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New results on synchronization of chaotic systems with time-varying delays via intermittent control

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Abstract This paper is concerned with the problem of exponential synchronization for chaotic systems with time-varying delays by using periodically intermittent control. Some new and useful synchronization criteria are obtained based on the differential inequality method and the analysis technique. It is noteworthy that the methods used in this paper are different from the techniques employed in the existing works, and the derived conditions are less conservative. Especially, a strong constraint on the control width that the control width should be large than the time delay imposed by the current references is released in this paper. Moreover, the new synchronization criteria do not impose any restriction on the size of time delay. Numerical examples are finally presented to illustrate the effectiveness of the theoretical results.

Keywords Exponential synchronization · Chaotic system · Time-varying delay · Intermittent control

1 Introduction

During the past decades, the control and synchronization of chaotic systems has been an active research topic due to its importance in theory and its potential applications in various areas such as mechanics, neural networks, biology, and secure communications [1–4]. Numerous methods have been developed for the control and synchronization in various chaotic systems which include linear and nonlinear feedback control, time-delay feedback control, adaptive control, impulsive control, intermittent control, among many others; see [1–23] and the references therein.

Intermittent control, a discontinuous feedback control which is activated during certain nonzero time intervals, but is off during other time intervals, has been widely used in engineering fields for its practical and easy implementation in engineering control. Recently, much effort has been devoted to study the issue of stabilization and synchronization of delayed chaotic systems and delayed dynamical networks by using intermittent control, and many important and interesting results have been obtained (see [14–23]). However, there exist different limitations in these results

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and such restrictions make these results less applicable. For example, in [14–20], two central differential inequalities and Lyapunov function method are used to investigate the stabilization and synchronization problems, but the results obtained in [14–20] require that the control width should be larger than the time delay. So, when the time delay has a relatively larger value, the results in [14–20] fail to check the stabilization and synchronization. And the results in [19, 20] also impose the restriction that the time delays are smaller than the noncontrol width. Since the results in [21] are delay-dependent, they cannot be applied to deal with exponential synchronization of delayed chaotic systems with any time delay. The results in [22] require the time-varying delays be differentiable and simultaneously their derivatives be not greater than 1. Moreover, the results in [23] focus on studying exponential lag synchronization of neural networks with discrete delays and distributed delays. Hence, the stabilization and synchronization problems of delayed chaotic systems and delayed dynamical networks under intermittent control have not been fully investigated yet. New techniques and methods for further improving the results mentioned above need to be explored and developed.

The problem of exponential synchronization for chaotic systems with time-varying delays by means of intermittent control is investigated in this paper. Some new and useful synchronization criteria are obtained by using the methods which are different from the techniques employed in the existing works, and the derived results are less conservative. Especially, the traditional assumption that the control width should be larger than the time delay imposed by the current references is removed in our results. Moreover, the new synchronization criteria are delay-independent in the sense that they are applicable for any bounded time-varying delays. Finally, two examples and their simulations are given to show the effectiveness of the theoretical results.

2 Problem formulation and preliminaries

Consider a class of chaotic systems with time-varying delays:

$$\begin{aligned} &\dot{x}(t) = Ax(t) + Bf\big[x(t)\big] + Cg\big[x\big(t-\tau(t)\big)\big], \quad t>0, \\ &x(t) = \varphi(t), \quad -\tau \le t \le 0, \end{aligned} \tag{1}$$

where $x(t) \in R^n$ is the state vector, A, B, and C are $n \times n$ constants matrices, the time delay $\tau(t)$ may be unknown (constant or time-varying) but is bounded by a known constant, i.e., $0 \le \tau(t) \le \tau$, $f, g : R^n \to R^n$ are nonlinear vector-valued functions satisfying $f(0) = 0$ and $g(0) = 0$. Moreover, we here always assume that the nonlinear vector-valued functions f, g satisfy uniform Lipschitz condition, i.e., for any $x, y \in R^n$, there exist positive constants L_f and L_g such that

$$\begin{aligned} &\big\|f(x) - f(y)\big\| \le L_f\|x-y\|, \\ &\big\|g(x) - g(y)\big\| \le L_g\|x-y\|. \end{aligned}$$

To realize synchronization by means of periodically intermittent control in master-slave configuration with system (1) as the master (drive) system, the slave (response) system is designed as

$$\begin{aligned} &\dot{y}(t) = Ay(t) + Bf\big[y(t)\big] + Cg\big[y\big(t-\tau(t)\big)\big] \\ &\qquad + K(t)\big[x(t) - y(t)\big], \quad t>0, \\ &y(t) = \phi(t), \quad -\tau \le t \le 0, \end{aligned} \tag{2}$$

where $y(t) \in R^n$ denotes the state vector of system (2), and $K(t)$ is the intermittent linear state feedback control gain defined as follows:

$$K(t) = \begin{cases} K, & n\mathrm{T} \le t < (n+\theta)\mathrm{T}, \\ 0, & (n+\theta)\mathrm{T} \le t < (n+1)\mathrm{T}, \end{cases} \tag{3}$$

where $K \in R^{n\times n}$ is a constant control gain, $\mathrm{T} > 0$ is the control period, $0 < \theta < 1$ is the rate of control duration called control rate, and $n = 0, 1, 2, \dots$. In this paper, we are concerned with how to design suitable θ, T, and K, such that the response system (2) exponentially synchronizes to the drive system (1).

