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Robust adaptive backstepping synchronization for a class of uncertain chaotic systems using fuzzy disturbance observer

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Abstract This paper proposes a robust adaptive backstepping synchronization method for a class of uncertain chaotic systems. Unknown factors including system uncertainties and external disturbances are estimated by a fuzzy disturbance observer. By use of the fuzzy disturbance observer, any prior information about the unknown factors is not need. The proposed method using the estimated values guarantees the global synchronization for chaotic systems with mismatched uncertainties in the sense of uniform ultimate boundedness. Finally, numerical examples are presented to show the effectiveness of the method.

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1 Introduction

Since the pioneering work of Pecora and Carroll [1], synchronization of chaotic systems has been one of interesting topics in various research fields including secure communication [2–4], neural networks [5], complex networks [6], biology [7], and mechanics [8]. Many researchers have investigated various synchronization methods such as state feedback control [9–12], delayed feedback control [13], adaptive control [14, 15], backstepping control [16–20], sliding mode control [21–23], $\mathcal{H}_\infty$ control [24, 25], and optimal control [26, 27]. Among the methods, backstepping control scheme has become one of the important and popular approaches for synchronization for chaotic systems. Because the scheme can guarantee global stability and good tracking performance. Moreover, it has an ability to overcome mismatched uncertainty by use of a recursive procedure with the choice of Lyapunov function. In [16], a backstepping synchronization method with parameter update laws was presented for uncertain chaotic systems. Wu and Chen [17] studied backstepping control to suppress the chaotic motion of a modified Chua's circuit system, guaranteeing the global exponential stability. Chaos synchronization was achieved by backstepping method using

single variable feedback [18]. In [19], chaotic systems with unknown bounded uncertainty were synchronized by use of an adaptive law for the upper bound of the uncertainty. In [20], dynamic fuzzy neural networks modeling was used to control uncertain chaotic systems via adaptive backstepping method.

Although the studies mentioned above can synchronize the chaotic systems, there are still some limitations. It is that they often need to know information about the unknown factors. In [16], knowledge of the structure of parameter uncertainty was required to synchronize the chaotic systems. The synchronization method proposed in [20] can be applied when upper bound of the existing uncertainty is assumed to be known. However, in many practical cases, the information about the parameter uncertainty or disturbance may not be available or can be difficult to use them. By using fuzzy logic system (FLS), Kim [28] proposed Fuzzy Disturbance Observer (FDO) to estimate the unknown factors and the estimated values were used to suppress the unknown factors without any prior information about them. In [29], a robust tracking control approach using a discrete-time FDO was studied for nonlinear sampled systems. Moreover, Yoo et al. [30] constructed a more precise FDO by modification of the adaptation law for the parameter vector and showed better performances, compared with the conventional one. Therefore, the method using FDO can be one of good schemes to synchronize uncertain chaotic systems. However, there exist few studies which use FDO for chaotic synchronization and the existing methods have difficulty to control mismatched uncertainty.

In this paper, a robust adaptive backstepping synchronization method using FDO is proposed for chaotic systems with mismatched uncertainty and disturbance. The FDO is used to estimate the unknown factors including the uncertainties and disturbances. Any knowledge about the factors such as their structure or bound is not required because the FLS is used. The estimated value is applied in the proposed controller to suppress the unknown factors. Moreover, the proposed method overcomes the mismatched uncertainty by adopting adaptive backstepping design procedure. The control and adaptation laws used in the control input are derived in based on the Lyapunov stability theory. Eventually, the proposed method guarantees that the synchronization error between the drive and the response system converges into any arbitrarily given small bound in the sense of uniform ultimate

boundedness (UUB). Numerical examples show the effectiveness of the proposed synchronization method.

Notations Throughout this paper, $\mathbb{R}^n$ and $\mathbb{R}^{n \times m}$ denote the sets of n component real vectors and $n \times m$ real matrices, respectively. For the vector $u \in \mathbb{R}^n$, u^T denotes its transpose. The notation $|\cdot|$ refers of the absolute value of a scalar function and $\|\cdot\|$ stands for the induced matrix 2-norm. $\mathcal{L}_2$ means the space of square integral vector functions on $[0, \infty)$ with norm $\|\cdot\| \equiv (\int_0^\infty \|\cdot\|^2 \, dt)^{1/2}$.

2 Problem statement and preliminary

The synchronization problem is described based on the drive-response framework. Consider the following chaotic system as the drive system:

$$\dot{x}_d(t) = f_d(x_d(t), t), \tag{1}$$

$$y_d(t) = x_{d1}(t), \tag{2}$$

where $x_d(t) = [x_{d1}, \ldots, x_{dm}] \in \mathbb{R}^m$ is state vector and $y_d(t) \in \mathbb{R}$ is output vector, and $f_d(x_d(t), t) = [f_{d1}, \ldots, f_{dm}] \in \mathbb{R}^m$ is a known continuous nonlinear vector function.

The response system is represented as

$$\dot{x}_1(t) = \left(f_1(\bar{x}_1(t), t) + \Delta f_1(\bar{x}_1(t), t) \right)$$
$$\quad + \left(g_1(\bar{x}_1(t), t) + \Delta g_1(\bar{x}_1(t), t) \right) x_2(t) + d_1(t)$$
$$\dot{x}_2(t) = \left(f_2(\bar{x}_2(t), t) + \Delta f_2(\bar{x}_2(t), t) \right)$$
$$\quad + \left(g_2(\bar{x}_2(t), t) + \Delta g_2(\bar{x}_2(t), t) \right) x_3(t) + d_2(t)$$
$$\vdots$$
$$\dot{x}_n(t) = \left(f_n(\bar{x}_n(t), t) + \Delta f_n(\bar{x}_n(t), t) \right)$$
$$\quad + \left(g_n(\bar{x}_n(t), t) + \Delta g_n(\bar{x}_n(t), t) \right) u(t) + d_n(t), \tag{3}$$

$$y(t) = x_1(t) \tag{4}$$

where $x(t) = [x_1(t), \ldots, x_n(t)]^T \in \mathbb{R}^n$, $u(t) \in \mathbb{R}$, $y(t) \in \mathbb{R}$ are, respectively, the state vector, the control input and output vector of the response system, $f_i(\bar{x}_i(t), t) \in \mathbb{R}$ and $g_i(\bar{x}_i(t), t) \in \mathbb{R} \neq 0$ with $\bar{x}_i(t) = [x_1(t), \ldots, x_i(t)]^T$ are known nominal functions, $\Delta f_i(\bar{x}_i(t), t)$, $\Delta g_i(\bar{x}_i(t), t)$, and $d_i(t)$ are uncertainty and disturbance, respectively.

For simplicity, let us denote $x_i(t) = x_i$, $u(t) = u$, $f_i(\bar{x}_i(t), t) = f_i$, $\Delta f_i(\bar{x}_i(t), t) = \Delta f_i$, $g_i(\bar{x}_i(t), t) = g_i$, $\Delta g_i(\bar{x}_i(t), t) = \Delta g_i$, and $d_i(t) = d_i$ for $i = 1, \ldots, n$.

