}\xi_{j,k+1}^4, \quad (59)$$

$$\sum_{k=1}^{m_j-1}\xi_{j,k+1}^3 B_{j,k+1}(\cdot) \le \frac{3}{4}\sum_{k=1}^{m_j-1}(\rho_{j,k}M_{j,k+1})^{\frac{4}{3}}\xi_{j,k+1}^4 + \frac{1}{4}\sum_{k=1}^{m_j-1}\frac{1}{\rho_{j,k}^4}, \quad (60)$$

$$\frac{3}{2}\sum_{k=1}^{m_j-1}\xi_{j,k+1}^2 Tr\{C_{j,k+1}(\cdot)^T C_{j,k+1}(\cdot)\}$$
$$\leq \frac{3}{4}\sum_{k=1}^{m_j-1}(\rho_{j,k}N_{j,k+1})^{\frac{4}{3}}\xi_{j,k+1}^4+\frac{1}{4}\sum_{k=1}^{m_j-1}\frac{1}{\rho_{j,k}^4} \quad (61)$$

where

$$|B_{j,k+1}(\cdot)| \leq M_{j,k+1}(\cdot),$$
$$|Tr\{C_{j,k+1}(\cdot)^T C_{j,k+1}(\cdot)\}| \leq N_{j,k+1}(\cdot).$$

Assumption 3 [33] *For a given $p_{j,i_j} > 0$, for all initial conditions satisfying $V_{j,i_j}(t) \leq p_{j,i_j}$, where*

$$V_{j,i_j}(t) = \frac{1}{2}\left(e_j^T P_j e_j\right)^2 + \frac{1}{4}\sum_{i_j=1}^{k}\chi_{j,i_j}^4 + \frac{1}{4}\sum_{i_j=1}^{k-1}\xi_{j,i_j+1}^4 + \frac{1}{2}\sum_{i_j=1}^{k}\frac{1}{\gamma_{j,i_j}}\tilde{\theta}_{j,i_j}^T\tilde{\theta}_{j,i_j} + \frac{\tilde{\varepsilon}_{j,1}^2}{2\bar{\gamma}_{j,1}} + \frac{1}{2}\sum_{i_j=2}^{k}\frac{\tilde{\omega}_{j,i_j}^2}{\bar{\gamma}_{j,i_j}}$$

Since for any $p_{j,i_j} > 0$, the sets $\prod_{j,k} = \{V_{j,i_j} \leq 2p_{j,i_j}\}$ $(j = 1,\dots,n, k = 2,\dots,i_j)$ is a compact set in $R^{\sum_{i_j=1}^{k}N_{i_j}+j+3k}$ where N_{i_j} is the dimension of $\tilde{\theta}_{i,i_j}$. Since $B_{j,i_j+1}(\cdot)$ and $Tr\{C_{j,i_j+1}(\cdot)^T C_{j,i_j+1}(\cdot)\}$ are continuous functions, there exists the positive constants $M_{j,i_j+1}(\cdot), N_{j,i_j+1}(\cdot)$ such that $|B_{j,k+1}(\cdot)| \leq M_{j,k+1}(\cdot)$, $|Tr\{C_{j,k+1}(\cdot)^T C_{j,k+1}(\cdot)\}| \leq N_{j,k+1}(\cdot)$ on $\prod_{j,k}$.

Substituting (55)–(61) into (54) results in

$$\begin{aligned}\ell V_{j,m_j} \leq& -p_{j,1}\|e_j\|^4 - \left(c_{j,1} - \frac{3}{4}\upsilon_{j,1}^{\frac{4}{3}} - \frac{3}{4}\rho_{j,1}^{\frac{4}{3}}\right)\chi_{j,1}^4\\ &- \sum_{k=2}^{m_j-1}\left(c_{j,k} - \frac{3}{4}\upsilon_{j,k}^{\frac{4}{3}} - \frac{1}{\upsilon_{j,k-1}^4} - \frac{3}{4}\rho_{j,k}^{\frac{4}{3}}\right)\chi_{j,k}^4 - \left(c_{j,m_j} - \frac{1}{\upsilon_{j,m_j-1}^4}\right)\chi_{j,m_j}^4\\ &- \sum_{k=1}^{m_j-1}\left(\frac{1}{\tau_{j,k+1}} - \frac{1}{\rho_{j,k}^4} - \frac{3}{4}(\rho_{j,k}M_{j,k+1})^{\frac{4}{3}} - \frac{3}{4}(\rho_{j,k}N_{j,k+1})^{\frac{4}{3}}\right)\xi_{j,k+1}^4\\ &- \sum_{k=1}^{m_j}\frac{\sigma_{j,k}\|\tilde{\theta}_{j,k}\|^2}{2\gamma_{j,k}} - \frac{\bar{\sigma}_{j,1}\tilde{\varepsilon}_{j,1}^2}{2\bar{\gamma}_{j,1}} - \sum_{k=2}^{m_j}\frac{\bar{\sigma}_{j,k}\tilde{\omega}_{j,k}^2}{2\bar{\gamma}_{j,k}} + \Xi_j + \frac{1}{2}\sum_{k=1}^{m_j-1}\frac{1}{\rho_{j,k}^4}\\ &+ \left(\varepsilon_{j,1}^* + \sum_{k=2}^{m_j}\omega_{j,k}^*\right)k_j' + \sum_{k=1}^{m_j}\frac{\sigma_{j,k}\|\theta_{j,k}^*\|^2}{2\gamma_{j,k}} + \frac{\bar{\sigma}_{j,1}\varepsilon_{i,1}^{*2}}{2\bar{\gamma}_{j,1}}\\ &+ \sum_{k=2}^{m_j}\frac{\bar{\sigma}_{j,k}\omega_{j,k}^{*2}}{2\bar{\gamma}_{j,k}} \end{aligned} \quad (62)$$