Let $e(t) = y(t) - x(t)$ be the synchronization error between the states of drive system (1) and response system (2). According to the control law (3), then we can derive the following error dynamical system:

$$\begin{cases} \dot{e}(t) = Ae(t) + Bf(x,y,t) + Cf(x_\tau, y_\tau, t-\tau(t)) \\ \qquad - Ke(t), \quad n\mathrm{T} \le t < (n+\theta)\mathrm{T}, \\ \dot{e}(t) = Ae(t) + Bf(x,y,t) + Cf(x_\tau, y_\tau, t-\tau(t)), \\ \quad (n+\theta)\mathrm{T} \le t < (n+1)\mathrm{T}, \end{cases} \tag{4}$$

where $f(x, y, t) = f[y(t)] - f[x(t)]$, and $f(x_\tau, y_\tau, t-\tau(t)) = g[y(t-\tau(t))] - g[x(t-\tau(t))]$. It is clear that if the zero solution of the error dynamical system (4) is globally exponentially stable, then exponential synchronization between the drive system (1) and the response system (2) is achieved.

To derive our main results is discussed in the next section. We need the following lemmas.

Lemma 1 [24] *For any two n-dimensional real vectors Y and Z, and a symmetrical positive definite matrix $Q \in R^{n\times n}$, the following matrix inequality holds*:

$$Y^T Z + Z^T Y \le Y^T QY + Z^T Q^{-1} Z.$$

Lemma 2 *Suppose that function $y(t)$ is continuous and nonnegative when $t \in [-\tau, \infty)$ and satisfies the following condition*:

$$\begin{cases} \dot{y}(t) \le -\gamma_1 y(t) + \gamma_2 y(t-\tau(t)), \\ \quad nT \le t < (n+\theta)T, \\ \dot{y}(t) \le \gamma_3 y(t) + \gamma_2 y(t-\tau(t)), \\ \quad (n+\theta)T \le t < (n+1)T, \end{cases} \tag{5}$$

where $\gamma_1, \gamma_2, \gamma_3$ are constants, and $n = 0, 1, 2, \ldots$. If

$$\begin{aligned} &\gamma_1 > \gamma_2 > 0, \qquad \delta = \gamma_1 + \gamma_3 > 0, \quad \textit{and} \\ &\varrho = \lambda - \delta(1-\theta) > 0, \end{aligned} \tag{6}$$

then

$$y(t) \le \sup_{-\tau \le s \le 0} y(s) \exp\{-\varrho t\}, \quad t \ge 0,$$

where $\lambda > 0$ is the unique positive solution of the equation $\lambda - \gamma_1 + \gamma_2 \exp\{\lambda\tau\} = 0$.

Proof Denote $f(\lambda) = \lambda - \gamma_1 + \gamma_2 \exp\{\lambda\tau\}$. Since $\gamma_1 > \gamma_2 > 0$, we have $f(0) < 0$, $f(+\infty) > 0$, and $f'(\lambda) > 0$. Using the continuity and the monotonicity of $f(\lambda)$, the equation $\lambda - \gamma_1 + \gamma_2 \exp\{\lambda\tau\} = 0$ has an unique positive solution $\lambda > 0$. Take $M_0 = \sup_{-\tau\le s\le 0} y(s)$, and $W(t) = \exp\{\lambda t\}y(t)$, where $t \ge 0$. Let $Q(t) = W(t) - hM_0$, where $h > 1$ is a constant. It is easy to see that

$$Q(t) < 0, \quad \text{for all } t \in [-\tau, 0]. \tag{7}$$

In the following, we will prove that

$$Q(t) < 0, \quad \text{for all } t \in [0, \theta T). \tag{8}$$

If this is not true, by (7) and the continuity of $y(t)$ as $t \in [-\tau, \infty)$, then there exists a $t_0 \in [0, \theta T)$ such that

$$Q(t_0) = 0, \qquad \dot{Q}(t_0) \ge 0, \tag{9}$$

$$Q(t) < 0, \quad -\tau \le t < t_0. \tag{10}$$

Using (5), (9) and (10), we obtain

$$\begin{aligned} \dot{Q}(t_0) &= \lambda W(t_0) + \exp\{\lambda t_0\}\dot{y}(t_0) \\ &\le (\lambda - \gamma_1)W(t_0) + \gamma_2 \exp\{\lambda\tau\} W\big(t_0 - \tau(t_0)\big) \\ &< \big(\lambda - \gamma_1 + \gamma_2 \exp\{\lambda\tau\}\big) hM_0 = 0. \end{aligned} \tag{11}$$

This contradicts the second inequality in (9), and so (8) holds.