By defining the overall disturbance $\Omega_i = \Delta f_i + \Delta g_i x_{i+1} + d_i$ for $i = 1, \ldots, n-1$ and $\Omega_n = \Delta f_n + \Delta g_n u + d_n$, (3) can be rewritten as

$$\dot{x}_1 = f_1 + g_1 x_2 + \Omega_1$$
$$\dot{x}_2 = f_2 + g_2 x_3 + \Omega_2$$
$$\vdots \qquad\qquad (5)$$
$$\dot{x}_{n-1} = f_{n-1} + g_{n-1} x_n + \Omega_{n-1}$$
$$\dot{x}_n = f_n + g_n u + \Omega_n.$$

Remark 1 The system (5) is called "strict-feedback" form, in which the backstepping design method can be applied. In practice, we can find many chaotic systems which can be represented as the form, for example, Duffing oscillator, Bonhoeffer–van der Pol oscillator, Rössler system, Chua's circuit, and so on.

The objective of this paper is to design a controller which makes the output of the response system (4) converge to the output of the drive system (2) with an arbitrarily small error. In order to accomplish our purpose, we need to suppress the overall disturbances existing in the system. It can be achieved by estimation of the disturbances and the estimated value can be obtained by fuzzy logic system (FLS). Here, the basic configuration of FLS [31] is described briefly. FLS performs a mapping from a compact set $X = X_1 \times \cdots \times X_n \subset \mathbb{R}^n$ to a compact set $V \subset \mathbb{R}$. The fuzzy rule base consists of a collection of M fuzzy IF-THEN rules:

$$R^{(l)} : \text{IF } x_1 \text{ is } A_1^l, \text{ and } \cdots \text{ and, } x_n \text{ is } A_n^l,$$
$$\text{Then } y \text{ is } G^l \qquad\qquad (6)$$

where $x = [x_1, \ldots, x_n]^T \in X$ and $y \in V$ are the input and output of the FLS, respectively, and A_i^l and G^l are labels of fuzzy sets in X_i and R for $l = 1, 2, \ldots, M$. By use of the product inference engine, center-average defuzzifier, and singleton fuzzifier, the output of the fuzzy system can be expressed as

$$y(x) = \frac{\sum_{l=1}^{M} y_l (\prod_{i=1}^{n} \mu_{A_i^l}(x_i))}{\sum_{l=1}^{M} (\prod_{i=1}^{n} \mu_{A_i^l}(x_i))} = \theta^T \xi(x) \qquad (7)$$

where $\mu_{A_i^l}(x_i)$ is membership function value of the fuzzy variable x_i, M is the number of fuzzy rules, $\theta = [y_1, y_2, \ldots, y_M]^T$ is adjustable parameter vector, and $\xi(x) = (\xi_1(x), \xi_2(x), \ldots, \xi_M(x))^T$ is a regressive vector defined as

$$\xi_l(x) = \frac{\prod_{i=1}^{n} \mu_{A_i^l}(x_i)}{\sum_{l=1}^{M} (\prod_{i=1}^{n} \mu_{A_i^l}(x_i))} \qquad (8)$$

which is named fuzzy basis functions (FBFs).

Based on the well-known "*universal approximation theorem*" [31], the fuzzy system (7) estimates unknown function with a small error called fuzzy approximation error. This characteristic was extended to estimate the disturbance, which was introduced as FDO in [28]. In this paper, FDO is applied to estimate the overall disturbances including uncertainties and disturbances in the response system (3). One of the major advantages of FDO is that it can be a good approach to obtain the estimated value when it is not available to know the bound or structure of the unknown factors. Eventually, the value will be used to compensate the actual disturbance in the proposed method in which the details will be presented in the next section.

3 Robust adaptive backstepping synchronization using FDO

In this section, we propose a synchronization method using robust adaptive backstepping procedure. First, we present a design method of FDO to construct the overall disturbance Ω_i that exists in the response system (3). Let us consider the following observer system as

$$\dot{\hat{x}}_1 = f_1 + g_1 x_2 + \hat{\Omega}_1 + p_1(x_1 - \hat{x}_1)$$
$$\dot{\hat{x}}_2 = f_2 + g_2 x_3 + \hat{\Omega}_2 + p_2(x_2 - \hat{x}_2)$$
$$\vdots \qquad\qquad (9)$$
$$\dot{\hat{x}}_n = f_n + g_n u + \hat{\Omega}_n + p_n(x_n - \hat{x}_n)$$

where $\hat{x} = [\hat{x}_1, \ldots, \hat{x}_n] \in \mathbb{R}^n$, p_i is a positive constant, $\hat{\Omega}_i = \theta_i^T \xi_i(x)$, $\theta_i \in \mathbb{R}^M$ is the fuzzy parameter vector, $\xi_i(x) \in \mathbb{R}^M$ is the fuzzy basis function vector. From this observer system, we can obtain a fuzzy logic system $\hat{\Omega}_i$ which guarantees that, by the universal ap-

proximation theorem [31],

$$|\Omega_i - \hat{\Omega}_i| < \bar{\varepsilon}_i \tag{10}$$

where $\bar{\varepsilon}_i \in \mathbb{R}$ is the upper bound of the fuzzy approximation error. Therefore, we can see that Ω_i is estimated by $\hat{\Omega}_i(x)$ with an error bound $\varepsilon_i \leq \bar{\varepsilon}_i \in \mathbb{R}$.

Define the *observation error* as follows:

$$\varphi_i = x_i - \hat{x}_i. \tag{11}$$

Then, from the response system (3) and the observer system (9), the error dynamics is described by

$$\dot{\varphi}_i = \dot{x}_i - \dot{\hat{x}}_i = \Omega_i - \hat{\Omega}_i - p_i(x_i - \hat{x}_i) = \varepsilon_i - p_i\varphi_i \tag{12}$$

where $\varepsilon_i = \Omega_i - \hat{\Omega}_i = \dot{\varphi}_i + P_i\varphi_i$. This *disturbance reconstruction error* ε_i can be rewritten as

$$\varepsilon_i = \Omega_i - \hat{\Omega}_i = \Omega_i - \hat{\Omega}_i^* + \hat{\Omega}_i^* - \hat{\Omega}_i \equiv l_i + m_i \tag{13}$$

where

$$l_i = \Omega_i - \hat{\Omega}_i^*, \tag{14}$$

$$m_i = \hat{\Omega}_i^* - \hat{\Omega}_i = \tilde{\theta}_i^T \xi_i(x), \tag{15}$$

$$\tilde{\theta}_i = \theta_i^* - \theta_i, \tag{16}$$

$$\hat{\Omega}_i^* = \hat{\Omega}_i(x|\theta_i^*) = \theta_i^{*T}\xi_i(x), \tag{17}$$

$$\theta_i^* = \arg\min_{\theta_i}\Big[\sup_x\big|\hat{\Omega}_i(x|\theta_i) - \Omega_i(x)\big|\Big].$$

The following theorem presents an adaptation law for θ_i to estimate the overall disturbance Ω_i.