Choose the design parameters $\eta_{j,0}, \eta_{j,1}, c_{j,i_j}, \upsilon_{j,i_j}$ and ρ_{j,i_j} $(j = 1,\dots,n; i_j = 1,\dots,m_j)$ such that

$$p_{j,1} = \lambda_j - \frac{3}{2}\eta_{j,0}^{\frac{4}{3}}\|P_j\|^{\frac{8}{3}} - 3m_j\sqrt{m_j}\eta_{j,0}^2\|P_j\|^4 - \frac{1}{4\eta_{j,1}^4} = \lambda_{j,0} > 0, \quad (63)$$

$$c_{j,1} - \frac{3}{4}\upsilon_{j,1}^{\frac{4}{3}} - \frac{3}{4}\rho_{j,1}^{\frac{4}{3}} = c_{j,1}^0 > 0, \quad (64)$$

$$c_{j,k} - \frac{3}{4}\upsilon_{j,k}^{\frac{4}{3}} - \frac{1}{\upsilon_{j,k-1}^4} - \frac{3}{4}\rho_{j,k}^{\frac{4}{3}} = c_{j,k}^0 > 0, \quad (65)$$

$$c_{j,m_j} - \frac{1}{\upsilon_{j,m_j-1}^4} = c_{j,m_j}^0 > 0, \quad (66)$$

$$\frac{1}{\tau_{j,k+1}} - \frac{1}{\rho_{j,k}^4} - \frac{3}{4}(\rho_{j,k}M_{j,k+1})^{\frac{4}{3}} - \frac{3}{4}(\rho_{j,k}N_{j,k+1})^{\frac{4}{3}} = l_{j,k+1} > 0 \quad (67)$$

Substituting (63)–(67) into (62), one has

$$\ell V_{j,m_j} \leq -\lambda_{j,0}\|e_i\|^4 - \sum_{k=1}^{m_j}c_{j,k}^0\chi_{j,k}^4 - \sum_{k=1}^{m_j-1}l_{j,k+1}\xi_{j,k+1}^4 - \sum_{k=1}^{m_j}\frac{\sigma_{j,k}\|\tilde{\theta}_{j,k}\|^2}{2\gamma_{j,k}}$$

$$
\begin{aligned}
&-\frac{\bar{\sigma}_{j,1}\tilde{\varepsilon}_{j,1}^2}{2\bar{\gamma}_{j,1}}-\sum_{k=2}^{m_j}\frac{\bar{\sigma}_{j,k}\tilde{\omega}_{j,k}^2}{2\bar{\gamma}_{j,k}}\\
&+\varXi_j+\frac{1}{2}\sum_{k=1}^{m_j-1}\frac{1}{\rho_{j,k}^4}+\left(\varepsilon_{j,1}^*+\sum_{k=2}^{m_j}\omega_{j,k}^*\right)k_j'\\
&+\sum_{k=1}^{m_j}\frac{\sigma_{j,k}\|\theta_{j,k}^*\|^2}{2\gamma_{j,k}}+\frac{\bar{\sigma}_{j,1}\varepsilon_{j,1}^{*2}}{2\bar{\gamma}_{j,1}}\\
&+\sum_{k=2}^{m_j}\frac{\bar{\sigma}_{j,k}\omega_{j,k}^{*2}}{2\bar{\gamma}_{j,k}}\\
\le{}&-\rho_j V_{j,m_j}+\mu_j
\end{aligned}
\tag{68}
$$

Denote $\lambda_{\max}(P_j)$ is the largest eigenvalue of P_j, and let

$$
\begin{aligned}
\rho_j&=\min\{(2\lambda_{j,0})/\lambda_{\max}^2(P_j),4c_{j,k}^0,4l_{j,i},\sigma_{j,k},\bar{\sigma}_{j,k}\}\\
&\quad j=1,2,\dots,n;k=1,\dots,m_j;i=2,\dots,m_j.\\
\mu_j&=\varXi_j+\frac{1}{2}\sum_{k=1}^{m_j-1}\frac{1}{\rho_{j,k}^4}+\left(\varepsilon_{j,1}^*+\sum_{k=2}^{m_j}\omega_{j,k}^*\right)k_j'\\
&\quad+\sum_{k=1}^{m_j}\frac{\sigma_{j,k}\|\theta_{j,k}^*\|^2}{2\gamma_{j,k}}+\frac{\bar{\sigma}_{j,1}\varepsilon_{j,1}^{*2}}{2\bar{\gamma}_{j,1}}+\sum_{k=2}^{m_j}\frac{\bar{\sigma}_{j,k}\omega_{j,k}^{*2}}{2\bar{\gamma}_{j,k}}
\end{aligned}
$$

then (68) becomes

$$
\ell V_{j,m_j}\le -\rho_j V_{j,m_j}+\mu_j \tag{69}
$$

Finally, choose the whole Lyapunov function candidate as

$$
V=\sum_{j=1}^{n}V_{j,m_j} \tag{70}
$$

Combining (69) and (70), one has

$$
\begin{aligned}
\ell V&=\sum_{j=1}^{n}\ell V_{j,m_j}\le-\sum_{j=1}^{n}(\rho_j V_{j,m_j})+\sum_{j=1}^{n}\mu_j\\
&\le-\rho V+\mu
\end{aligned}
\tag{71}
$$

where $\rho=\min\{\rho_1,\rho_2\cdots\rho_n\}$, $\mu=\mu_1+\mu_2+\cdots+\mu_n$.