Now, we prove that for $t \in [\theta T, T)$

$$H(t) = W(t) - hM_0 \exp\big\{\delta(t - \theta T)\big\} < 0. \tag{12}$$

Otherwise, there exists a $t_1 \in [\theta T, T)$ such that

$$H(t_1) = 0, \qquad \dot{H}(t_1) \ge 0, \tag{13}$$

$$H(t) < 0, \quad \theta T \le t < t_1. \tag{14}$$

For $\tau(t) > 0$, if $\theta T \le t_1 - \tau(t_1) < t_1$, it follows from (14) that

$$W\big(t_1 - \tau(t_1)\big) < hM_0 \exp\big\{\delta(t_1 - \theta T)\big\},$$

and if $-\tau \le t_1 - \tau(t_1) < \theta T$, from (7) and (8), we have

$$W\big(t_1 - \tau(t_1)\big) < hM_0 \le hM_0 \exp\big\{\delta(t_1 - \theta T)\big\}.$$

Hence, for $\tau(t) > 0$, we always have

$$W\big(t_1 - \tau(t_1)\big) < hM_0 \exp\big\{\delta(t_1 - \theta T)\big\}.$$

Then

$$\begin{aligned} \dot{H}(t_1) &= \lambda W(t_1) + \exp\{\lambda t_1\}\dot{y}(t_1) \\ &\quad - \delta hM_0 \exp\big\{\delta(t_1 - \theta T)\big\} \\ &\le (\lambda + \gamma_3)W(t_1) + \gamma_2 \exp\{\lambda\tau\} W\big(t_1 - \tau(t_1)\big) \\ &\quad - \delta hM_0 \exp\big\{\delta(t_1 - \theta T)\big\} \\ &< \big(\lambda - \gamma_1 + \gamma_2 \exp\{\lambda\tau\}\big) hM_0 \\ &\quad \times \exp\big\{\delta(t_1 - \theta T)\big\} = 0, \end{aligned} \tag{15}$$

which contradicts the second inequality in (13). Hence, (12) holds. That is, for $t \in [\theta T, T)$,

$$W(t) < hM_0 \exp\big\{\delta(t - \theta T)\big\} \le hM_0 \exp\big\{\delta(1-\theta)T\big\}.$$

On the other hand, it follows from (7) and (8) that for $t \in [-\tau, \theta\mathrm{T})$

$$W(t) < hM_0 < hM_0 \exp\{\delta(1-\theta)\mathrm{T}\}.$$

So

$$W(t) < hM_0 \exp\{\delta(1-\theta)\mathrm{T}\}, \quad \text{for all } t \in [-\tau, \mathrm{T}). \tag{16}$$

Similarly, we can prove that for $t \in [\mathrm{T}, (1+\theta)\mathrm{T})$,

$$W(t) < hM_0 \exp\{\delta(1-\theta)\mathrm{T}\},$$

and for $t \in [(1+\theta)\mathrm{T}, 2\mathrm{T})$

$$W(t) < hM_0 \exp\{\delta(t-2\theta)\mathrm{T}\}.$$

By induction, we can derive the following estimation of $V(t)$ for any integer n.

For $n\mathrm{T} \le t < (n+\theta)\mathrm{T}$,

$$W(t) < hM_0 \exp\{n\delta(1-\theta)\mathrm{T}\}, \tag{17}$$

and for $(n+\theta)\mathrm{T} \le t < (n+1)\mathrm{T}$,

$$W(t) < hM_0 \exp\{\delta[t-(n+1)\theta\mathrm{T}]\}. \tag{18}$$

Since for any $t \ge 0$, there exists a nonnegative integers k, such that $k\mathrm{T} \le t < (k+1)\mathrm{T}$, we can deduce the following estimation of $V(t)$ for any t by (17) and (18).

For $k\mathrm{T} \le t < (k+\theta)\mathrm{T}$,

$$W(t) < hM_0 \exp\{k\delta(1-\theta)\mathrm{T}\} \le hM_0 \exp\{\delta(1-\theta)t\}$$

and for $(k+\theta)\mathrm{T} \le t < (k+1)\mathrm{T}$,

$$\begin{aligned} W(t) &< hM_0 \exp\{\delta(t-(k+1)\theta\mathrm{T})\} \\ &\le hM_0 \exp\{\delta(1-\theta)t\}. \end{aligned}$$

Let $h \to 1$, from the definition of $V(t)$, we obtain

$$y(t) \le M_0 \exp\{-[\lambda - \delta(1-\theta)]t\} = M_0 \exp\{-\varrho t\}, \quad t \ge 0.$$

This completes the proof of Lemma 2. □

Throughout this paper, $P > 0 (< 0, \le 0, \ge 0)$ denotes a symmetrical positive (negative, seminegative, semipositive) definite matrix P; also $P^\top$, $\lambda_{\max(\min)}(P)$ are the transpose and the maximum (minimum) eigenvalue of a square matrix P, respectively. The vector (or matrix) norm is taken to be Euclidian, denoted by $\|\cdot\|$.

