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A method for Lyapunov spectrum estimation using cloned dynamics and its application to the discontinuously-excited FitzHugh–Nagumo model

Diogo C. Soriano · Filipe I. Fazanaro · Ricardo Suyama · José Raimundo de Oliveira · Romis Attux · Marconi K. Madrid

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Abstract This work presents a new method to calculate the Lyapunov spectrum of dynamical systems based on the time evolution of initially small disturbed copies ("clones") of the motion equations. In this approach, it is not necessary to construct the tangent space associated with the time evolution of linearized versions of motion equations, being the Lyapunov exponents directly estimated in terms of the rate of convergence or divergence of these disturbed clones with respect to the fiducial trajectory, there being periodic correction via the Gram–Schmidt Reorthonormalization procedure. The proposed method offers the possibility of partial estimation of the Lyapunov spectrum and can also be applied to nonsmooth dynamics, since the linearization procedure is no longer required. The idea is tested for representative continuous- and discrete-time dynamical systems and validated by means of comparison with the classical method to perform this calculation. To illustrate its applicability in the nonsmooth context, the largest Lyapunov exponent of the FitzHugh–Nagumo neuronal model under discontinuous periodic excitation is calculated taking the amplitude of stimulation as control parameter. This analysis reveals some complex behaviours for this simple neuronal model, which motivates relevant discussions about the possible role of chaos in the cognitive process.

Keywords Chaos · Lyapunov spectrum estimation · FitzHugh–Nagumo neuronal model · Discontinuous excitation · Nonsmooth models

D.C. Soriano (✉) · R. Attux
Laboratory of Signal Processing for Communications (DSPCom), Department of Computer Engineering and Industrial Automation (DCA), School of Electrical and Computer Engineering (FEEC), UNICAMP, Av. Albert Einstein, 400, 13083-852, Campinas, SP, Brazil
e-mail: soriano@dca.fee.unicamp.br

F.I. Fazanaro · J.R. de Oliveira
Modular Robotic Systems Laboratory (LSMR), DCA, FEEC, UNICAMP, Campinas, SP, Brazil

R. Suyama
Engineering, Modeling and Applied Social Sciences Center, UFABC, Santo André, Brazil

M.K. Madrid
LSMR, Department of Energy Control and Systems (DSCE), FEEC, UNICAMP, Campinas, SP, Brazil

1 Introduction

Dynamical systems can be understood in terms of state mappings usually described by a set of differential equations. When these mappings are defined by nonlinear functions of the state variables, in general, a rich scenario of oscillatory behaviors can be achieved, which includes convergence to stationary points (fixed points), periodic solutions (limit-cycles), quasiperiodicity, and chaos [4, 19, 29, 30].

The analysis of the motion equations in the phase space is closely related to the system stability, being the average growth rate of initially small deviations a manner to quantify it, an idea that is strongly supported by Lyapunov's seminal work (see [29] for some interesting historical notes). Thus, it is possible to verify how small perturbations evolve under the system motion in the phase space by studying the linearized versions of the state equations in successive time steps, which allows conclusions to be drawn about the nature of the system behavior [19].

Lyapunov's framework can be illustrated by considering a discrete-time system given by the motion equation $\mathbf{x}(n+1)=\mathbf{F}(\mathbf{x}(n))$, where n is the discrete time index. Considering δ_{x0} a small magnitude value, $\mathbf{e}_1$ is a basis vector specifying a direction, for instance, $\mathbf{e}_1=[1;0;\dots;0]_{k\times 1}$ for a k-dimensional system. Then, if a small perturbation of the form $\boldsymbol{\Delta}(n)=\delta_{x0}\mathbf{e}_1$ is applied to a specific system state variable, its dynamical evolution can be described as [2, 4, 10]:

$$\begin{aligned}\mathbf{x}(n+1)+\boldsymbol{\Delta}(n+1) &= \mathbf{F}\big(\mathbf{x}(n)+\boldsymbol{\Delta}(n)\big)\\ &\approx \mathbf{J}\big(\mathbf{x}(n)\big)\cdot\boldsymbol{\Delta}(n)+\mathbf{F}\big(\mathbf{x}(n)\big),\end{aligned} \tag{1}$$

where $\mathbf{J}(\mathbf{x}(n))$ is the Jacobian matrix of $\mathbf{F}(\mathbf{x}(n))$, with elements given by $\partial F_i(\mathbf{x}(n))/\partial x_j$, which leads to:

$$\begin{aligned}\boldsymbol{\Delta}(n+1) &\approx \mathbf{J}\big(\mathbf{x}(n)\big)\cdot\boldsymbol{\Delta}(n)\\ &= \mathbf{J}\big(\mathbf{x}(n)\big)\cdot\delta_{x0}\cdot\mathbf{e}_1.\end{aligned} \tag{2}$$

In this case, dynamical stability can be analyzed in terms of the evolution of the perturbation vector $\boldsymbol{\Delta}(n+1)$, which can be obtained L steps ahead by applying the chain rule to the Jacobian matrix [2, 10]:

$$\begin{aligned}\boldsymbol{\Delta}(n+L) &\approx \mathbf{J}\big(\mathbf{x}(n+L-1)\big)\cdot\mathbf{J}\big(\mathbf{x}(n+L-2)\big)\cdots\\ &\quad\cdot\mathbf{J}\big(\mathbf{x}(n)\big)\cdot\boldsymbol{\Delta}(n)\\ &= \mathbf{J}^L\big(\mathbf{x}(n)\big)\cdot\boldsymbol{\Delta}(n)\\ &= \mathbf{J}^L\big(\mathbf{x}(n)\big)\cdot\delta_{x0}\cdot\mathbf{e}_1,\end{aligned} \tag{3}$$

where $\mathbf{J}^L(\mathbf{x}(n))$ is the composition of L Jacobian matrices during the time evolution. In addition to that, (2)–(3) are satisfied by the exponential function described in (4):

$$\big\|\boldsymbol{\Delta}(n+L)\big\| = \big\|\boldsymbol{\Delta}(n)\big\|\cdot\exp(\lambda L) \tag{4}$$