By Lemma 1 and inequality (74), and using the same arguments as [23, 26, 28], one can obtain that all the signals of the closed-loop system are bounded by μ/ρ, that is, e_j and χ_{j,i_j} are SUUB in probability. $\tilde{\theta}_{j,i_j}$, $\tilde{\varepsilon}_{j,1}$, and $\tilde{\omega}_{j,k}$ are also SUUB in probability ($j=1,2,\dots,n, i_j=1,2,\dots,m_j, k=2,\dots,m_j$). Moreover, choosing appropriate design parameters, the states observer errors and the outputs of the control system can be made as small as the desired [25, 26].

The above design procedures and stable analysis are summarized in the following theorem.

Theorem 1 *For stochastic nonlinear system (1), under Assumptions 1–3, the state observer (15) and the controller (51), with the intermediate control (32), (45) and parameter laws (33)–(34), (46)–(47), and (52)–(53) guarantee that all the signals in the closed-loop system is semiglobally uniformly ultimately bounded in probability. Moreover, the states observer errors and the outputs of the control system can be made as small as the desired by choosing appropriate design parameters.*

5 Simulation example

In this section, the proposed adaptive fuzzy control approach is applied to the following example to verify its effectiveness.

Example Consider a two-continuous stirred tank reactor process with stochastic disturbances, which is described by the following differential equation [16, 34]:

$$
\begin{cases}
\dot{x}_{1,1}=b_{11}x_{1,2}+y_1^3dw\\
\dot{x}_{1,2}=b_{12}u_1+\frac{1}{2}y_1^2dw\\
y_1=x_{1,1}\\
\dot{x}_{2,1}=b_{21}x_{2,2}+\phi_{21}(x_{1,1},x_{2,1})+\varPhi x_{2,1}+y_2^2dw\\
\dot{x}_{2,2}=b_{22}u_2+\phi_{22}(x_{2,1},x_{2,2})+y_2^2\cos(y_2^2)dw\\
y_2=x_{2,1}\\
\dot{x}_{3,1}=b_{31}x_{3,2}+\phi_{31}(x_{1,1},x_{2,1},x_{2,2},x_{3,1})\\
\qquad+\varPsi\omega+2y_3^2dw\\
\dot{x}_{3,2}=b_{32}u_3+\phi_{32}(x_{3,1},x_{3,2})+y_3^5dw\\
y_3=x_{3,1}
\end{cases}
\tag{72}
$$

as the described [34], cooling water is added to the cooling jackets around both reactors at flow rates F_{j1} and F_{j2}, temperatures $T_{j,1}$ and $T_{j,2}$, respectively. Denote $x_{1,1}=C_{A2}-C_{A2}^d$, $x_{1,2}=F_2$, $x_{2,1}=T_2-T_2^d$, $x_{2,2}=T_{j2}-T_{j2}^d$, $x_{3,1}=T_1-T_{j1}^d$, with $V_{j1}=V_{j2}=V_j$, $V_1=V_2=V$, $F_0=F_2=F$. w is an independent r-dimensional standard Wiener process, and the parameters in (72) are

Table 1 The values of the process parameters

$\alpha = 7.08 \times 10^{10}\ \mathrm{h}^{-1}$	$\rho = 800.9189\ \mathrm{kg/m^3}$	$T_0^d = 703.7°\mathrm{C}$
$E = 3.1644 \times 10^7\ \mathrm{J/mol}$	$\rho_j = 997.9450\ \mathrm{kg/m^3}$	$T_1^d = 750°\mathrm{C}$
$R = 1679.2\ \mathrm{J/mol°C}$	$c_\rho = 1395.3\ \mathrm{J/kg°C}$	$T_2^d = 737.5°\mathrm{C}$
$\lambda = -3.1644 \times 10^7\ \mathrm{J/mol}$	$c_j = 1860.3\ \mathrm{J/kg°C}$	$T_{j1}^d = 740.8°\mathrm{C}$
$U = 1.3625 \times 10^6\ \mathrm{J/h\,m^2\,°C}$	$F = 2.8317\ \mathrm{m^3/h}$	$T_{j2}^d = 727.6°\mathrm{C}$
$C_{A0}^d = 18.3728\ \mathrm{mol/m^3}$	$F_{j1} = 1.4130\ \mathrm{m^3/h}$	$V_j = 0.1090\ \mathrm{m^3}$
$C_{A1}^d = 12.3061\ \mathrm{mol/m^3}$	$F_{j2} = 1.4130\ \mathrm{m^3/h}$	$V = 1.3592\ \mathrm{m^3}$
$C_{A2}^d = 10.4178\ \mathrm{mol/m^3}$	$F_R = 1.4158\ \mathrm{m^3}/h$	$A = 23.2\ \mathrm{m^3}$
$T_{j10}^d = 629.2°\mathrm{C}$	$T_{j20}^d = 608.2°\mathrm{C}$	

$$b_{11} = 1, \quad b_{12} = 1, \quad b_{21} = \frac{UA}{\rho c_\rho V}, \quad b_{22} = \frac{F_{j2}}{V_j},$$

$$b_{31} = \frac{UA}{\rho c_\rho V}, \quad b_{32} = \frac{F_{j1}}{V_j}, \quad \Psi = \frac{F_0}{V},$$