Theorem 1 *Consider the system (3) and the observer system (9). If the update law of the parameter vector θ_i for $\hat{\Omega}_i(x|\theta_i)$ is chosen as*

$$\dot{\theta}_i = \gamma_{i1}\xi_i(x)(\varphi_i + \gamma_{i0}\varepsilon_i) \tag{18}$$

where γ_{i0} and γ_{i1} are positive constants, then the unknown factor Ω_i is estimated by $\hat{\Omega}_i(x|\theta_i) = \theta_i^T\xi_i(x)$ guaranteeing the following robust performance

$$\sum_{i=1}^n \int_0^T P_i\varphi_i^2\, dt + \sum_{i=1}^n \int_0^T \gamma_{i0}m_i^2\, dt$$

$$\leq \sum_{i=1}^n \varphi_i^2(0) + \sum_{j=1}^n \frac{1}{\gamma_{i1}}\tilde{\theta}_i^T(0)\tilde{\theta}_i(0)$$

$$+ \sum_{i=1}^n \int_0^T \left(\gamma_{i0} + \frac{1}{p_i}\right)l_i^2\, dt \tag{19}$$

where $\Gamma_{i0} = \gamma_{i0}I_{n\times n} > 0$ and γ_{i1} is the positive constant for $i = 1, \ldots, n$.

Proof Choose the following Lyapunov function candidate:

$$V_F = \frac{1}{2}\sum_{i=1}^n \varphi_i^2 + \frac{1}{2}\sum_{i=1}^n \frac{1}{\gamma_{i1}}\tilde{\theta}_i^T\tilde{\theta}_i \tag{20}$$

where γ_{i1} is a positive constant.

Differentiating V_F along the error dynamics (12) and using (17)–(18) yield

$$\dot{V}_F = \sum_{i=1}^n \varphi_i\dot{\varphi}_i + \sum_{i=1}^n \frac{1}{\gamma_{i1}}\tilde{\theta}_i^T\dot{\tilde{\theta}}_i$$

$$= \sum_{i=1}^n \varphi_i(-p_i\varphi_i + \Omega_i - \hat{\Omega}_i) + \sum_{i=1}^n \frac{1}{\gamma_{i1}}\tilde{\theta}_i^T\dot{\tilde{\theta}}_i$$

$$= \sum_{i=1}^n \varphi_i\big(-p_i\varphi_i + \Omega_i - \hat{\Omega}_i^* + \hat{\Omega}_i^* - \hat{\Omega}_i\big)$$

$$\quad + \sum_{i=1}^n \frac{1}{\gamma_{i1}}\tilde{\theta}_i^T\dot{\tilde{\theta}}_i$$

$$= -\sum_{i=1}^n p_i\varphi_i^2 + \sum_{i=1}^n \varphi_i^T l_i + \sum_{i=1}^n \varphi_i\big(\hat{\Omega}_i^* - \hat{\Omega}_i\big)$$

$$\quad - \sum_{i=1}^n \tilde{\theta}_i^T\xi_i(\varphi_i + \gamma_{i0}\varepsilon_i)$$

$$= \sum_{i=1}^n \big[-p_i\varphi_i^2 + \varphi_i l_i + \tilde{\theta}_i^T\xi_i\varphi_i - \tilde{\theta}_i^T\xi_i(\varphi_i + \gamma_{i0}\varepsilon_i)\big]$$

$$= \sum_{i=1}^n \big[-p_i\varphi_i^2 + \varphi_i l_i - \gamma_{i0}m_i^2 - \gamma_{i0}m_i l_i\big]. \tag{21}$$

Applying the following inequalities

$$\varphi_i l_i \leq \frac{1}{2}p_i\varphi_i^2 + \frac{1}{2p_i}l_i^2 \quad \text{and} \quad -m_i l_i \leq \frac{1}{2}m_i^2 + \frac{1}{2}l_i^2, \tag{22}$$

we have the following inequality:

$$\dot{V}_F \le \sum_{i=1}^{n} \left(-p_i\varphi_i^2 - \gamma_{i0}m_i^2 + \frac{1}{2}\gamma_{i0}m_i^2 + \frac{1}{2}\gamma_{i0}l_i^2 \right.$$
$$\left. + \frac{1}{2}p_i\varphi_i^2 + \frac{1}{2p_i}l_i^2 \right)$$
$$= \sum_{i=1}^{n} \left(-\frac{1}{2}p_i\varphi_i^2 - \frac{1}{2}\gamma_{i0}m_i^2 + \frac{1}{2}\left(\gamma_{i0} + \frac{1}{p_i}\right)l_i^2 \right).$$

(23)

By integrating both sides of (23) from 0 to T, we can lead to

$$\frac{1}{2}\sum_{i=1}^{n}\left[\int_0^T p_i\varphi_i^2\,dt + \int_0^T \gamma_{i0}m_i^2\,dt \right]$$
$$\le V_F(0) - V_F(T) + \frac{1}{2}\sum_{i=1}^{n}\int_0^T \left(\gamma_{i0} + \frac{1}{p_i}\right)l_i^2\,dt.$$

(24)

This inequality (24) is equivalent to inequality (19) in Theorem 1, since $V_1(T) > 0$. This completes the proof. $\square$

Based on Barbalat's lemma [32, 33], the robust performance of inequality (19) can be explained. If $l_i \in \mathcal{L}_2$, i.e., $\int_0^\infty l_i^2\,dt < \infty$, then $\varphi_i \in \mathcal{L}_2$ and $m_i \in \mathcal{L}_2$. This means that $\lim_{t\to\infty} \|\varphi_i(t)\| = 0$ and $\lim_{t\to\infty} \|m_i(t)\| = 0$. Moreover, even though $l_i \notin \mathcal{L}_2$, φ_i^2 is bounded by l_i^2. Hence, we can make the observation error arbitrarily small by setting the predetermined positive weighting constant $\gamma_{i0} + \frac{1}{p_i}$. Hence, we can conclude that $\hat{\Omega}_i$ can estimate Ω_i with arbitrarily small error.

Remark 2 Yoo et al. in [30] improved the performance of the conventional FDO proposed by Kim [28] by modifying the adaptive law for the parameter vector. However, the control methods considered the situation where the matched uncertainty only exists in the system. In this paper, we enable to overcome the mismatched uncertainty by using backstepping scheme with the FDO.

We can obtain the estimated value of the overall disturbance Ω_i in (5) by Theorem 1. This means that the value can be used to compensate the overall disturbance Ω_i. Now, using the estimated value, we propose a robust adaptive backstepping controller design

method to achieve the synchronization between the outputs of the drive system and response system. The design method is presented in the following theorem.