being λ the average largest growing rate (in the L interval) of the dynamics, also called largest Lyapunov exponent. Using (3)–(4), it is possible to obtain λ for the whole attractor in the state space (a rigorous mathematical proof can be found in [6]):

$$\begin{aligned}\lambda &= \lim_{\delta_{x0}\to 0}\lim_{L\to\infty}\frac{1}{L}\ln\left(\left\|\frac{\boldsymbol{\Delta}(n+L)}{\delta_{x0}}\right\|\right)\\ &= \lim_{L\to\infty}\frac{1}{L}\ln\big(\big\|\mathbf{J}^L\big(\mathbf{x}(n)\big)\cdot\mathbf{e}_1\big\|\big)\end{aligned} \tag{5}$$

and a theorem stated by Oseledec in [18] ensures that this limit exists for almost all initial conditions in the same basin of attraction.

When small perturbations are initially applied to all orthogonal directions of the phase space, it is possible to estimate all Lyapunov exponents, each being related to the average growth rate in a given direction. The set of Lyapunov exponents for all directions is called Lyapunov spectrum of the dynamical system, and once the largest Lyapunov exponent is known, it is possible to characterize the system dynamics. In particular, if $\lambda>0$ (being the solution within a compact space), the dynamics will present, at least, one unstable direction in the phase space, which implies chaotic behavior [4, 19]. In this case, the system will exhibit interesting oscillatory characteristics such as aperiodicity and sensitivity to initial conditions [2, 4, 8, 10, 19, 29, 30], which can be extremely relevant for the physical process under study.

In this context, Lyapunov exponents are an invariant measure of the dynamics, i.e., a measure independent of a specific orbit or initial condition in the same basin of attraction. Invariances are of fundamental importance to characterize chaotic behavior [10, 15], and, in particular, the focused one provides a measure of the predictability of the system, since, in practice, it is impossible to infer its initial state with infinite precision [4, 8, 23, 29, 30].

Given the state equations, there are fundamentally two methods to estimate the Lyapunov spectrum. The first is to use the multiplicative ergodic theorem stated by Oseledec [18] to form the Oseledec matrix and extract its eigenvalues by applying a recursive QR decomposition. This avoids its ill-conditioned behavior when L is large [1, 2, 10, 19]. This ill-conditioned behavior refers to the alignment of the directions defined by the linearized system in its most expansive direction, which can cause the numerical collapse of composition of the Jacobian matrices. The second method was introduced independently in [7] and in [24], being revisited in [30]. In these works, the exponents are obtained by establishing a reference trajectory (the fiducial trajectory, which is the solution of the dynamical system) and constructing the tangent space associated with the dynamics. The main axes of the tangent space are related to a set of variational equations, which govern the time evolution of the linearized versions of state equations. The exponents are then obtained by estimating the average divergence rate provided by the application of this tangent map to an orthogonal basis anchored to the fiducial trajectory. In this case, the collapse of the tangent space after long term evolution can be avoided by applying the Gram–Schmidt Reorthonormalization (GSR) procedure to the vectors obtained after tangent map application [19, 22, 30]. There are several modifications of these two methods intending faster and more robust calculations (see [22] for a comparison).

The present work proposes an alternative method for estimating the Lyapunov spectrum which overcomes certain limitations in the usual procedures. The main idea bears some resemblance with an early work by Bennettin and coauthors in [6] where the largest exponent is obtained by quantifying the expanding or the contracting behavior of a difference state vector built from the original system and a copy initially disturbed by a small value. This approach is analogous to the one developed in (3)–(5), estimating the ratio $\|\boldsymbol{\Delta}(n+L)/\delta_{x0}\|$ instead of determining the system Jacobian via the solution of the variational equations. The present paper extends the ideas developed in [6] by calculating the Lyapunov spectrum using disturbed copies (called "clones") of the original dynamics for each direction associated with its respective exponents, avoiding the collapse of all clones in the most expansive direction by applying the GSR procedure, as is done in the classical tangent map approach introduced in [7, 24, 30]. This work does not aim to present a rigorous mathematical proof as a generalization of [6] to all directions of phase space, but to offer a consistent and practical algorithm to obtain the desired spectra based on the fundamental concept of evolution of small perturbations combined to classical numerical corrections. This new strategy to perform the calculation can provide relevant advantages in specific cases (for instance, nonsmooth and hyperchaotic dynamical systems), and it is carefully tested and compared here to the tangent map approach for representative continuous- and discrete-time models.

The method offers the possibility of partial estimation of the Lyapunov spectrum, which allows the integration (or iteration) of a smaller number of differential (or difference) equations to obtain, for instance, only the positive exponents of the dynamics, which may be interesting for hyperchaotic models. In addition to that, as the proposal does not require linearization of the state equations, it is suitable for applications in nonsmooth dynamical systems. This applicability is illustrated with the aid of the FitzHugh–Nagumo neuronal model [12] excited with rectangular pulses, a scenario of difficult theoretical treatment employing usual methods, but frequently adopted in practice [9]. Rigorously, discontinuous rectangular pulses do not exist in experimental procedures, but they are useful as approximations of real-world stimuli. In this context, the application of the developed method to neuronal models seems to be specially attractive, as there are evidences relating chaotic behaviour to information transmission, coding and storage (memory) in biological systems [16].

This work has the following organization: in Sect. 2, a brief review of the classical procedure to compute Lyapunov exponents using tangent maps is presented, followed by the introduction of the cloned dynamics method. In Sect. 3, the proposed method is tested for representative discrete- and continuous-time chaotic systems. Furthermore, the cloned dynamics approach is then applied to analyze the oscillatory behavior of the FitzHugh–Nagumo neuronal model under periodic discontinuous stimulation. Finally, in Sect. 4, some discussions and comments about the contributions and perspectives are exposed.