3 Main results

In this section, with the help of Lemma 2, some novel exponential synchronization criteria through intermittent linear state feedback are rigorously derived. The main results are stated as follows.

Theorem 1 *Suppose that there exist a symmetric positive define matrix $P > 0$ and positive constants $\gamma_1 > \gamma_2$, γ_3, μ_1, and μ_2 such that the following conditions hold*:

(i) $P(A-K) + (A-K)^\top P + \mu_1 PBB^\top P + \mu_1^{-1} L_f^2 I_n + \mu_2 PCC^\top P + \gamma_1 P \le 0$,
(ii) $\mu_2^{-1} L_g^2 I_n - \gamma_2 P \le 0$,
(iii) $PA + A^\top P + \mu_1 PBB^\top P + \mu_1^{-1} L_f^2 I_n + \mu_2 PCC^\top P - (\gamma_3 - \gamma_1) P \le 0$,
(iv) $\varrho = \epsilon - \gamma_3(1-\theta) > 0$,

where $\epsilon > 0$ is the unique positive solution of the equation $\epsilon - \gamma_1 + \gamma_2 \exp\{\epsilon\tau\} = 0$. Then the drive-response systems (1) and (2) are exponentially synchronized.

Proof Consider the following Lyapunov function:

$$V(t) = e^\top(t) P e(t). \tag{19}$$

According to Lemma 1 and conditions (i)–(iii), the derivative of $V(t)$ with respect to time t along the trajectory of the error system (4) can be calculated as follows:

When $n\mathrm{T} \le t < (n+\theta)\mathrm{T}$, for $n = 0, 1, 2, \ldots$,

$$\begin{aligned} \dot V(t) &= 2e^\top(t)P[Ae(t) + Bf(x,y,t) \\ &\quad + Cf(x_\tau, y_\tau, t-\tau(t)) - Ke(t)] \\ &\le e^\top(t)[P(A-K) + (A-K)^\top P]e(t) \\ &\quad + \mu_1 e^\top(t) PBB^\top Pe(t) + \mu_1^{-1} L_f^2 e^\top(t)e(t) \\ &\quad + \mu_2 e^\top(t) PCC^\top Pe(t) \\ &\quad + \mu_2^{-1} L_g^2 e^\top(t-\tau(t)) e(t-\tau(t)) \\ &= e^\top(t)[P(A-K) + (A-K)^\top P \\ &\quad + \mu_1 PBB^\top P + \mu_1^{-1} L_f^2 I_n \\ &\quad + \mu_2 PCC^\top P + \gamma_1 P]e(t) \\ &\quad - \gamma_1 e^\top(t)Pe(t) + e^\top(t-\tau(t))[\mu_2^{-1} L_g^2 I_n \\ &\quad - \gamma_2 P]e(t-\tau(t)) \end{aligned}$$

$$
\begin{aligned}
&\quad + \gamma_2 e^\top\big(t-\tau(t)\big)Pe\big(t-\tau(t)\big) \\
&\le -\gamma_1 e^\top(t)Pe(t) + \gamma_2 e^\top\big(t-\tau(t)\big)Pe\big(t-\tau(t)\big) \\
&= -\gamma_1 V(t) + \gamma_2 V\big(t-\tau(t)\big).
\end{aligned}
$$

and when $(n+\theta)\mathrm{T} \le t < (n+1)\mathrm{T}$, for $n = 0, 1, 2, \ldots,$

$$
\begin{aligned}
\dot V(t) &= 2e^\top(t)P\big[Ae(t) + Bf(x,y,t) \\
&\quad + Cf\big(x_\tau, y_\tau, t-\tau(t)\big)\big] \\
&\le e^\top(t)\big[PA + A^\top P\big]e(t) + \mu_1 e^\top(t)PBB^\top Pe(t) \\
&\quad + \mu_1^{-1} L_f^2 e^\top(t)e(t) \\
&\quad + \mu_2 e^\top(t)PCC^\top Pe(t) + \mu_2^{-1} L_g^2 e^\top \\
&\quad \times \big(t-\tau(t)\big)e\big(t-\tau(t)\big) \\
&= e^\top(t)\big[PA + A^\top P + \mu_1 PBB^\top P + \mu_1^{-1} L_f^2 I_n \\
&\quad + \mu_2 PCC^\top P - (\gamma_3 - \gamma_1)P\big]e(t) \\
&\quad + (\gamma_3 - \gamma_1)e^\top(t)Pe(t) + e^\top\big(t-\tau(t)\big) \\
&\quad \times \big[\mu_2^{-1} L_g^2 I_n - \gamma_2 P\big]e\big(t-\tau(t)\big) \\
&\quad + \gamma_2 e^\top\big(t-\tau(t)\big)Pe\big(t-\tau(t)\big) \\
&\le (\gamma_3 - \gamma_1)e^\top(t) \\
&\quad \times Pe(t) + \gamma_2 e^\top\big(t-\tau(t)\big)Pe\big(t-\tau(t)\big) \\
&= \gamma_3 V(t) + \gamma_2 V\big(t-\tau(t)\big).
\end{aligned}
$$