Theorem 2 *Consider the drive system (1) and the response system (3). Let us apply the coordinate transformation*

$$z_1 = x_1 - x_{d1},$$
$$z_i = x_i - x_{di} - \alpha_{i-1} \quad for\ i = 2, \ldots, n$$

(25)

with

$$\alpha_1 = \frac{1}{g_1}(-c_1z_1 - f_1 + f_{d1} - g_1x_{d2} - \beta_1 - \hat{\Omega}_1),$$

(26)

$$\alpha_i = \frac{1}{g_i}(-c_iz_i - f_i + f_{di} - g_ix_{d(i+1)} - g_{i-1}z_{i-1}$$
$$+ \dot{\alpha}_{i-1} - \beta_i - \hat{\Omega}_i), \quad i = 2, \ldots, n-1.$$

(27)

If the following control law is chosen as

$$u = \frac{1}{g_n}(-c_nz_n - f_n + f_{dn} - g_{n-1}z_{n-1} + \dot{\alpha}_{n-1}$$
$$- \beta_n - \hat{\Omega}_n)$$

(28)

where c_i is the positive constant,

$$\beta_i = k_iw_i(z_i),$$

(29)

$$\dot{k}_i = \rho_i\left[z_iw_i(z_i) - \sigma_i\left(k_i - k_i^0\right)\right],$$

(30)

$$w_i(z_i) = \tanh\left(\frac{z_i}{\delta_i}\right),$$

(31)

$$\hat{\Omega}_i = \theta_i^T\xi_i(x),$$

(32)

σ_i, k_i^0, and δ_i are positive constants for $i = 1, \ldots, n$, then the error z_1 between the outputs of the drive system (1) and the response system (3) converges to a small neighborhood of the origin.

Proof The backstepping design procedure consists of n steps. At each step ($1 \le i \le n-1$), a stabilizing function α_i with $\hat{\Omega}_i$ is developed with respect to Lyapunov function V_i and the control law u is obtained in the last step.

Step 1: From (1) and (3), the derivative of $z_1 = x_1 - x_{d1}$ is given by

$$\dot{z}_1 = f_1 + g_1 x_2 + \Omega_1 - f_{d1}$$
$$= f_1 - f_{d1} + g_1(z_2 + x_{d2} + \alpha_1) + \Omega_1. \qquad (33)$$

In order to design the stabilizing function α_1, choose the Lyapunov function candidate

$$V_1 = \frac{1}{2}z_1^2 + \frac{1}{2\rho_1}\tilde{k}_1^2 \qquad (34)$$

where $\tilde{k}_1 = k_1 - k_1^*$.

Differentiating (34) lead to

$$\dot{V}_1 = z_1 \big[f_1 - f_{d1} + g_1(z_2 + x_{d2} + \alpha_1) + \Omega_1 \big]$$
$$+ \frac{1}{\rho_1}\tilde{k}_1 \dot{k}_1. \qquad (35)$$

By applying the stabilizing function (26) and (29)–(31), we can have

$$\dot{V}_1 = z_1(-c_1 z_1 + g_1 z_2 - \beta_1 - \hat{\Omega}_1 + \Omega_1) + \frac{1}{\rho_1}\tilde{k}_1 \dot{k}_1$$
$$= -c_1 z_1^2 + g_1 z_1 z_2 - z_1 k_1 w_1(z_1) + z_1 \varepsilon_1$$
$$+ \tilde{k}_1 \big[z_1 w_1(z_1) - \sigma_1 (k_1 - k_1^0) \big]$$
$$= -c_1 z_1^2 + g_1 z_1 z_2 - z_1 k_1 w_1(z_1) + z_1 \varepsilon_1$$
$$+ \tilde{k}_1 z_1 w_1(z_1) - \sigma_1 \tilde{k}_1 (k_1 - k_1^0)$$
$$= -c_1 z_1^2 + g_1 z_1 z_2 - k_1^* z_1 w_1(z_1) + z_1 \varepsilon_1$$
$$- \sigma_1 \tilde{k}_1 (k_1 - k_1^0). \qquad (36)$$

Using the following equation

$$-\sigma_1 (k_1 - k_1^*)(k_1 - k_1^0)$$
$$= -\frac{1}{2}\sigma_1 (k_1 - k_1^*)^2 - \frac{1}{2}\sigma_1 (k_1 - k_1^0)^2$$
$$+ \frac{1}{2}\sigma_1 (k_1^* - k_1^0)^2, \qquad (37)$$

(36) can be represented as

$$\dot{V}_1 = -c_1 z_1^2 + g_1 z_1 z_2 - k_1^* z_1 w_1(z_1) + z_1 \varepsilon_1 - \frac{1}{2}\sigma_1 \tilde{k}_1^2$$
$$- \frac{1}{2}\sigma_1 (k_1 - k_1^0)^2 + \frac{1}{2}\sigma_1 (k_1^* - k_1^0)^2$$
$$\le -c_1 z_1^2 + g_1 z_1 z_2 - k_1^* z_1 \tanh\left(\frac{z_1}{\delta_1}\right) + z_1 \varepsilon_1$$
$$- \frac{1}{2}\sigma_1 \tilde{k}_1^2 + \frac{1}{2}\sigma_1 (k_1^* - k_1^0)^2. \qquad (38)$$

Based on the universal approximation theorem [31], choose k_1^* such that $\bar{\varepsilon}_1 \le k_1^*$ where $\bar{\varepsilon}_1$ is the upper bound of $\varepsilon_1 = \Omega_1 - \hat{\Omega}_1$. Then we obtain

$$\dot{V}_1 \le -c_1 z_1^2 + g_1 z_1 z_2 + k_1^* \left[|z_1| - z_1 \tanh\left(\frac{z_1}{\delta_1}\right) \right]$$
$$- \frac{1}{2}\sigma_1 \tilde{k}_1^2 + \frac{1}{2}\sigma_1 (k_1^* - k_1^0)^2. \qquad (39)$$

Now, we introduce the following claim to progress this proof.

Claim *The following inequality holds for any $\delta > 0$ and $y \in \mathbb{R}$:*

$$0 \le |y| - y \tanh\left(\frac{y}{\delta}\right) \le \eta \delta \qquad (40)$$

where η is a constant that satisfies $\eta = e^{-(\eta+1)}$, i.e., $\eta = 0.2785$.

The proof of this claim follows after straightforward algebraic manipulation, and is therefore omitted. Then, by application of the claim, we have

$$\dot{V}_1 \le -c_1 z_1^2 + g_1 z_1 z_2 + \frac{1}{2}k_1^* \delta_1$$
$$- \frac{1}{2}\sigma_1 \tilde{k}_1^2 + \frac{1}{2}\sigma_1 (k_1^* - k_1^0)^2$$
$$\le -r_1 V_1 + \lambda_1 + g_1 z_1 z_2 \qquad (41)$$

where $r_1 = \min\{2c_1, \sigma_1 \rho_1\} > 0$ and $\lambda_1 = \frac{1}{2}k_1^* \delta_1 + \frac{1}{2}\sigma_1(k_1^* - k_1^0)^2$. It should be noted that the coupling term $g_1 z_1 z_2$ is cancelled in the next step.