2 Methods

2.1 The tangent map (TanMap) approach

Given an n-dimensional dynamical system $\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}, t)$ with initial condition $\mathbf{x}_0$, the first step to evaluate the Lyapunov exponents is to establish n orthogonal vectors initially defined as

$$\{\boldsymbol{\delta}_{1x}, \boldsymbol{\delta}_{2x}, \ldots, \boldsymbol{\delta}_{nx}\} = \{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\} = \mathbf{I}_n, \tag{6}$$

with $\mathbf{I}_n$ the n-dimensional identity matrix, anchored on the fiducial trajectory. These vectors will be transformed by successive applications of the tangent map associated with the motion equations. The principal axes of the tangent map are determined by the variational equations, which rule the time evolution of linearized versions of the state equations, analytically described by

$$\dot{\boldsymbol{\Phi}}(\mathbf{x},t) = \mathbf{J}(\mathbf{x},t)\cdot\boldsymbol{\Phi}(\mathbf{x},t), \tag{7}$$

where $\mathbf{J}(\mathbf{x},t)$ is the Jacobian of $\mathbf{F}(\mathbf{x},t)$, and its elements are given by

$$J_{ij}(\mathbf{x},t) = \frac{\partial F_i(\mathbf{x},t)}{\partial x_j(t)}. \tag{8}$$

The divergence rate is then evaluated by integrating the whole system (original motion and the variational equations) for an interval T starting from $\mathbf{x}_0$ with $\boldsymbol{\Phi}(\mathbf{x}_0) = \mathbf{I}_n$. After this process, is possible to update the vectors anchored in the fiducial trajectory transformed by the tangent map, which for the most expansive direction (largest Lyapunov exponent), is given by $\boldsymbol{\delta}_{1x}^{(1)} = \boldsymbol{\Phi}(\mathbf{x},T)\cdot\mathbf{u}_1^{(0)}$, being $\mathbf{u}_1^{(0)} = \boldsymbol{\delta}_{1x}^{(0)}/\|\boldsymbol{\delta}_{1x}^{(0)}\|$ (where the superscript denotes the current iteration). Repeating the integration and normalization procedure K times (for K large enough to take into account the entire attractor behavior), the largest Lyapunov exponent is given by [19, 30]:

$$\lambda_1 = \lim_{K\to\infty}\frac{1}{KT}\cdot\sum_{k=1}^{K}\ln\|\boldsymbol{\delta}_{1x}^{(k)}\|. \tag{9}$$

Since the system continuously changes its orientation, it is impossible to define a specific axis of the phase space as either expansive or contractive. Moreover, the vectors $\boldsymbol{\delta}_{1x}, \boldsymbol{\delta}_{2x}, \ldots, \boldsymbol{\delta}_{nx}$ tend to align in the most expansive direction as the dynamical system evolves, which leads to numerical errors and can cause the collapse of the tangent map into a single direction. Having this fact in view, the Gram–Schmidt Reorthonormalization (GSR) can be employed to subtract the contribution of the most expansive direction from the others, which allows the correct estimation of λ_2 to λ_n. The GSR procedure that leads to the corrected version of the vectors $\boldsymbol{\delta}_{1x}, \boldsymbol{\delta}_{2x}, \ldots, \boldsymbol{\delta}_{nx}$ given by $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ and their normalized versions $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$ can be analytically described as [19, 30]:

$$\begin{aligned}
\mathbf{v}_1^{(k)} &= \boldsymbol{\delta}_{1x}^{(k)},\\
\mathbf{u}_1^{(k)} &= \frac{\mathbf{v}_1^{(k)}}{\|\mathbf{v}_1^{(k)}\|},\\
\mathbf{v}_2^{(k)} &= \boldsymbol{\delta}_{2x}^{(k)} - \langle\boldsymbol{\delta}_{2x}^{(k)}, \mathbf{u}_1^{(k)}\rangle, \mathbf{u}_1^{(k)},\\
\mathbf{u}_2^{(k)} &= \frac{\mathbf{v}_2^{(k)}}{\|\mathbf{v}_2^{(k)}\|},\\
&\vdots\\
\mathbf{v}_n^{(k)} &= \boldsymbol{\delta}_{nx}^{(k)} - \langle\boldsymbol{\delta}_{nx}^{(k)}, \mathbf{u}_1^{(k)}\rangle\mathbf{u}_1^{(k)} - \cdots\\
&\quad - \langle\boldsymbol{\delta}_{nx}^{(k)}, \mathbf{u}_{n-1}^{(k)}\rangle\mathbf{u}_{n-1}^{(k)},\\
\mathbf{u}_n^{(k)} &= \frac{\mathbf{v}_n^{(k)}}{\|\mathbf{v}_n^{(k)}\|},
\end{aligned} \tag{10}$$

where $\langle\mathbf{a},\mathbf{b}\rangle$ denotes the inner product of vectors $\mathbf{a}$ and $\mathbf{b}$. In the Kth iteration, the Lyapunov spectrum is given by

$$\lambda_n = \lim_{K\to\infty}\frac{1}{KT}\cdot\sum_{k=1}^{K}\ln\|\mathbf{v}_n^{(k)}\|. \tag{11}$$

After that, the tangent map given by $\boldsymbol{\Phi}(\mathbf{x},T)$ is set as the identity matrix, to correctly evaluate the divergence (or convergence) rates for the next iteration. The same procedure can be applied to discrete-time dynamical systems, being also necessary to perform the GSR correction for every iteration.