$$\Phi = \frac{F + F_R}{V},$$

$$\begin{aligned}\phi_{21} = {} & \frac{F + F_R}{V} T_1^d - \frac{F + F_R}{V}\left(x_{2,1} + T_2^d\right) \\ & - \frac{\alpha\lambda}{\rho c_\rho}\left(x_{1,1} + C_{A2}^d\right) e^{-\frac{E}{R(x_{2,1} + T_2^d)}} \\ & - \frac{UA}{\rho c_\rho V}\left(x_{2,1} + T_2^d - T_{j2}^d\right),\end{aligned}$$

$$\begin{aligned}\phi_{31} = {} & \frac{F_0}{V} T_0^d - \frac{F + F_R}{V}\left(x_{3,1} + T_1^d\right) \\ & + \frac{F_R}{V}\left(x_{2,1} + T_2^d\right) \\ & - \frac{\alpha\lambda}{\rho c_\rho} C_{A1} e^{-\frac{E}{R(x_{3,1} + T_1^d)}} \\ & - \frac{UA}{\rho c_\rho V}\left(x_{3,1} + T_1^d - T_{j1}^d\right),\end{aligned}$$

$$\begin{aligned}\phi_{22} = {} & \frac{F_{j2}}{V_j}\left(T_{j20}^d - x_{2,2} - T_{j2}^d\right) \\ & + \frac{UA}{\rho_j c_j V_j}\left(x_{2,1} + T_2^d - x_{2,2} - T_{j2}^d\right),\end{aligned}$$

$$\omega = e^{-0.15t} \sin(t),$$

$$\begin{aligned}\phi_{32} = {} & \frac{F_{j1}}{V_j}\left(T_{j10}^d - x_{3,2} - T_{j1}^d\right) \\ & + \frac{UA}{\rho_j c_j V_j}\left(x_{3,1} + T_1^d - x_{3,2} - T_{j1}^d\right),\end{aligned}$$

$$\begin{aligned}C_{A1} = {} & \frac{V}{F + F_R}\Bigg(x_{1,2} + \frac{F + F_R}{V}\left(x_{1,1} + C_{A2}^d\right) \\ & + \alpha\left(x_{1,1} + C_{A2}^d\right) e^{-\frac{E}{R(x_{2,1} T_2^d)}}\Bigg)\end{aligned}$$

where α, E, and λ denote the reaction rate constant, activation energy, and heat generation rate; ρ and ρ_j are the densities of liquid in the reactors and in the jackets; c_p and c_j stand for heat capacities. The values of the process parameters are provided in Table 1.

The objective is to control C_{A2}, T_1, and T_2 by manipulating C_{A0}, T_{j10}, and T_{j20}. The deviation $T_0 - T_0^d$ of the inlet temperature T_0 from the steady-state value T_0^d is assumed to be an unmeasurable disturbance.

Define the following coordinate changes: $\bar{x}_{1,1} = x_{1,1}, \bar{x}_{1,2} = b_{11}x_{1,2}, \bar{x}_{2,1} = x_{2,1}, \bar{x}_{2,2} = b_{21}x_{2,2}, \bar{x}_{3,1} = x_{3,1}$ and $\bar{x}_{3,2} = b_{31}x_{3,2}$, then the system (72) is of the same form as in system (1)

$$\begin{cases} \dot{\bar{x}}_{1,1} = \bar{x}_{1,2} + y_1^3 dw \\ \dot{\bar{x}}_{1,2} = \bar{u}_1 + \frac{1}{2} y_1^2 dw \\ y_1 = \bar{x}_{1,1} \\ \dot{\bar{x}}_{2,1} = \bar{x}_{2,2} + \bar{\phi}_{21}(\bar{x}_{1,1}, \bar{x}_{2,1}) + \Phi \bar{x}_{2,1} + y_2^2 dw \\ \dot{\bar{x}}_{2,2} = \bar{u}_2 + \bar{\phi}_{22}(\bar{x}_{2,1}, \bar{x}_{2,2}) + y_2^2 \cos(y_2^2) dw \\ y_2 = \bar{x}_{2,1} \\ \dot{\bar{x}}_{3,1} = \bar{x}_{3,2} + \bar{\phi}_{31}(\bar{x}_{2,1}, \bar{x}_{3,1}) + \Psi \omega + 2y_3^2 dw \\ \dot{\bar{x}}_{3,2} = \bar{u}_3 + \bar{\phi}_{32}(\bar{x}_{3,1}, \bar{x}_{3,2}) + y_3^5 dw \\ y_3 = \bar{x}_{3,1} \end{cases} \tag{73}$$

where $\bar{u}_1 = b_{11}b_{12}u_1, \bar{u}_2 = b_{21}b_{22}u_2, \bar{\phi}_{21}(\bar{x}_{2,1}, \bar{x}_{2,1}) = \phi_{21}(x_{1,1}, x_{2,1}), \bar{\phi}_{22}(\bar{x}_{2,1}, \bar{x}_{2,2}) = b_{21}\phi_{22}(x_{2,1}, x_{2,2}),$ $\bar{u}_3 = b_{31}b_{32}u_3$ and $\bar{\phi}_{32}(\bar{x}_{3,1}, \bar{x}_{3,2}) = b_{31}\phi_{32}(x_{3,1}, b_{31}x_{3,2})$.