Step i ($2 \le i \le n-1$): The derivative of $z_i = x_i - x_{di} - \alpha_{i-1}$ is written by

$$\dot{z}_i = f_i - f_{ri} + g_i(z_{i+1} + x_{d(i+1)} + \alpha_i) + \Omega_i - \dot{\alpha}_{i-1}. \qquad (42)$$

Let us choose the Lyapunov function candidate to obtain the stabilizing function α_i

$$V_i = V_{i-1} + \frac{1}{2}z_i^2 + \frac{1}{2\rho_i}\tilde{k}_i^2 \qquad (43)$$

where $\tilde{k}_i = k_i - k_i^*$.

By using the stabilizing function (27) and (29)–(31), we obtain the derivative of V_i as follows:

$$\dot{V}_i = \dot{V}_{i-1} + z_i \dot{z}_i + \frac{1}{\rho_i} \tilde{k}_i \dot{k}_i$$

$$\leq -r_{i-1} V_{i-1} + \lambda_{i-1} + g_{i-1} z_{i-1} z_i$$

$$+ z_i \big[f_i - f_{ri} + g_i (z_{i+1} + x_{d(i+1)} + \alpha_i) \big]$$

$$+ \Omega_i - \dot{\alpha}_{i-1} \big] + \frac{1}{\rho_i} \tilde{k}_i \dot{k}_i$$

$$= -r_{i-1} V_{i-1} + \lambda_{i-1} - c_i z_i^2 + g_i z_i z_{i+1}$$

$$- k_i^* z_i w_i(z_i) + z_i \varepsilon_i - \sigma_i \tilde{k}_i (k_i - k_i^0). \tag{44}$$

Using the following equation

$$-\sigma_i (k_i - k_i^*)(k_i - k_i^0)$$

$$= -\frac{1}{2} \sigma_i (k_i - k_i^*)^2 - \frac{1}{2} \sigma_i (k_i - k_i^0)^2$$

$$+ \frac{1}{2} \sigma_i (k_i^* - k_i^0)^2 \tag{45}$$

and choosing k_i^* such that $\bar{\varepsilon}_i \leq k_i^*$ where $|\varepsilon_i| = |\Omega_i - \hat{\Omega}_i| \leq \bar{\varepsilon}_i$, the inequality (44) can be represented as

$$\dot{V}_i \leq -r_{i-1} V_{i-1} + \lambda_{i-1} - c_i z_i^2 + g_i z_i z_{i+1}$$

$$+ k_i^* \left[|z_i| - z_i \tanh\left(\frac{z_i}{\delta_i}\right) \right] - \frac{1}{2} \sigma_i \tilde{k}_i^2$$

$$+ \frac{1}{2} \sigma_i (k_i^* - k_i^0)^2. \tag{46}$$

Then, with the claim mentioned above, we have

$$\dot{V}_i \leq -r_{i-1} V_{i-1} + \lambda_{i-1} - c_i z_i^2 + g_i z_i z_{i+1}$$

$$+ \frac{1}{2} k_i^* \delta_i - \frac{1}{2} \sigma_i \tilde{k}_i^2 + \frac{1}{2} \sigma_i (k_i^* - k_i^0)^2$$

$$\leq -r_i V_i + \lambda_i + g_i z_i z_{i+1} \tag{47}$$

where $r_i = \min\{2c_i, \sigma_i \rho_i \text{ for } i = 1, \ldots, i\} > 0$ and $\lambda_i = \sum_{j=1}^{i} [\frac{1}{2} k_j^* \delta_j + \frac{1}{2} \sigma_j (k_j^* - k_j^0)^2]$.

Step n: In this final step, we derive the control law $u(t)$ to achieve the synchronization. The derivative of $z_n = x_n - x_{dn} - \alpha_{n-1}$ is written by

$$\dot{z}_n = f_n - f_{dn} + g_n u + \Omega_n - \dot{\alpha}_{n-1}. \tag{48}$$

In the similar way with the previous steps, the Lyapunov function candidate is chosen as

$$V_n = V_{n-1} + \frac{1}{2} z_n^2 + \frac{1}{2\rho_n} \tilde{k}_n^2 \tag{49}$$

where $\tilde{k}_n = k_n - k_n^*$ and its derivative is obtained with the control input (28) and (28)–(31) as follows:

$$\dot{V}_n = \dot{V}_{n-1} + z_n \dot{z}_n + \frac{1}{\rho_n} \tilde{k}_n \dot{k}_n$$

$$\leq -r_{n-1} V_{n-1} + \lambda_{n-1} + g_{n-1} z_{n-1} z_n$$

$$+ z_n [f_n - f_{dn} + g_n u + \Omega_n - \dot{\alpha}_{n-1}] + \frac{1}{\rho_n} \tilde{k}_n \dot{k}_n$$

$$= -r_{n-1} V_{n-1} + \lambda_{n-1} - c_n z_n^2 - k_n^* z_n w_n(z_n)$$

$$+ z_n \varepsilon_n - \sigma_n \tilde{k}_n (k_n - k_n^0). \tag{50}$$

Using the following equation

$$-\sigma_n (k_n - k_n^*)(k_n - k_n^0)$$

$$= -\frac{1}{2} \sigma_n (k_n - k_n^*)^2 - \frac{1}{2} \sigma_n (k_n - k_n^0)^2$$

$$+ \frac{1}{2} \sigma_n (k_n^* - k_n^0)^2, \tag{51}$$

choosing k_n^* such that $|\varepsilon_n| = |\Omega_n - \hat{\Omega}_n| \leq \bar{\varepsilon}_n \leq k_n^*$, and applying the claim yield

$$\dot{V}_n \leq -r_{n-1} V_{n-1} + \lambda_{n-1} - c_n z_n^2 + \frac{1}{2} k_n^* \delta_n$$

$$- \frac{1}{2} \sigma_n \tilde{k}_n^2 + \frac{1}{2} \sigma_n (k_n^* - k_n^0)^2$$

$$\leq -r_n V_n + \lambda_n \tag{52}$$

where $r_n = \min\{2c_n, \sigma_n \rho_n \text{ for } i = 1, \ldots, n\} > 0$ and $\lambda_n = \sum_{j=1}^{n} [\frac{1}{2} k_j^* \delta_j + \frac{1}{2} \sigma_j (k_j^* - k_j^0)^2]$.

It is obvious that the inequality (52) is equivalent to

$$0 \leq V_n(t) \leq \frac{\lambda_n}{r_n} + \left[V_n(0) - \frac{\lambda_n}{r_n} \right] e^{-r_n t}. \tag{53}$$

This means that z_i and k_i are globally uniformly ultimately bounded (UUB). Therefore, we can conclude that the output of the response system (3) converges to one of the drive system (1) with a small error, that is, the synchronization is achieved by the proposed control law. Moreover, if the design constants δ_i, σ_i, and ρ_i, are appropriately chosen, then it is possible to make the error as small as desired. This completes the proof. □

Remark 3 It is clear that the stabilizing function α_i for $i = 1, \ldots, n - 1$ are differentiable because all of

terms of α_i are differentiable. Therefore, the differential terms included in α_i for $i = 2, \ldots, n - 1$ and u can be implemented.