Namely, we have

$$
\begin{cases}
\dot V(t) \le -\gamma_1 V(t) + \gamma_2 V(t-\tau(t)), \\
\quad n\mathrm{T} \le t < (n+\theta)\mathrm{T}, \\
\dot V(t) \le (\gamma_3 - \gamma_1)V(t) + \gamma_2 V(t-\tau(t)), \\
\quad (n+\theta)\mathrm{T} \le t < (n+1)\mathrm{T}.
\end{cases}
\tag{20}
$$

Using Lemma 2 and condition (iv), we obtain

$$
V(t) \le \sup_{-\tau\le s\le 0} V(s)\exp\{-\varrho t\}, \qquad t \ge 0. \tag{21}
$$

By (19) and (21), we have

$$
\big\|e(t)\big\| \le \sqrt{\frac{\sup_{-\tau\le s\le 0} V(s)}{\lambda_{\max}(P)}}\exp\left\{-\frac{\varrho}{2}t\right\}, \qquad t \ge 0.
$$

This implies the conclusion and the proof is complete. □

In the following, we shall establish some numerically tractable synchronization criteria. To this end, let $P = I_n$, $K = kI_n$, $\mu_1 = L_f$, $\mu_2 = L_g$ in Theorem 1, then the following results can be obtained readily from Theorem 1.

Corollary 1 *Suppose that there exist positive constants $\gamma_1 > \gamma_2 \ge L_g$, γ_3 such that the following conditions hold*:

(i) $A + A^\top + L_f BB^\top + LgCC^\top + (L_f + \gamma_1 - 2k)I_n \le 0,$
(ii) $A + A^\top + L_f BB^\top + L_g CC^\top + (L_f - \gamma_3 + \gamma_1)I_n \le 0,$
(iii) $\varrho = \epsilon - \gamma_3(1-\theta) > 0,$

where $\epsilon > 0$ is the unique positive solution of the equation $\epsilon - \gamma_1 + \gamma_2 \exp\{\epsilon\tau\} = 0$. Then the drive-response systems (1) and (2) are exponentially synchronized.

Letting $m_0 = \lambda_{\max}(A + A^\top + L_f BB^\top + L_g CC^\top + L_f I_n)$, and selecting $\gamma_2 = L_g$, $\gamma_3 = (m_0 + \gamma_1) > 0$, then conditions (ii) in Corollary 1 hold. Thus, we can reduce the above corollary to the following:

Corollary 2 *The system (1) and system (2) achieve exponential synchronization if there exists positive constant $\gamma_1 > L_g$ such that the following conditions hold*:

(i) $0 < \frac{m_0+\gamma_1}{2} \le k,$
(ii) $\varrho = \epsilon - (\gamma_1 + m_0)(1-\theta) > 0,$

where $\epsilon > 0$ is the unique positive solution of the equation $\epsilon - \gamma_1 + L_g \exp\{\epsilon\tau\} = 0$.

Remark 1 Although some important and interesting results have been obtained on the exponential stabilization and synchronization for delayed chaotic systems and delayed dynamical networks via intermittent control [14–22], there exist different limitations in these results and such restrictions make these results less applicable. More specifically, the condition $\theta = 0.5$ is assumed in [14]. The results in [15–20] require that the control width should be larger than the time delay, i.e., $\theta\mathrm{T} > \tau$. So, when systems have a relatively larger value of delay, the results in [15–20] fail to check the stabilization and synchronization. And the results in [19, 20] also impose the restriction that the time delay is smaller than the noncontrol width, i.e., $\tau < \mathrm{T} - \theta\mathrm{T}$. In addition, the assumption $\inf_{t\in[0,+\infty)}\{1 - \dot\tau(t)\} > 0$ is assumed in [22], that is, the time-varying delays are differentiable and simultaneously their derivatives should be not greater

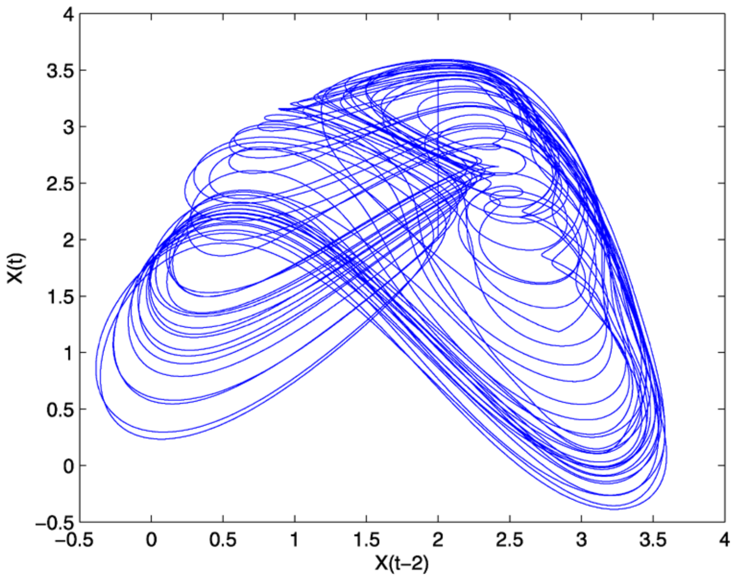

Fig. 1 The chaotic attractor of the Ikeda system (22)