In the simulation study, eleven fuzzy set are defined over interval $[-10, 10]$ for all $\bar{x}_{1,1}, \bar{x}_{1,2}, \bar{x}_{2,1}, \bar{x}_{2,2}, \bar{x}_{3,1}$, and $\bar{x}_{3,2}$, and by choosing partitioning points as $-10, -8, -6, -4, 0, 2, 4, 6, 8$, and 10, their fuzzy membership functions are given as follows:

$$\mu_{F_{ij}^1}(x_{i,j}) = e^{-0.5(x_{i,j}+10)^2},$$

$$\mu_{F_{ij}^2}(x_{i,j}) = e^{-0.5(x_{i,j}+8)^2},$$

$$\mu_{F_{ij}^3}(x_{i,j}) = e^{-0.5(x_{i,j}+6)^2},$$

$$\mu_{F_{ij}^4}(x_{i,j}) = e^{-0.5(x_{i,j}+4)^2},$$

$$\mu_{F_{ij}^5}(x_{i,j}) = e^{-0.5(x_{i,j}+2)^2},$$

$$\mu_{F_{ij}^6}(x_{i,j}) = e^{-0.5(x_{i,j})^2},$$

$$\mu_{F_{ij}^7}(x_{i,j}) = e^{-0.5(x_{i,j}-2)^2},$$

$$\mu_{F_{ij}^8}(x_{i,j}) = e^{-0.5(x_{i,j}-4)^2},$$

$$\mu_{F_{ij}^9}(x_{i,j}) = e^{-0.5(x_{i,j}-6)^2},$$

$$\mu_{F_{ij}^{10}}(x_{i,j}) = e^{-0.5(x_{i,j}-8)^2},$$

$$\mu_{F_{ij}^{11}}(x_{i,j}) = e^{-0.5(x_{i,j}-10)^2}$$

Let

$$\varphi_{2,1}^k = \frac{\mu_{F_{11}^k}(\bar{x}_{1,1})\mu_{F_{21}^k}(\bar{x}_{2,1})\mu_{F_{31}^k}(\bar{x}_{3,1})}{\sum_{k=1}^{11}(\mu_{F_{11}^k}(\bar{x}_{1,1})\mu_{F_{21}^k}(\bar{x}_{2,1})\mu_{F_{31}^k}(\bar{x}_{3,1}))},$$

$$\varphi_{2,2}^k = \frac{\prod_{i=1}^2\prod_{j=1}^2 \mu_{F_{ij}^k}(\bar{x}_{i,j})\mu_{F_{31}^k}(\bar{x}_{3,1})}{\sum_{k=1}^{11}(\prod_{i=1}^2\prod_{j=1}^2 \mu_{F_{ij}^k}(\bar{x}_{i,j})\mu_{F_{31}^k}(\bar{x}_{3,1}))},$$

$$\varphi_{3,1}^k = \frac{\prod_{j=1}^2 \mu_{F_{1j}^k}(\bar{x}_{1,j})\mu_{F_{21}^k}(\bar{x}_{2,1})\mu_{F_{31}^k}(\bar{x}_{3,1})}{\sum_{k=1}^{11}(\prod_{j=1}^2 \mu_{F_{1j}^k}(\bar{x}_{1,j})\mu_{F_{21}^k}(\bar{x}_{2,1})\mu_{F_{31}^k}(\bar{x}_{3,1}))},$$

$$\varphi_{32}^k = \frac{\prod_{i=1}^3\prod_{j=1}^2 \mu_{F_{ij}^k}(\bar{x}_{i,j})}{\sum_{k=1}^{11}(\prod_{i=1}^3\prod_{j=1}^2 \mu_{F_{ij}^k}(\bar{x}_{i,j}))}.$$

Then

$$\varphi_{2,1}^k = [\varphi_{2,1}^1, \varphi_{2,1}^2 \cdots \varphi_{2,1}^{11}]^T, \varphi_{2,2} = [\varphi_{2,2}^1, \varphi_{2,2}^2 \cdots \varphi_{2,2}^{11}]^T,$$

$$\varphi_{3,1}^k = [\varphi_{3,1}^1, \varphi_{3,1}^2 \cdots \varphi_{3,1}^{11}]^T, \varphi_{3,2}^k = [\varphi_{3,2}^1, \varphi_{3,2}^2 \cdots \varphi_{3,2}^{11}]^T$$

The fuzzy controllers and parameters of adaptive law are constructed as

$$u_1 = -c_{1,2}\chi_{1,2} + k_{1,2}(\hat{\bar{x}}_{1,1} - y_1) - \frac{1}{\tau_{1,2}}(z_{1,2} - \alpha_{1,1}),$$

$$u_2 = -c_{2,2}\chi_{2,2} + k_{2,2}(\hat{\bar{x}}_{2,1} - y_2) - \theta_{2,2}^T\varphi_{2,2}(\hat{X}_{2,2}) - \frac{1}{\tau_{2,2}}(z_{2,2} - \alpha_{2,1}) - \hat{\omega}_{2,2}\tanh(\chi_{2,2}^3/k),$$

$$u_3 = -c_{3,2}\chi_{3,2} + k_{3,2}(\hat{\bar{x}}_{3,1} - y_3) - \theta_{3,2}^T\varphi_{3,2}(\hat{X}_{3,2}) - \frac{1}{\tau_{3,2}}(z_{3,2} - \alpha_{3,1}) - \hat{\omega}_{3,2}\tanh(\chi_{3,2}^3/k),$$

$$\dot{\theta}_{2,1} = \gamma_{2,1}\varphi_{2,1}(\hat{X}_{2,1})\chi_{2,1}^3 - \sigma_{2,1}\theta_{2,1}$$

$$\dot{\hat{\varepsilon}}_{2,1} = \bar{\gamma}_{2,1}\chi_{2,1}^3\tanh\left(\frac{\chi_{2,1}^3}{k}\right) - \bar{\sigma}_{2,1}\hat{\varepsilon}_{2,1}$$