Remark 4 The hyperbolic function $\tanh(\cdot)$ is often used as an approximation of the discontinuous function $\text{sgn}(\cdot)$. In fact, as δ_i approaches zero, $w_i(z_i)$ approaches the sign function. This replacement makes the control law continuous without the loss of the robustness.

4 Numerical examples

In order to verify the effectiveness of the proposed method, two examples are presented in this section. In the first example, the drive system and the response system are selected as Bonhoeffer–van der Pol (BVP) oscillator and Duffing oscillator, respectively. The second example presents the chaos synchronization between Lorenz system and Duffing oscillator. Namely, the drive and response systems in each example have different dynamics [21, 30]. Because, in many practical cases, we cannot easily guarantee that response system works by the same dynamics with drive system. Moreover, by replacing the known function f_i with f_{di}, we can easily extend our proposed method to the situation where the systems are identical. The simulations are conducted in Simulink (MATLAB) using a fixed step fourth- order Runge–Kutta solver with a sample period of $T_s = 0.005$ s.

Example 1 Consider Bonhoeffer–van der Pol (BVP) oscillator as the drive system

$$\dot{x}_{d1}(t) = x_{d2}(t)$$
$$\dot{x}_{d2}(t) = h\big(x_{d1}(t)\big) - 0.5x_{d2}(t) - 0.05x_{d2}^3(t) \qquad (54)$$
$$\qquad + \sin\big(x_{d1}(t)\big) + 35.5\sin(2t + 6)\sin\big(x_{d1}(t)\big)$$

where

$$h\big(x_{d1}(t)\big) = -10^2 \frac{(1 - \cos x_{d1})^2}{\sin^3 x_{d1}}$$
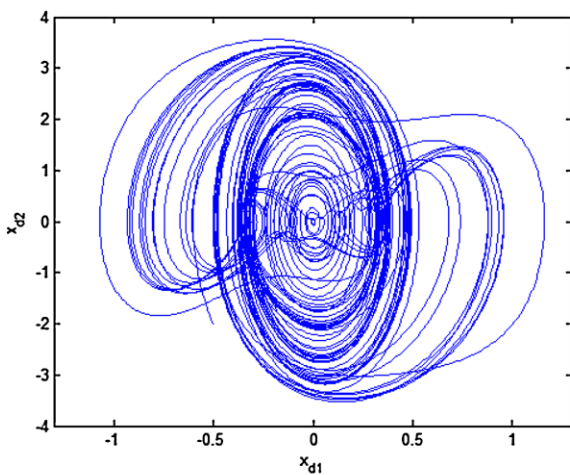
and Duffing oscillator as the response system

$$\dot{x}_1(t) = f_1 + \Delta f_1 + (g_1 + \Delta g_1)x_2(t) + d_1(t)$$
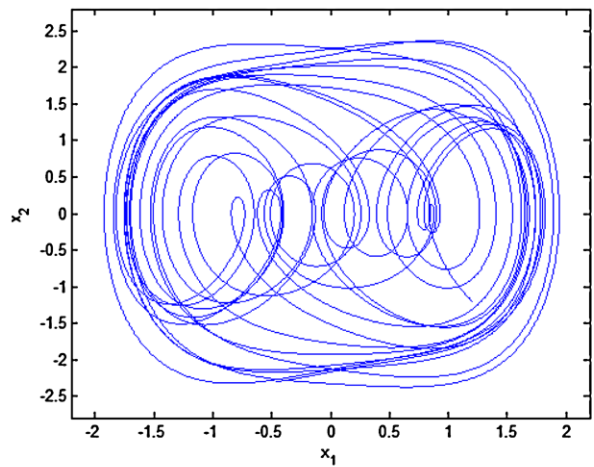$$\dot{x}_2(t) = f_2 + \Delta f_2 + (g_2 + \Delta g_2)u(t) + d_2(t) \qquad (55)$$

where the nominal system, uncertainty, and disturbance are $f_1 = 0$, $g_1 = 1$, $f_2 = -0.4x_2(t) + 1.1x_1(t) - x_1^3(t) + 1.8\cos(1.8t)$, $g_2 = 1$, $\Delta f_1 = 0$, $\Delta g_1 = 0.1$, $\Delta f_2 = -0.05x_1^3(t) + 0.2x_1(t)$, $\Delta g_2 = 0$, $d_1(t) = 0.1\cos(t)$, $d_2(t) = 0.2\sin(t)$.

Then the overall disturbance is defined as $\Omega_1 = \Delta f_1 + \Delta g_1 + d_1(t) = 0.1x_2(t) + 0.1\cos(t)$ and $\Omega_2 = \Delta f_2 + \Delta g_2 + d_2(t) = -0.05x_1^3(t) + 0.2x_1(t) + 0.2\sin(t)$.

To construct the FDO, we use the input vector of the FLS as $X = [x_1, x_2]^T$ where $x_l \in [-2, 2]$, $l = 1, 2$. We choose 3 centers of the Gaussian membership function $\mu_l = \exp[-(x_l - c_{ml})^2/\sigma_\mu^2]$ where $\sigma_\mu = 1.2$ for each