than 1, which is a very strict condition. However, all these constraints are removed in our results. Moreover, the results in [21] are delay-dependent, which are only valid for the time delay less than a positive threshold. However, our results are delay-independent in the sense that they are valid for any bounded time-varying delays $\tau(t) \in (0, +\infty)$. Therefore, our results are less conservative and more practically applicable than those in the literature [14–22].

Remark 2 It should be pointed out that the previous results presented in [14–20] were obtained by constructing Lyapunov functions and using two central differential inequalities. Because of considering the control width and the noncontrol width in a control period separately by using two central differential inequalities, a strong constraint that the control width should be larger than the time delay is required in [14–20]. In this paper, a novel differential inequality (see Lemma 2) releasing the constraint mentioned above is established, and then based on which some new and useful results are derived. Evidently, the methods proposed in this paper are totally different from the existing works [14–23].

Remark 3 From the above results, we can see that the synchronization criteria derived in the paper depend on the control rate θ, but not the control period T. For practical problems, we can thus choose randomly the control period T for achieving chaos synchronization.

Note that there is only one parameter γ_1 in Corollary 2. Suppose that γ_1 is given as $\gamma^* > L_g$. Substituting $\gamma_1 = \gamma^*$ into the equation $\epsilon - \gamma_1 + L_g \exp\{\epsilon\tau\} = 0$ yields $\epsilon = \Phi(\gamma^*)$, then the following result is immediate from Corollary 2.

Corollary 3 *Suppose that γ_1 is given as $\gamma^* > L_g$. The system (1) and system (2) achieve exponential synchronization if the following conditions hold*:

(i′) $k \geq \frac{m_0+\gamma^*}{2} > 0$,
(ii′) $1 - \frac{\Phi(\gamma^*)}{m_0+\gamma^*} < \theta < 1$.

Remark 4 Corollary 3 allows us to determine the control gain k and the control rate θ in a simple way, based on which intermittent controller can be designed. In order to illuminate how to design the intermittent controller, the following procedures are preformed: (1) Given γ_1, and then compute $\Phi(\gamma^*)$; (2) Select k, θ such that conditions (i′) and (ii′) are satisfied; (3) Select randomly a control period T; (4) According to above chosen k, θ, T design the intermittent controller.

4 Numerical example

In this section, two examples are presented to show the effectiveness of the obtained new results.

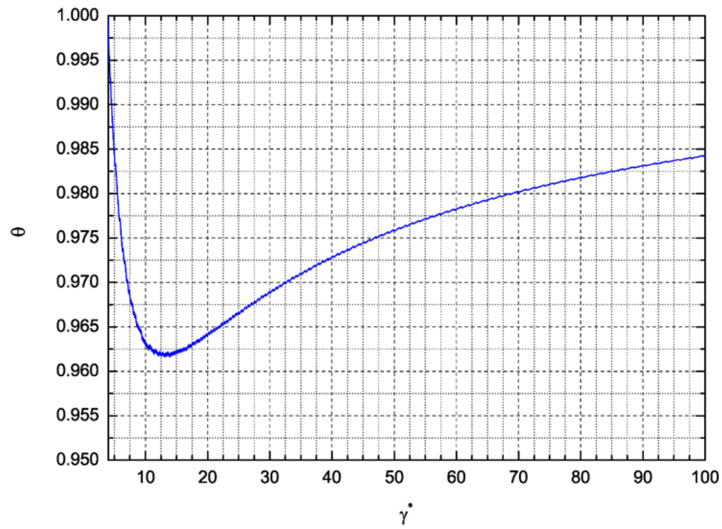

Fig. 2 The relationship curve between the parameter γ^* and the control rate θ in Example 1