Finally, it is important to remark that at least one positive Lyapunov exponent is a necessary but not a sufficient condition for a system to exhibit chaotic behaviour. In fact, the solution of the dynamical system has also to remain within a compact space and, if dissipative dynamical systems are considered, contraction has to outweigh expansion [10, 19]. In this case, any limit set in an autonomous continuous-time system, except for an equilibrium point, has at least one null Lyapunov exponent (see [19] for a mathematical proof) and the signals of the others exponents will define the topological characteristics of the attractor. For instance, a dissipative third-dimensional continuous-time dynamical system having the Lyapunov spectrum described by $(\lambda_1, \lambda_2, \lambda_3)$ can exhibit chaotic behavior when those signals are $(+, 0, -)$ being, $\lambda_3 < -\lambda_1$. A two-torus can also be obtained when the exponents are $(0, 0, -)$, and a limit cycle is produced when they assume the combination equal to $(0, -, -)$. Finally, it is possible to characterize a fixed point for $(\lambda_1, \lambda_2, \lambda_3) = (-, -, -)$. It is important to observe that autonomous continuous-time dynamical systems require at least three state variables to exhibit chaos, a restriction that does not hold for discrete-time dynamical systems. In the latter case, unidimensional motion equations can exhibit a positive Lyapunov exponent and chaotic behavior, but this oscillating pattern is only achieved if the solution of the dynamical system also satisfies the requirement of remaining within a compact space, defining an attracting limit set in order to characterize a strange attractor [10, 19]. For instance, a geometric progression $x_{n+1} = rx_n$ with ratio $r > 1$ has a positive Lyapunov exponent, but it is not chaotic since its solution does not remain within a compact space.

2.2 The cloned dynamics (CLDYN) approach

The essence of the proposed method is to analyze the evolution of the difference state vectors defined as the distance between the fiducial trajectory and the clones of these motion equations initially disturbed by small values in orthogonal directions. Therefore, given an n-dimensional dynamical system, n clones are created (if one wishes to estimate all n exponents):

$$\begin{aligned} \dot{\mathbf{x}}_{c1} &= \mathbf{F}(\mathbf{x}_{c1}, t), \\ \dot{\mathbf{x}}_{c2} &= \mathbf{F}(\mathbf{x}_{c2}, t), \\ &\vdots \\ \dot{\mathbf{x}}_{cn} &= \mathbf{F}(\mathbf{x}_{cn}, t). \end{aligned} \tag{12}$$

Each clone receives the initial condition of the reference system disturbed by a small value δ_{x0} along a specific orthogonal direction, which means that:

$$\begin{aligned} \mathbf{x}_{0c1} &= \mathbf{x}_0 + \boldsymbol{\delta}_{1x}^{(0)}, \\ \mathbf{x}_{0c2} &= \mathbf{x}_0 + \boldsymbol{\delta}_{2x}^{(0)}, \\ &\vdots \\ \mathbf{x}_{0cn} &= \mathbf{x}_0 + \boldsymbol{\delta}_{nx}^{(0)}, \end{aligned} \tag{13}$$

being $\{\boldsymbol{\delta}_{1x}, \boldsymbol{\delta}_{2x}, \ldots, \boldsymbol{\delta}_{nx}\}$ an orthogonal basis initially defined as $\delta_{x0}\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\} = \delta_{x0}\mathbf{I}_n$. Note that, in this case, the identity matrix is now multiplied for the magnitude of the applied perturbation, which implies in an important difference from the TANMAP approach. In the CLDYN approach, the value of δ_{x0} has to be numerically chosen as small enough to be considered infinitesimal when compared to the attractor size. More details about the effect of this on the numerical estimation of the Lyapunov spectrum and how to suitably choose δ_{x0} will be provided in the next section.

In the posed framework, each clone will correspond to a specific direction associated with the Lyapunov exponent to be evaluated. Thus, the original motion equations and the clones are then integrated (or iterated for discrete-time dynamical systems) for an interval T, and at the end of this process, the perturbation vectors are estimated by the difference of the final states achieved by the fiducial trajectory and the cloned trajectories (defining the difference state vectors) in the form:

$$\begin{aligned} \boldsymbol{\delta}_{1x}^{(1)} &= \mathbf{x}(T) - \mathbf{x}_{c1}(T), \\ \boldsymbol{\delta}_{2x}^{(1)} &= \mathbf{x}(T) - \mathbf{x}_{c2}(T), \\ &\vdots \\ \boldsymbol{\delta}_{nx}^{(1)} &= \mathbf{x}(T) - \mathbf{x}_{cn}(T). \end{aligned} \tag{14}$$

To avoid the same numerical problems previously commented in the explanation of the TANMAP approach, the GSR procedure is applied as described in (10). After that, and before starting a new iteration, the clones are displaced in the neighborhood of the fiducial trajectory, receiving new "initial conditions" in the orthogonal frame spanned by the $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$ vectors:

$$\begin{aligned} \mathbf{x}_{0c1}^{(1)} &= \mathbf{x}(T) + \delta_{x0}\mathbf{u}_1^{(1)}, \\ \mathbf{x}_{0c2}^{(1)} &= \mathbf{x}(T) + \delta_{x0}\mathbf{u}_2^{(1)}, \\ &\vdots \\ \mathbf{x}_{0cn}^{(1)} &= \mathbf{x}(T) + \delta_{x0}\mathbf{u}_n^{(1)} \end{aligned} \tag{15}$$

and so the small disturbances will always stand in the specific direction of the Lyapunov exponent being estimated. Finally, after the Kth iteration, the Lyapunov exponents are given by:

$$\lambda_n = \lim_{\delta_{x0}\to 0} \lim_{K\to\infty} \frac{1}{KT} \sum_{k=1}^{K} \ln \left\| \frac{\mathbf{v}_n^{(k)}}{\delta_{x0}} \right\|. \tag{16}$$

Figure 1 illustrates a typical iteration of the CLDYN method for disturbances initially applied in two orthogonal directions of the phase space ($\boldsymbol{\delta}_{1x}^{(0)}$ and $\boldsymbol{\delta}_{2x}^{(0)}$).