(a) Parametrically-excited gyroscope   (b) Duffing oscillator
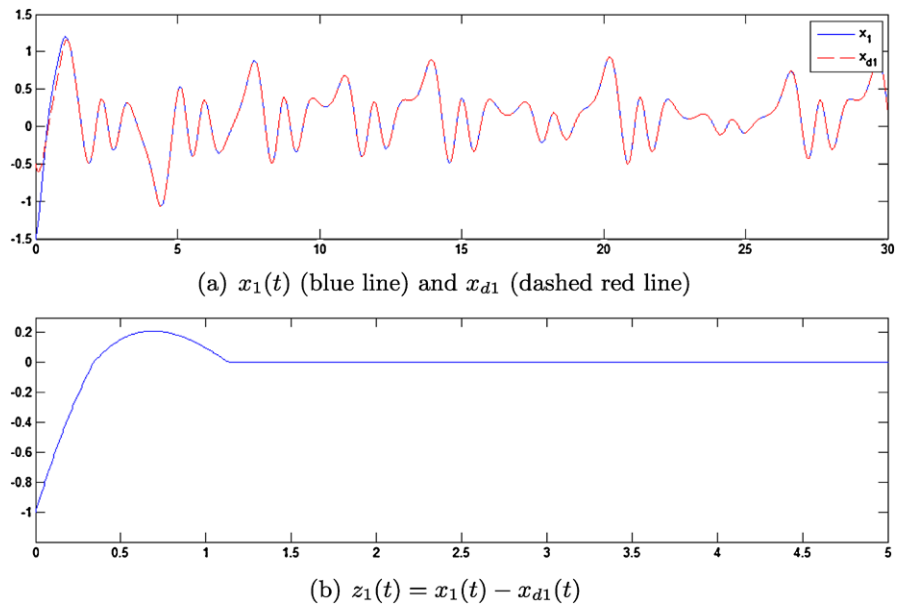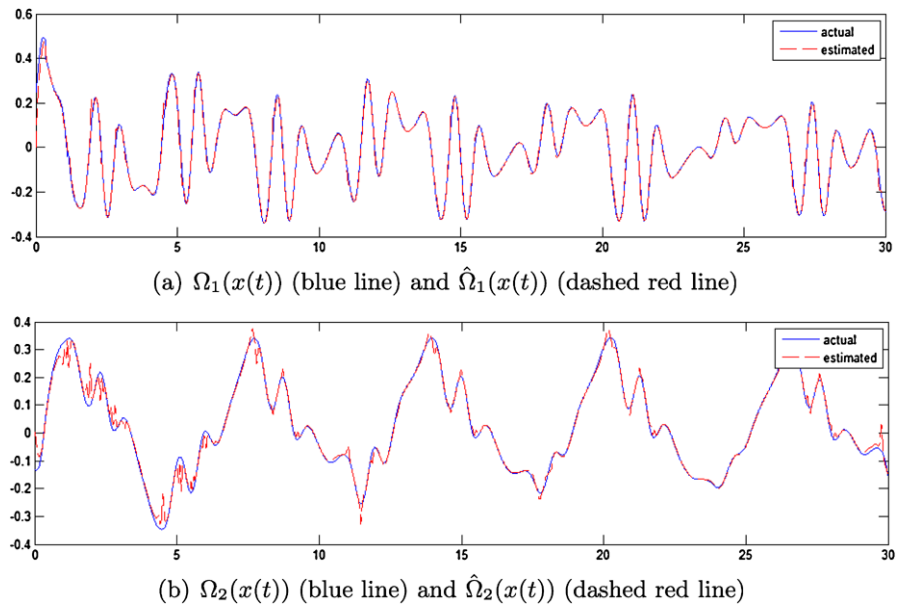
Fig. 1 The drive system (**a**) and response system (**b**) in Example 1

Fig. 2 (Color online) The output trajectories of the drive and response system (**a**) and the synchronization error (**b**) in Example 1

(a) $x_1(t)$ (blue line) and x_{d1} (dashed red line)

(b) $z_1(t) = x_1(t) - x_{d1}(t)$

Fig. 3 (Color online) The actual overall disturbance $\Omega(x(t))$ and FDO $\hat{\Omega}(x(t))$ in Example 1

(a) $\Omega_1(x(t))$ (blue line) and $\hat{\Omega}_1(x(t))$ (dashed red line)

(b) $\Omega_2(x(t))$ (blue line) and $\hat{\Omega}_2(x(t))$ (dashed red line)

FLS input, i.e. $C_m = [c_{m1}, c_{m2}]$ for $m = 1, 2, \ldots, 9$ with uniform distance. The used parameters are $c_1 = c_2 = 1$, $\gamma_0 = \gamma_1 = 10$, $p_i = 10$, $\delta_i = 0.01$, $\rho_i = 3$, $\sigma_i = 1$, $k_i^0 = 1$ for $i = 1, 2$. Initial values are chosen as $x_{d0} = [-0.5; -2]$, $x_0 = [-1.5; 1.5]$, $\hat{x}_0 = [0; 0]$, $\theta_{i0} = 0$, $k_{i0} = 0$.

The drive system (54) and response system (55) are depicted in Fig. 1(a) and (b), respectively. Figure 2(a) shows the output trajectories of the drive system and response system and the synchronization error

is shown in Fig. 2(b). From the figure, we can easily see that the synchronization is achieved by the proposed method. In Fig. 3, the actual overall disturbance existing in the response system and the estimated values obtained by FDO are presented. It is shown that the FDO can estimate the overall disturbance well and the proposed method synchronizes the chaotic systems.

Example 2 In this example, to show the effectiveness of FDO, we compare the simulation results of two sit-

uations; when the FDO is used and not. The proposed method can be applied for the synchronization, even though the dimension of the drive system and response system is different. Let us choose Lorenz system as the drive system, which is a three-dimensional system

$$\dot{x}_{d1}(t) = 10\big(x_{d2}(t) - x_{d1}(t)\big)$$
$$\dot{x}_{d2}(t) = x_{d1}(t)\big(28 - x_{d3}(t)\big) - x_{d2}(t) \tag{56}$$
$$\dot{x}_{d3}(t) = x_{d1}(t)x_{d2}(t) - \frac{8}{3}x_{d3}(t)$$

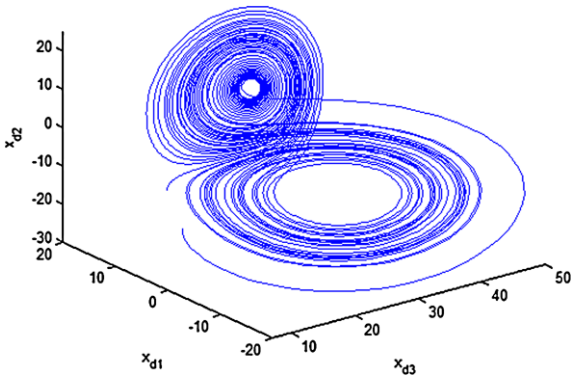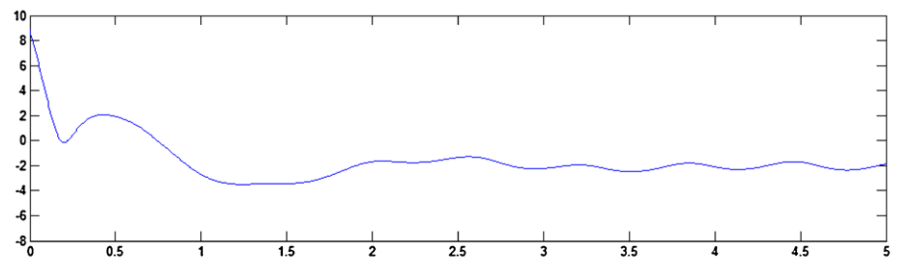and the response system is chosen as the same system, Duffing oscillator, with Example 1.

Fig. 4 Lorenz system as the drive system in Example 2

We use the input vector of the FLS as $X = [x_1, x_2]^T$ where $x_1 \in [-20, 20]$ and $x_2 \in [-50, 50]$. We choose 5 centers of the Gaussian membership function $\mu_l = \exp[-(x_l - c_{ml})^2/\sigma_{\mu l}^2]$ where $\sigma_{\mu 1} = 6$ and $\sigma_{\mu 2} = 15$ for each FLS input, i.e., $C_m = [c_{m1}, c_{m2}]$ for $m = 1, 2, \ldots, 25$ with uniform distance. The used parameters are $c_i = 1$, $\gamma_0 = \gamma_1 = 20$, $p_i = 20$, $\delta_i = 0.01$, $\rho_i = 1$, $\sigma_i = 1$, $k_i^0 = 1$ for $i = 1, 2$. Initial values are chosen as $x_{d0} = [-10; -5; 1]$, $x_0 = [-1.2; 1.2]$, $\hat{x}_0 = [0; 0]$, $\theta_{i0} = 0$, $k_{i0} = 0$.