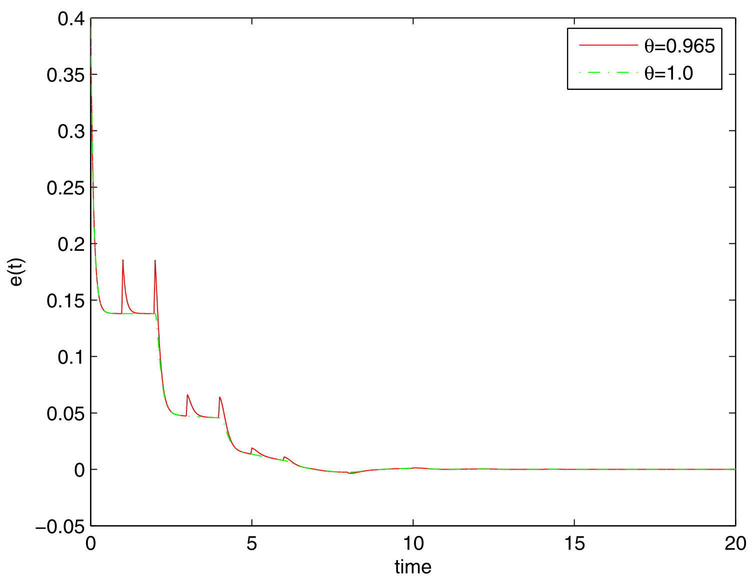

Fig. 3 (Color online) Synchronization errors $e(t)$ with control parameters $k = 10$, $\theta = 0.965$, $\mathrm{T} = 1.0$ (*red*), and $k = 10$, $\theta = 1.0$, $\mathrm{T} = 1.0$ (*green*) in Example 1

Example 1 Consider the following Ikeda oscillator of the form

$$\dot{x}(t) = -ax(t) + c\sin\big(x(t-\tau)\big). \tag{22}$$

This system exhibits chaotic behavior when $a = 1$, $c = 4$, and $\tau = 2$, which can be seen from Fig. 1. The corresponding response system is given by

$$\dot{y}(t) = -ay(t) + c\sin\big(y(t-\tau)\big) + K(t)\big[x(t) - y(t)\big],$$

where

$$K(t) = \begin{cases} k & n\mathrm{T} \le t < (n+\theta)\mathrm{T}, \\ 0 & (n+\theta)\mathrm{T} + \delta \le t < (n+1)\mathrm{T}, \\ & n = 0, 1, 2, \ldots. \end{cases} \tag{23}$$

Note that $A = -1$, $B = 0$, $C = 1$, $L_f = 0$ and $L_g = 4$, one has $m_0 = 2$. Based on Corollary 3 (ii′), we can plot the relationship curve between the control rate θ and the parameter γ^* in Fig. 2. In this example, we select $\gamma^* = 18$, then we obtain $k \ge 10$, and $0.963 < \theta < 1$ from conditions (i′) and (ii′) of Corollary 3. For numerical simulation, we take $k = 10$, $\theta = 0.965$, $\mathrm{T} =$

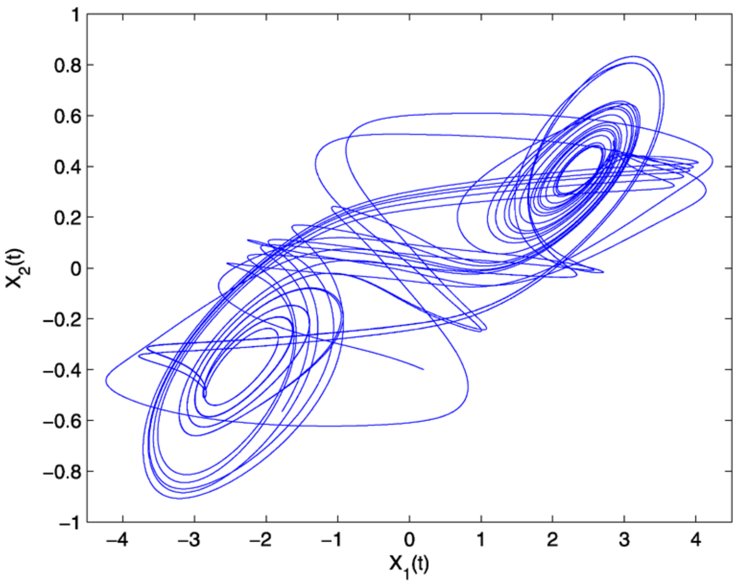

Fig. 4 The chaotic attractor of the Lu oscillator (24)

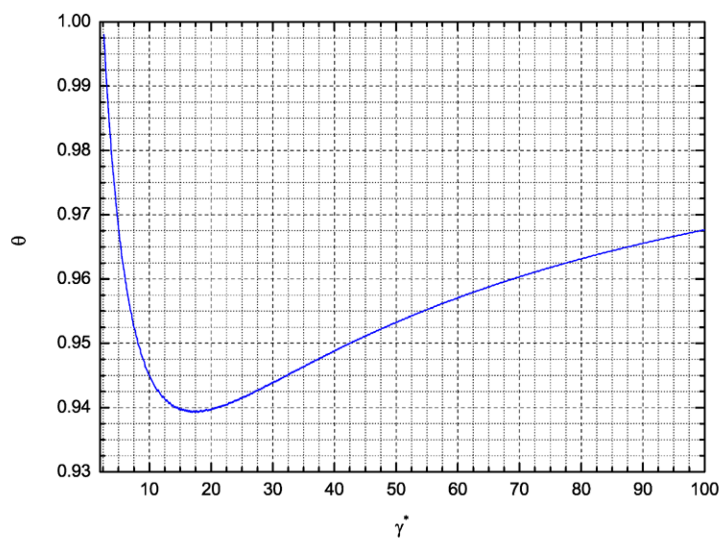

Fig. 5 The relationship curve between the parameter γ^* and the control rate θ in Example 2

1.0 and plot the synchronization error curve, as shown in Fig. 3. Since $\theta\mathrm{T} = 0.965 < \tau = 2$, the results in [16] cannot be used in this case. Hence, our results is more practically applicable.