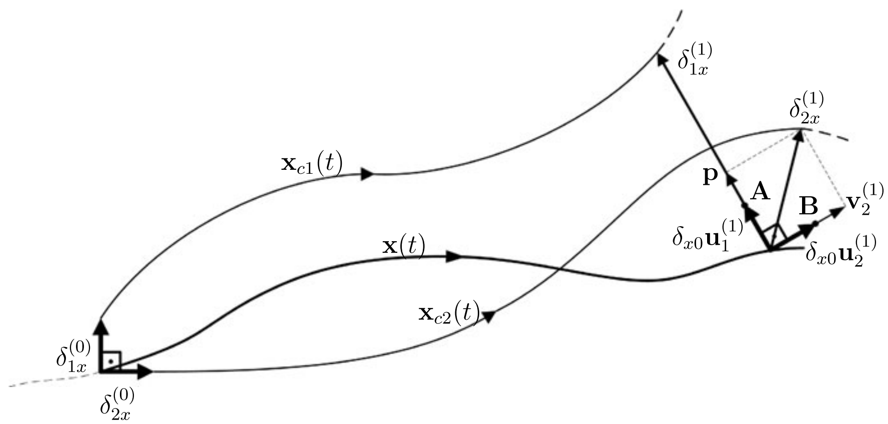


Fig. 1 Illustration of a typical CLDYN iteration. $\boldsymbol{\delta}_{1x}^{(0)}$ and $\boldsymbol{\delta}_{2x}^{(0)}$ are the initial difference state vectors given by $\delta_{x0}\{\mathbf{e}_1, \mathbf{e}_2\}$. $\mathbf{p}$ is the projection of $\boldsymbol{\delta}_{2x}^{(1)}$ in $\boldsymbol{\delta}_{1x}^{(1)}$ used to obtain $\mathbf{v}_2$ vector. **A** and **B** represent the initial conditions for the next iteration of the procedure. The value of δ_{x0} was exaggerated here for the sake of the illustration

As time passes, for each iteration, the difference state vectors are updated as in (14), being the tendency of alignment with the most expansive direction corrected by the GSR procedure. Before the next iteration begins, the clones are displayed in the neighborhood of the fiducial trajectory in an orthogonal manner (as it stated in (15)). Finally, the next iteration begins with the clones starting from points **A** and **B**. This process is repeated until the average behaviour of the whole attractor is taken into account.

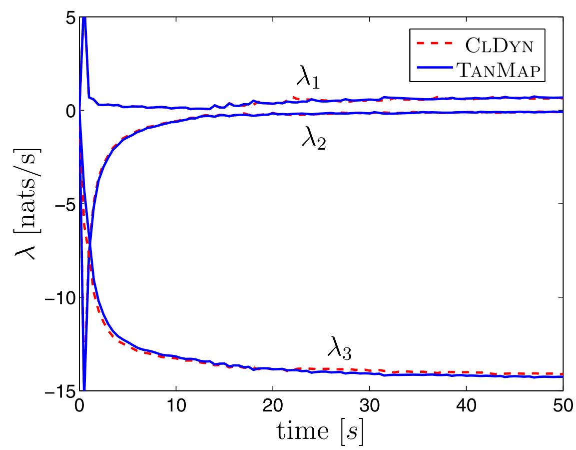


Fig. 2 Time evolution of Lorenz Lyapunov spectrum using TANMAP and CLDYN methods

3 Results

3.1 Analyzing the performance of the CLDYN approach for classical dynamical models

Figure 2 brings the time evolution of the Lyapunov spectrum for the classical Lorenz system obtained by the TANMAP and CLDYN methods (with $T = 0.5$ s and $\delta_{x0} = 10^{-4}$). It can be noted that both algorithms have very similar convergence behaviors and yield almost identical exponent values in "steady state" conditions. The Lorenz model is a natural choice since there is a great deal of "numerical experimentation" available in the literature regarding it. In particular, the numerical values for the Lyapunov exponents found here are in perfect accordance with these works (see [22] for values provided by different methods).

Table 1 shows the numerical values of the Lyapunov exponents obtained after convergence ($t_{\text{final}} = 10000$ s) for three emblematic continuous-time chaotic systems: the already mentioned Lorenz system, the chaotic and the hyperchaotic Rössler system (the models are described in [22] and also in Appendix). The results show that the proposed method is a reliable algorithm to perform this calculation, even for the hyperchaotic Rössler system, which displays two positive exponents and two exponents close to zero (a difficult scenario in terms of numerical estimation) [22]. In the latter case, the possibility of partial spectrum estimation allowed the calculation of the positive exponents by solving 12 differential equations (the original motion equations and two clones), while the TANMAP approach required the construction of the entire tangent space, being necessary the integration of 20 differential equations. Moreover, Table 1 also shows that the standard deviations of the Lyapunov exponents when t_{final} is attained tend to zero as time evolves. This result illustrates the convergence of the Lyapunov exponents

Table 1 Lyapunov spectra ± standard deviation (σ_i) obtained using the TanMap and ClDyn methods

Dynamics	Method	$\lambda_1 \pm \sigma_1$	$\lambda_2 \pm \sigma_2$	$\lambda_3 \pm \sigma_3$	$\lambda_4 \pm \sigma_4$	t_{sim}
Lorenz	TanMap	0.9037 ± 0.0527	0.0011 ± 0.1376	-14.5672 ± 0.1651	–	689.9463
	ClDyn	0.9025 ± 0.0519	-0.0014 ± 0.1340	-14.5293 ± 0.1528	–	564.3825
Rössler	TanMap	0.0886 ± 0.0020	0.0002 ± 0.0059	-9.8009 ± 0.1011	–	239.9587
	ClDyn	0.0895 ± 0.0031	0.0002 ± 0.0059	-9.8079 ± 0.1013	–	135.0057
Rösslerh	TanMap	0.1083 ± 0.0032	0.0228 ± 0.0052	-0.0007 ± 0.1102	-25.4881 ± 0.6586	1179.8421
	ClDyn	0.1128 ± 0.0053	0.0324 ± 0.0053	-0.0247 ± 01102	-23.9892 ± 0.6353	452.6063
Logistic	TanMap	0.3634 ± 0.0119	–	–	–	0.0021
	ClDyn	0.3637 ± 0.0119	–	–	–	0.0024
Hénon	TanMap	0.4173 ± 0.0181	-1.6213 ± 0.0314	–	–	0.1767
	ClDyn	0.4173 ± 0.0181	-1.6213 ± 0.0314	–	–	0.1695