$$\theta_{2,2} = \gamma_{2,2}\varphi_{2,2}(\hat{X}_{2,2})\chi_{2,2}^3 - \sigma_{2,2}\theta_{2,2},$$

$$\dot{\hat{\omega}}_{2,2} = \bar{\gamma}_{2,2}\chi_{2,2}^3\tanh\left(\frac{\chi_{2,2}^3}{k}\right) - \bar{\sigma}_{2,2}\hat{\omega}_{2,2},$$

$$\dot{\theta}_{3,1} = \gamma_{3,1}\varphi_{3,1}(\hat{X}_{3,1})\chi_{3,1}^3 - \sigma_{3,1}\theta_{3,1},$$

$$\dot{\hat{\varepsilon}}_{3,1} = \bar{\gamma}_{3,1}\chi_{3,1}^3\tanh\left(\frac{\chi_{3,1}^3}{k}\right) - \bar{\sigma}_{3,1}\hat{\varepsilon}_{3,1},$$

$$\theta_{3,2} = \gamma_{3,2}\varphi_{3,2}(\hat{X}_{3,2})\chi_{3,2}^3 - \sigma_{3,2}\theta_{3,2},$$

$$\dot{\hat{\omega}}_{3,2} = \bar{\gamma}_{3,2}\chi_{3,2}^3\tanh\left(\frac{\chi_{3,2}^3}{k}\right) - \bar{\sigma}_{3,2}\hat{\omega}_{3,2},$$

$$\alpha_{1,1} = -c_{1,1}\chi_{i,1} - \frac{3}{4}\eta_{1,1}^{\frac{4}{3}}\chi_{1,1} - \frac{3m\sqrt{m}}{\eta_{1,0}^2}\chi_{1,1}\|\psi_1(y_1)\|^4 - \frac{3}{2}\chi_{1,1}\psi_{1,1}(y_1)^T\psi_{1,1}(y_1)$$

$$\begin{aligned}\alpha_{2,1} = & -c_{2,1}\chi_{2,1} - \frac{3}{4}\eta_{2,1}^{\frac{4}{3}}\chi_{2,1} \\ & - \theta_{2,1}^{T}\varphi_{2,1}(\hat{X}_{2,1}) \\ & - \hat{\varepsilon}_{2,1}\tanh\left(\chi_{2,1}^{3}/k\right) \\ & - \frac{3m\sqrt{m}}{\eta_{2,0}^{2}}\chi_{2,1}\left\|\psi_{2}(y_{2})\right\|^{4} \\ & - \frac{3}{2}\chi_{2,1}\psi_{2,1}(y_{2})^{T}\psi_{2,1}(y_{2}),\end{aligned}$$

$$\begin{aligned}\alpha_{3,1} = & -c_{3,1}\chi_{3,1} - \frac{3}{4}\eta_{3,1}^{\frac{4}{3}}\chi_{3,1} \\ & - \theta_{3,1}^{T}\varphi_{3,1}(\hat{X}_{3,1}) - \hat{\varepsilon}_{3,1}\tanh\left(\chi_{3,1}^{3}/k\right) \\ & - \frac{3m\sqrt{m}}{\eta_{3,0}^{2}}\chi_{3,1}\left\|\psi_{3}(y_{3})\right\|^{4} \\ & - \frac{3}{2}\chi_{3,1}\psi_{3,1}(y_{3})^{T}\psi_{3,1}(y_{3})\end{aligned}$$

The design parameters are chosen as

$$\begin{aligned}& c_{1,1} = 10, \quad c_{1,2} = 4, \quad c_{2,1} = 10, \\ & c_{2,2} = 10, \quad c_{3,1} = 10, \quad c_{3,2} = 10, \\ & k_{1,1} = 5, \quad k_{1,2} = 5, \quad k_{2,1} = 10, \\ & k_{2,2} = 10, \quad k_{3,1} = 5, \quad k_{3,2} = 5, \quad r_{2,1} = 1, \\ & r_{2,2} = 1, \quad r_{3,1} = 1, \quad r_{3,2} = 1, \quad \bar{r}_{2,1} = 1, \\ & \bar{r}_{2,2} = 1, \quad \bar{r}_{3,1} = 1, \quad \bar{r}_{3,2} = 1, \\ & \tau_{1,2} = 0.5, \quad \tau_{2,2} = 0.1, \quad \tau_{3,2} = 0.1, \\ & \eta_{1,0} = 10, \quad \eta_{1,1} = 0.5, \quad \eta_{2,0} = 10, \quad \eta_{2,1} = 0.4, \\ & \eta_{3,0} = 10, \quad m = 2, \\ & \sigma_{2,1} = 0.01, \quad \sigma_{2,2} = 0.01, \quad \sigma_{3,1} = 0.01, \\ & \sigma_{3,2} = 0.01, \quad \bar{\sigma}_{2,1} = 0.01, \\ & \bar{\sigma}_{2,2} = 0.01, \quad \bar{\sigma}_{3,1} = 0.01, \\ & \bar{\sigma}_{3,2} = 0.01, \quad k = 0.01\end{aligned}$$

The initial conditions are chosen as

$$\begin{aligned}& \bar{x}_{1,1}(0) = 1.5, \quad \bar{x}_{1,2}(0) = 0, \quad \bar{x}_{2,1}(0) = 0.5, \\ & \bar{x}_{2,2}(0) = 1, \quad \bar{x}_{3,1}(0) = 0.5,\end{aligned}$$