The Lorenz system (56) is depicted in Fig. 4. The effectiveness of the FDO is shown by comparing two cases: (1) when the FDO is not applied (Fig. 5) and (2) when the proposed method including the FDO is applied (Fig. 6). Both figures show the output trajectories and the synchronization error for each case. From these figures, we can easily see that the synchronization performance is better when the FDO is used. This is why the proposed method effectively suppress the overall disturbance by using the estimated value obtained by the FDO. The estimated values are shown with the actual overall disturbance existing in the response system in Fig. 7. From these results, we conclude that the proposed method successfully synchronizes although uncertainty and disturbance exist in the system.

Fig. 5 (Color online) The output trajectories of the drive and response system (**a**) and the synchronization error (**b**) when the FDO is not applied, $\hat{\Omega}_i = 0$ (Example 2)
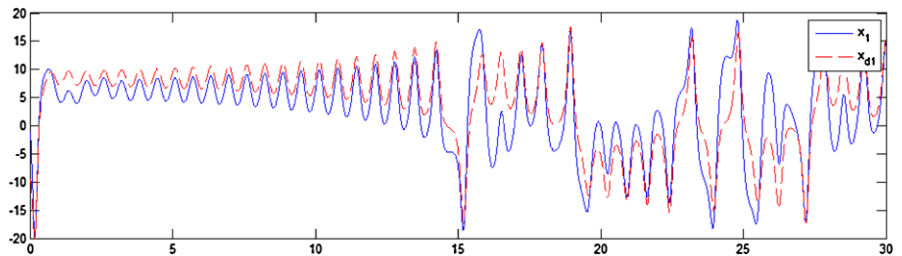
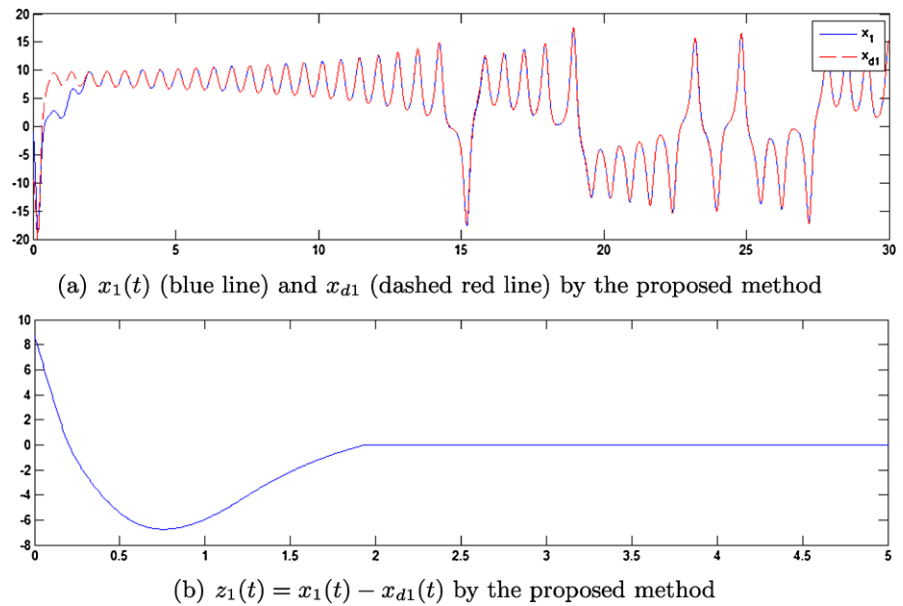(a) $x_1(t)$ (blue line) and x_{d1} (dashed red line) when the FDO is not applied

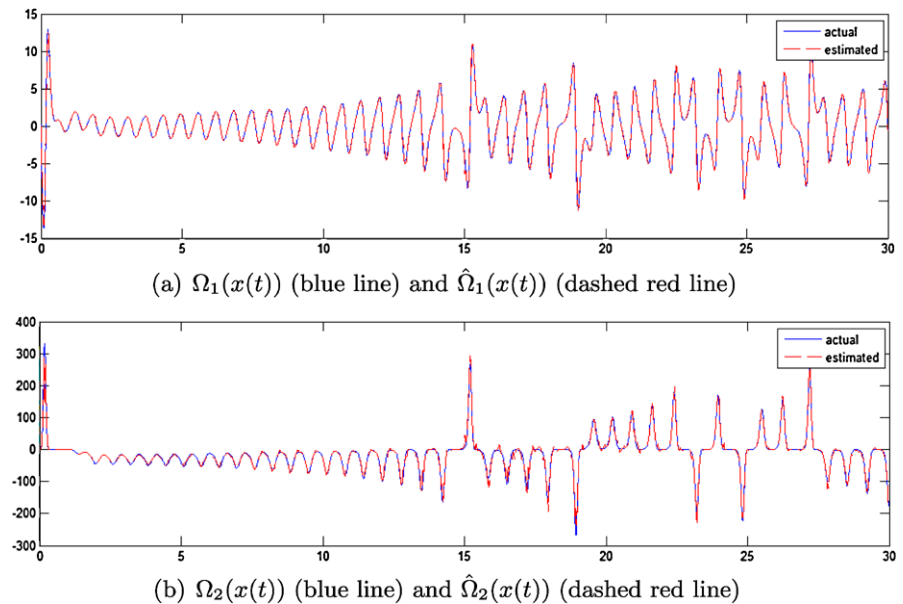(b) $z_1(t) = x_1(t) - x_{d1}(t)$ when the FDO is not applied

Fig. 6 (Color online) The output trajectories of the drive and response system (**a**) and the synchronization error (**b**) by the proposed method in Example 2

(a) $x_1(t)$ (blue line) and x_{d1} (dashed red line) by the proposed method

(b) $z_1(t) = x_1(t) - x_{d1}(t)$ by the proposed method

Fig. 7 (Color online) The actual overall disturbance $\Omega(x(t))$ and FDO $\hat{\Omega}(x(t))$ in Example 2

(a) $\Omega_1(x(t))$ (blue line) and $\hat{\Omega}_1(x(t))$ (dashed red line)

(b) $\Omega_2(x(t))$ (blue line) and $\hat{\Omega}_2(x(t))$ (dashed red line)

5 Conclusions

We proposed a robust adaptive backstepping synchronization method using FDO for chaotic systems with mismatched uncertainties and disturbances. The unknown factors were estimated by the FDO without requiring any prior information about them. The mismatched uncertainties in the response system were overcome by the proposed adaptive backstepping design procedure with the estimated values. The adaptive laws used in the procedure were obtained based on Lyapunov stability theorem. Finally, the proposed method guaranteed the global chaotic synchronization in the sense of UUB. Numerical examples showed the effectiveness of the proposed method.

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