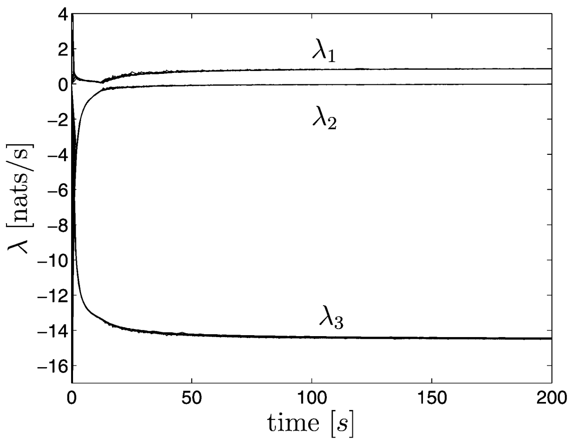
In all simulations, $\delta_{x0} = 10^{-4}$ and $t_{final} = 10000$ s for continuous-time cases (exponents given in [nats/s]). Both methods utilized the following parameters: Lorenz, $\mathbf{x}_0 = [1\ 0\ 1]^T$, $T = 0.5$ s, Rössler, $\mathbf{x}_0 = [1\ 0\ 1]^T$, $T = 1.0$ s; Rössler hyperchaos (Rösslerh), $\mathbf{x}_0 = [-20\ 0\ 0\ 15]^T$ and $T = 0.1$ s. For discrete-time models (exponents given in [nats/iteration]), $\delta_{x0} = 10^{-4}$, $N = 10000$ iterations. Logistic map: $\mathbf{x}_0 = 0.49$, $T = 1$ iteration; Hénon map: $\mathbf{x}_0 = [0\ 0]^T$, $T = 1$ iteration. The upper index T denotes matrix transpose. The *last column* presents the required time in seconds for the simulation of each one of the dynamical systems

estimation process and it is in completely agreement with the result provided by [1], obtained by applying a recursive QR decomposition approach.

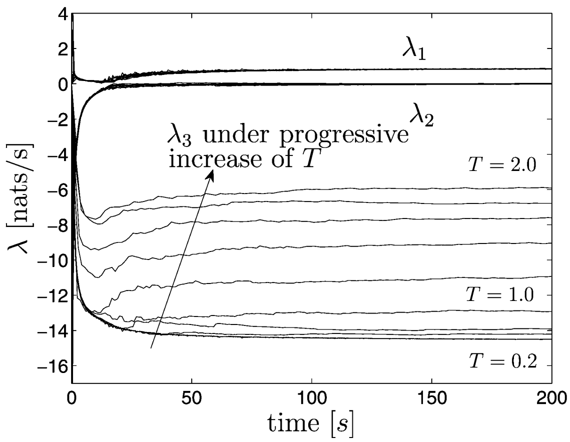
Two well-known discrete-time systems were also considered in Table 1: the Logistic map and the Hénon map (see Appendix). The one-dimensional Logistic map has the attractiveness that is not difficult to derive a expression for its Jacobian, which allows a simple formula to be reached for the TanMap approach [29]. In this sense, the values reached by the ClDyn method indicated that the proposal is a consistent tool to calculate Lyapunov spectrum. Furthermore, the Hénon map generalizes the calculations to a two-dimensional discrete-time case, in which there is also agreement with the numerical values obtained via the classical procedure.

In almost all performed simulations, the ClDyn method has required a shorter simulation time than the TanMap approach (last column—Table 1). Although this difference of performance seems to be significative in some cases, as observed for the hyperchaotic Rössler system, it may be related to the required time to solve the set of differential equations provided by the different approaches using a specific numerical method. In such a case, it is difficult to say a priori which method will require minimal computational resources.

Although the numerical experiments gave rise to very similar results obtained using both methods. It should be stressed that a certain degree of caution is required when it comes to choosing some parameters of the ClDyn method. As has already been exposed in [30], the TanMap approach presents a good robustness with respect to the integration time (T), which specifies the GSR interval. Meanwhile, increasing T implies decreasing the computational cost to calculate the exponents; on the other hand, this can introduce numerical oscillations in the spectrum or even cause the tangent map to collapse, if T is taken to be extremely large. Figure 3(a) illustrates this robustness by repeating the calculation of the Lorenz Lyapunov spectrum 10 times, increasing T progressively (from 0.2 s to 2 s with steps of 0.2 s), which does not affect the obtained exponent values. When this simulation is repeated for the ClDyn method (see Fig. 3(b)), it can be observed that λ_1 and λ_2 are similar to those obtained via the TanMap, but an overestimation of λ_3 for larger values of T is detected. This imprecision for the lowest exponent is related to the loss of information due to the fast convergence of the cloned trajectory to the fiducial one in the most contractive flux direction, which requires a more conservative choice of T for the ClDyn method. A systematic approach to determine T would be to perform the whole spectrum



(a) TanMap method.



(b) CLDyn method.

Fig. 3 The Lorenz Lyapunov spectrum for the TanMap and CLDyn approaches under a progressive increase of T (from 0.2 s to 2 s, with steps of 0.2 s)

calculation reducing this parameter on each simulation for achieving a lower bound for λ_3.

Overestimates of λ_3 in the CLDyn method can also occur when the magnitude of the initial perturbation δ_{x0} is not small enough. Figure 4 shows the Lorenz's Lyapunov spectrum for a progressive increase in δ_{x0} (from 10^{-5} to 10^{-1} with a geometric ratio of 10). It can be observed that a significant overestimate of λ_3 starts to occur from $\delta_{x0} = 10^{-2}$, while λ_1 and λ_2 are negligibly affected. A small perturbation of 10^{-4} was adequate for the strange attractors analyzed here, and a progressive reduction of δ_{x0} can also be carried out for achieving a lower bound to λ_3 and ensuring more reliable calculations.