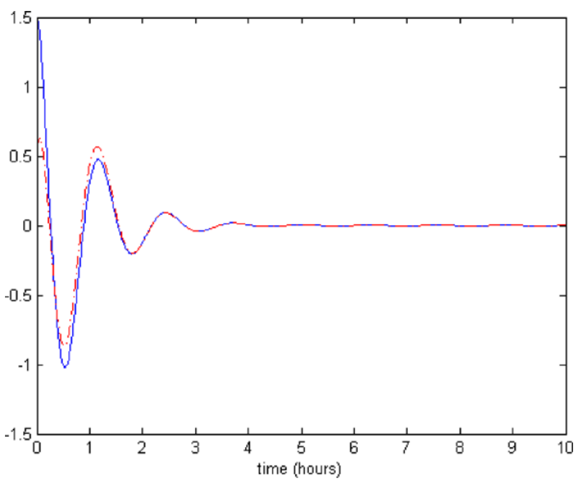

Fig. 1 The trajectories of $x_{1,1}$ "*solid line*" and $\hat{x}_{1,1}$ "*dash-dotted*"

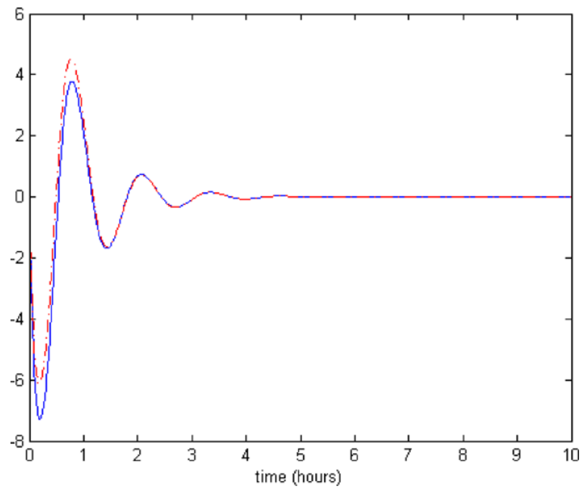

Fig. 2 The trajectories of $x_{1,2}$ "*solid line*" and $\hat{x}_{1,2}$ "*dash-dotted*"

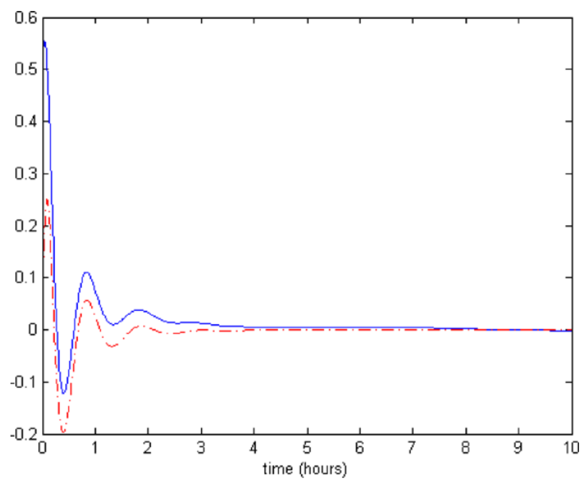

Fig. 3 The trajectories of $x_{2,1}$ "*solid line*" and $\hat{x}_{2,1}$ "*dash-dotted*"

$\bar{x}_{3,2}(0) = \hat{\bar{x}}_{1,1}(0) = 0,$

$\hat{\bar{x}}_{1,2}(0) = 0.5, \quad \hat{\bar{x}}_{2,1}(0) = 0.5, \quad \hat{\bar{x}}_{2,2}(0) = 0,$

$\hat{\bar{x}}_{3,1}(0) = 0, \quad \hat{\bar{x}}_{3,2}(0) = 0,$

$\theta_{2,1}(0) = 0, \quad \theta_{22}(0) = 0,$

$\theta_{31}(0) = 0, \quad \theta_{32}(0) = 0, \quad \hat{\varepsilon}_{2,1}(0) = 0, \quad \hat{\varepsilon}_{3,1}(0) = 0,$

$\hat{\omega}_{2,2}(0) = 0, \quad \hat{\omega}_{3,2}(0) = 0$

The simulation results are shown in Figs. 1–9.

From the above simulation results, it is clear that even though the exact information on the nonlinear functions in the system is not available and the state variables are immeasurable, the proposed adaptive fuzzy output feedback control approaches guarantee the stability of the closed-loop adaptive control system and achieve good control performance.

6 Conclusions

In this paper, an observer-based adaptive fuzzy output feedback control approach has been proposed for a class of uncertain MIMO stochastic nonlinear system with immeasurable states. Fuzzy logic systems are used to approximate the unknown nonlinear functions and a fuzzy state observer is designed to estimate

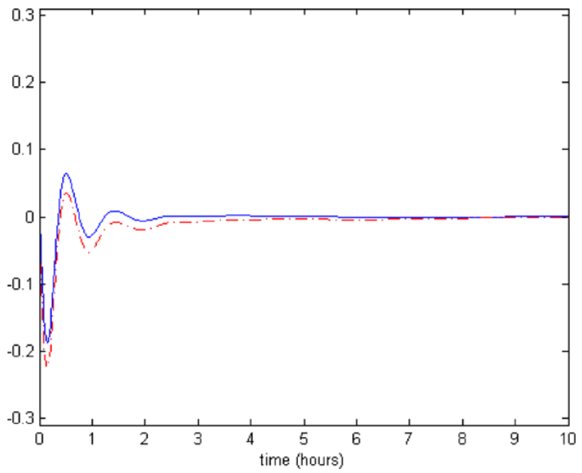

Fig. 4 The trajectories of $x_{2,2}$ "*solid line*" and $\hat{x}_{2,2}$ "*dash-dotted*"