It is noteworthy that when the control rate $\theta = 1.0$, the periodically intermittent control will reduce to the general continuous linear feedback control. For comparison, the synchronization error curve with control parameters $k = 10$, $\theta = 1.0$, $\mathrm{T} = 1.0$ is also plotted in Fig. 3. The simulation results show that the intermittent control is more cost effective than the linear feedback control since the latter activates the control all the times.

Example 2 Consider the Lu oscillator [25]

$$\dot{x}(t) = -Ax(t) + f\big[x(t)\big] + g\big[x(t-\tau)\big], \tag{24}$$

where $x(t) = (x_1(t), x_2(t))^\top \in R^2$, $\tau = 1$, and

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad f(x) = \begin{bmatrix} 3.0 & 5.0 \\ 0.1 & 2.0 \end{bmatrix} \begin{bmatrix} \tanh(x_1) \\ \tanh(x_2) \end{bmatrix},$$

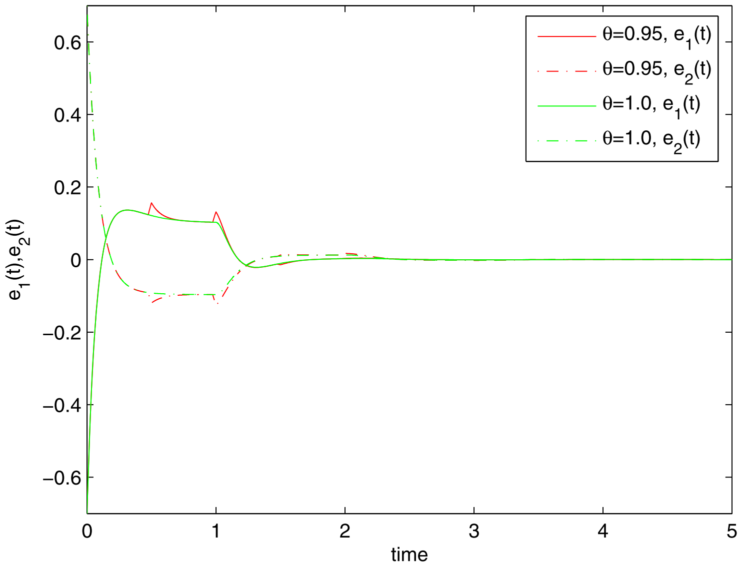

Fig. 6 (Color online) Synchronization errors $e_1(t)$, $e_2(t)$ with control parameters $k = 12$, $\theta = 0.95$, $\mathrm{T} = 0.5$ (*red*), and $k = 12$, $\theta = 1.0$, $\mathrm{T} = 0.5$ (*green*) in Example 2

$$g(y) = \begin{bmatrix} -2.5 & 0.2 \\ 0.1 & -1.5 \end{bmatrix} \begin{bmatrix} \tanh(y_1) \\ \tanh(y_2) \end{bmatrix}.$$

The model has a chaotic attractor, as shown in Fig. 4. The corresponding response system is of the form

$$\dot{y}(t) = -Ay(t) + f\big[y(t)\big] + g\big[y(t-\tau)\big] + K(t)\big[x(t) - y(t)\big],$$

where $K(t)$ is defined by (23).

Note that $L_f = 6.0989$ and $L_g = 2.5226$, one has $m_0 = 12.7204$. From Corollary 3 (ii′), the relationship curve between the control rate θ and the parameter γ^* is plotted in Fig. 5. In this example, we take $\gamma^* = 10$, then we obtain $k \geq 11.3602$, and $0.945 < \theta < 1$ from Conditions (i′) and (ii′) of Corollary 3. For numerical simulation, we select $k = 12$, $\theta = 0.95$, $\mathrm{T} = 0.5$ and plot the synchronization error curve, as shown in Fig. 6. For comparison, the synchronization error curve with control parameters $k = 12$, $\theta = 1.0$, $\mathrm{T} = 0.5$ is also plotted in Fig. 6. We can see from the simulation results that the intermittent control is more cost effective than the linear feedback control since the latter activates the control all the times.

5 Conclusion

In this paper, new sufficient conditions for exponential synchronization of chaotic systems with time-varying delays via intermittent control have been proposed. The obtained results remove the traditional assumption that the control width should be larger than the time delay. Moreover, the new synchronization criteria do not impose any restriction on the size of time delays. Two numerical simulations are given to show the effectiveness of the theoretical results.

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