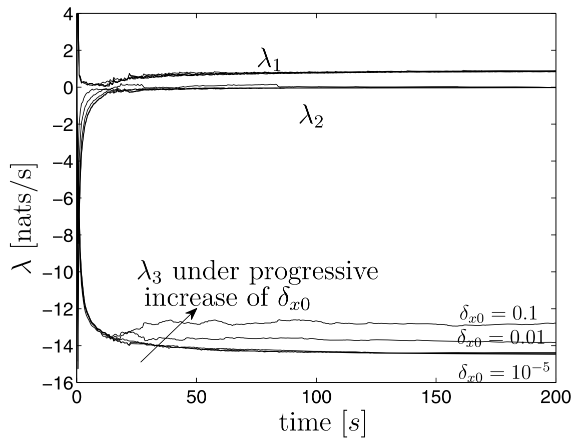
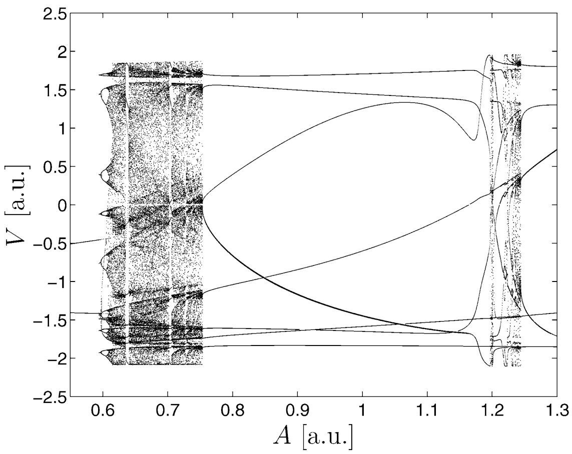


Fig. 4 Effect of a progressive increase in the magnitude of the initial perturbation δ_{x0} (from 10^{-5} to 10^{-1} with a geometric ratio 10) in the Lorenz Lyapunov spectrum for CLDyn method

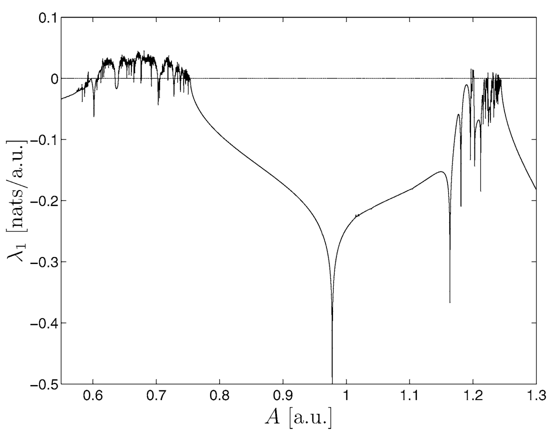
3.2 The CLDyn approach applied to the FitzHugh–Nagumo model with discontinuous inputs

An advantage of the proposed method is the possibility of evaluating the Lyapunov spectrum without constructing the tangent space, which can be useful in a wide range of applications, such as the analysis of dynamics with discontinuous inputs or states. To illustrate this, the CLDyn method is employed to analyze the oscillatory behaviour of the neuronal FitzHugh–Nagumo model [12] for nonsmooth inputs (rectangular pulses of frequency ω and amplitude A—see Appendix). This system consists of a modified version of the Van der Pol's equations to describe relaxation oscillators, aiming to capture the characteristics of neuronal oscillations. Although rectangular pulses are commonly used to extract relevant neuronal characteristics such as refractory period or to obtain the strength *versus* duration curve. The mathematical treatment of the neuronal FitzHugh–Nagumo model is far from trivial due to their discontinuous nature [9].

Since in this case it is impossible to use the TanMap method to compute the exponents and establish a comparison with the CLDyn approach, an stroboscopic map was built to provide an independent technique for validation. The stroboscopic map is obtained by periodic sampling of the state variables (when the transients are assumed to have vanished) taking the amplitude of stimulation as control parameter. In this analysis, aperiodic oscillations, like chaotic oscillations, display a number of points that tends to infinity,



(a)



(b)

Fig. 5 (**a**) Is the stroboscopic map taking the amplitude A [a.u.] as control parameter of the neuronal FitzHugh–Nagumo model; V [a.u.] is the state variable membrane potential. (**b**) Represents the largest Lyapunov exponent associated to the respective control parameter

while periodic solutions tend to produce a finite number of points for each value of A. Thus, the map corresponds to a geometrical approach that gives a qualitative view of the topological structure of the system solution [4, 19, 29]. It is possible to observe in Fig. 5(b) the richness of dynamical behaviors that this apparently simple oscillator can achieve. For instance, one may cite the presence of smooth transitions from periodic to chaotic oscillations, as well as of abrupt collapse and merging of strange attractors for smooth changes in the control parameter. Such flip, tangent, and crisis bifurcations have already been verified for the FitzHugh–Nagumo model excited by smooth inputs [28], and can give rise to experimentally observed oscillatory behaviors [3], as intermittent behavior [20], defined as a bursting of action potentials of irregular length, which is also observed in the simulations performed here for points near tangent bifurcations (as that which occurs for $A = 0.6348$).

Figure 5(b) shows the largest Lyapunov exponent obtained after a significant time (3000 arbitrary time units [a.u.]) with $T = 0.5$ [a.u.], $\delta_{x0} = 10^{-4}$ and A varying from 0.55 to 1.3 with steps of 2×10^{-4} [a.u.]. It can be observed that the λ_1 values are in agreement with the captured oscillatory patterns (with sampling rate of 1 rad/a.u.) by the stroboscopic map, that is, positive exponents are associated with control parameters that produce apparently chaotic behaviours, while negative exponents are obtained for values of A that lead to apparently periodic oscillations.

4 Discussions and final conclusions

The main idea of estimating the Jacobian matrix of a dynamical system by the difference of states in a neighborhood of a given attractor point is already known and used to calculate Lyapunov exponents from experimental time series [10, 11]. This concept was adapted here in order to incorporate a priori information of knowledge of motion equations, creating initially small disturbed clones to analyze, in a more practical and intuitive way, the stability of dynamical systems, in accordance with what has been developed by [6] for the estimation of the largest Lyapunov exponent, and extended here to the entire spectrum. This new method opens interesting perspectives of applications, specially to hyperchaotic systems and nonsmooth dynamics.