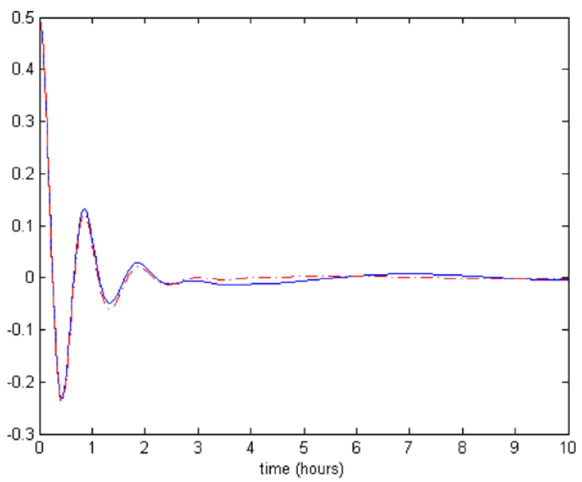

Fig. 5 The trajectories of $x_{3,1}$ "*solid line*" and $\hat{x}_{3,1}$ "*dash-dotted*"

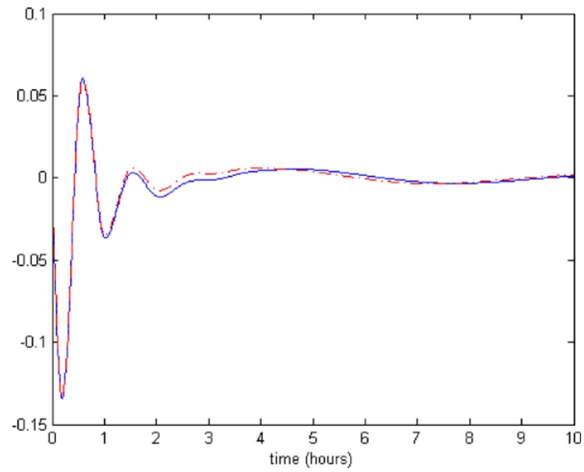

Fig. 6 The trajectories of $x_{3,2}$ "*solid line*" and $\hat{x}_{3,2}$ "*dash-dotted*"

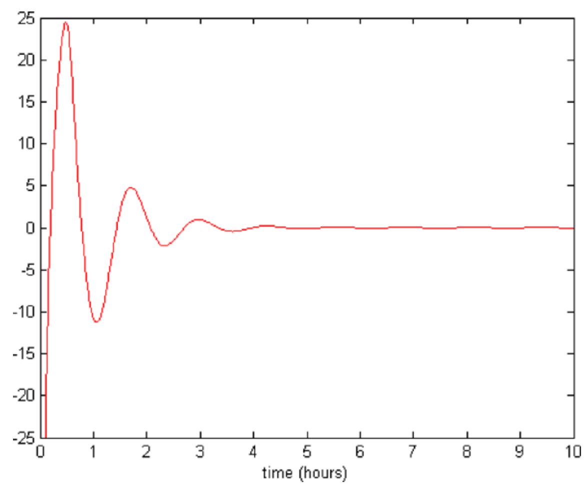

Fig. 7 The trajectory of u_1

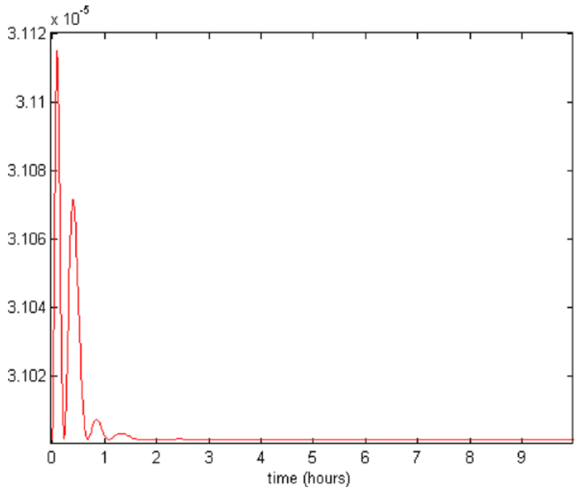

Fig. 8 The trajectory of u_2

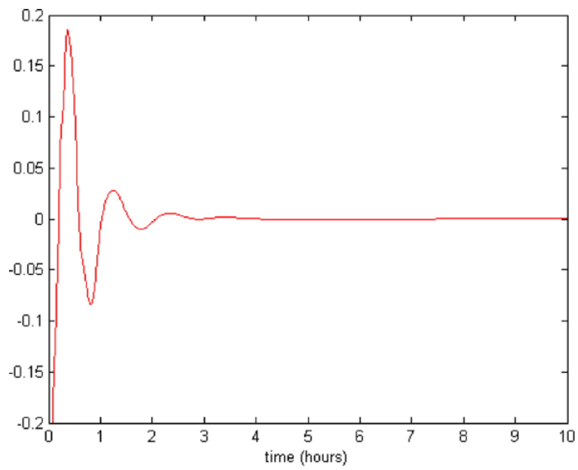

Fig. 9 The trajectory of u_3

those immeasurable states. By combining the adaptive backstepping design with the DSC technique, a novel adaptive fuzzy output feedback backstepping control approach is developed. It is proved that all the signals of the closed-loop control system are semiglobally uniformly ultimately bounded (SUUB) in probability; the observer errors and the system outputs can be made as small as the desired by appropriate choice of the design parameters. Simulation results are provided to show the effectiveness of the proposed approach.

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