An extension of [6] has already been presented in [13], providing the existence of a relation between λ_1 and the coupling factor between oscillators when a synchronous pattern is achieved. In particular, [13] presented a procedure for computing the largest exponent that, under specific conditions, is analogous to the CLDYN procedure, but is still restricted to computing the largest exponent and not the whole spectrum.

Furthermore, the method proposed in [13] was adapted in [26] in order to calculate λ_1 for the Duffing's oscillator with impacts (a classical case of nonsmooth dynamics) based in the synchronism of identical systems. This approach has some similarities with the proposed method, since it uses a cloned version to calculate the largest exponent. However, it should be stressed that the CLDYN method is not based on any synchronism principle, but it is related to the main idea introduced by the stability theory of dynamical systems, attempting to evaluate how small perturbations evolve. The strategy of monitoring a difference state vector built from disturbed clones of the dynamics with the aid of the numerical corrections, also employed in the usual TANMAP procedure, consists of a different way to address this problem, which allows not only computation of the largest Lyapunov exponent but of the whole Lyapunov spectrum, without performing an exhaustive search for parameters that synchronize dynamics, or even treating discontinuous points (in the case of nonsmooth systems) as exceptions, forcing state transitions, as exposed in [17]. In this last case, nonsmooth functions that appear in the motion equations will also appear in the clones, which is not prohibitive in the process of obtaining the difference state vectors for calculating the Lyapunov exponents.

Even in the case of smooth systems, there are some advantages in using the CLDYN method. For instance, as the Lyapunov spectrum can be partially estimated, the largest exponent (that is sufficient to characterize the oscillatory behavior) of an n-dimensional dynamical system can be obtained by integrating $2n$ differential equations, while the TANMAP method requires the construction of whole tangent space, which implies integrating $n(n+1)$ equations. Thus, for high-dimensional dynamics or for state equations that are mathematically hard to be linearized, the CLDYN method seems to be a convenient tool to be employed.

In this work, the performance of the proposed method was analyzed with the aid of extensive tests based on representative continuous- and discrete-time dynamical systems, which provided strong numerical support to the developed proposal. Moreover, the possibility of using the CLDYN method for nonsmooth dynamics was illustrated for the classical FitzHugh–Nagumo neuronal model excited by rectangular pulses. In such case, the computation of Lyapunov exponents can offer valuable bases for studying biological information processing in the light of dissipative nonlinear dynamical systems theory, by its close connection to information theory [15, 23]. It is already known that chaotic processes can lead to an efficient way of transmitting and codifying information, as was shown for some classical dynamical systems [5], and has been, in a few steps, related to cognitive processes through the analysis of low- and high-dimensional neuronal systems in real and artificial paradigms [16, 21, 25].

Finally, complex systems constituted of nonlinear coupled neuronal oscillators have been intensively studied in an attempt to explain the emergence of biopotential patterns which can be related to memory formation, learning and recognition [25] due to nonlinear phenomenons as transient synchronism [27] and chaotic itinerancy [14]. The characterizations of such mechanisms are closely related to the convergence of Lyapunov exponents (e.g., in the fluctuations of largest Lyapunov exponent that characterize chaotic itinerancy [14]), there being a possibility that the proposed method be capable of leading to contributions.

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Appendix

The Lorenz model is described by as:

$$\begin{aligned}\dot{x} &= \sigma(y-x),\\ \dot{y} &= -xz + rx - y,\\ \dot{z} &= xy - bz,\end{aligned} \tag{17}$$

with the following employed parameters: $\sigma = 10$, $r = 28$, $b = \frac{8}{3}$. The units of the state variables are arbitrary and time is assumed to be in seconds.

The Rössler system is described as:

$$\begin{aligned}\dot{x} &= -y - z,\\ \dot{y} &= x + ay,\\ \dot{z} &= b + z(x-c),\end{aligned} \tag{18}$$

with the following employed parameters: $a = 0.15$, $b = 0.2$, $c = 10$. The units of state variables are arbitrary and time is assumed to be in seconds. The Hyperchatic Rössler system is described as:

$$\begin{aligned}\dot{x} &= -y - z,\\ \dot{y} &= x + ay + w,\\ \dot{z} &= b + xz,\\ \dot{w} &= cw - dz,\end{aligned} \tag{19}$$

with the following employed parameters: $a = 0.25$, $b = 3.0$, $c = 0.05$, $d = 0.5$. The units of the state variables are arbitrary and time is assumed to be in seconds.

The FitzHugh–Nagumo system is described as:

$$\begin{aligned} \dot{V} &= V - \frac{V^3}{3} - W + I(t), \\ \dot{W} &= c(V + a - bW), \\ I(t) &= A \cdot \mathrm{square}(\omega t), \end{aligned} \tag{20}$$

with the following employed parameters: $a = 0.7$, $b = 0.8$, $c = 0.1$. The function $A \cdot$ square represents a train of rectangular pulses of amplitude A, frequency $\omega = 1$ rad/a.u. (arbitrary units) and duty cycle of 50%. V represents the membrane potential and W its refractoriness, in arbitrary units. Time is given in arbitrary unit, as well.

The logistic map is described as

$$X_{n+1} = rX_n(1 - X_n), \tag{21}$$

with the employed parameter $r = 3.75$. The state variable assumes arbitrary units.

The Hénon system is described as:

$$\begin{aligned} X_{n+1} &= 1 - aX_n^2 + bY_n, \\ Y_{n+1} &= X_n \end{aligned} \tag{22}$$

with the following employed parameters: $a = 1.4$ and $b = 0.3$. The state variables have arbitrary units.

All continuous-time dynamical systems were integrated using a 4th-order variable step Runge–Kutta method with relative and absolute precision equal to 10^{-10}. All the simulations were performed using a computer with a CPU Intel Core 2 Quad Q8200 @ 2.33 GHz (4 cores/4 threads), 4 GB RAM DDR2 @ 800 MHz, Microsoft Windows XP x86 and MATLAB R2009a (v7.8.0.347).

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