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Stochastic resonance in a bias monostable system driven by a periodic rectangular signal and uncorrelated noises

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Abstract The stochastic resonance in a bias monostable system driven by a periodic rectangular signal and uncorrelated noises is investigated by using the theory of signal-to-noise (SNR) in the adiabatic limit. The analytic expression of the SNR is obtained for arbitrary signal amplitude without being restricted to small amplitudes. The SNR is a nonmonotonic function of intensities of multiplicative and additive noises and the noise intensity ratio $R = D/Q$, so stochastic resonance exhibits in the bias monostable system. We investigate the effect of any system parameter (such as D, Q, R, r) on the SNR. It is shown that the SNR is a nonmonotonic function of the static asymmetry r, also; the SNR is decreased when $|r|$ is increased. Moreover, the SNR is increased when the noise intensity ratio $R = D/Q$ is increased.

Keywords Stochastic resonance · Bias monostable system · Signal-to-noise ratio

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1 Introduction

Stochastic resonance (SR) is one of the most studied and utilized fundamental physical phenomenon and has already been observed in experimental studies. The original work on SR is mentioned by Benzi et al. [1] for explaining the periodic recurrences of the earth's ice ages. The study of stochastic resonance in bistable system with several periodic forces has attracted great attention [2–5]. Landa and McClitock [2] found the vibrational resonance in an over-damped bistable system only subject to two periodic fields. Gitterman [3] developed the theoretical results of a bistable oscillator driven by two periodic forces. Grigorenko et al. [5] investigated the response of a bistable system with a frequency mixing force. Strier et al. [6] presented an analytical study of the enhancement of the signal-to-noise ratio (SNR) in a monostable non-harmonic potential. They made use of the exact expression for the diffusion for propagator obtained in a previous work, and found a monotonically increasing response with the noise amplitude. For the first time, they provided a cut-off to such an increase, which prevents a probability leakage out of the system. Conventional SR is a nonlinear effect that accounts for the optimum response of a dynamical system to an external force at certain noise intensity. The SR in a broad sense means the nonmonotonic behavior of the output signal as a function of some characteristics of the noise (noise intensity or noise correlation time) or of a periodic force (amplitude or frequency).

There are a lot of monostable systems [7–15] in actual systems, including chemical, electronic, physical, and biological systems. These systems have been widely studied in theory and experiment. Dykman et al. [7] and Evstigneev et al. [9] investigated the SR in a monostable over-damped system based on linear response theory. Stocks et al. investigated the zero-dispersion stochastic resonance in a monostable system [14, 15], for which the dependence of eigenfrequency upon energy has an extremum. It is well known that multiplicative noise is more familiar in real physical systems and often plays a different role on the output of a system, with respect to the additive noise. Therefore, the investigation of the response of a monostable system driven by multiplicative noise is of great significance. In this paper, based on adiabatic approximation theory, we study the SR in a bias monostable system driven by a periodic rectangular signal and uncorrelated noises.

2 The monostable system and its signal-to-noise ratio

Consider an over-damped monostable system [8] driven by a periodic rectangular signal and uncorrelated noises described by the following Langevin equation:

$$\dot{x} = -ax^3 + x\xi(t) + \eta(t) + AG(t) + r, \tag{1}$$

where $a > 0$ is a system parameter, and r is a constant force, denoting the bias of the monostable system. The multiplicative colored noise $\xi(t)$ and additive white noise $\eta(t)$ are uncorrelated with the following statistical properties:

$$\begin{aligned} &\langle\xi(t)\rangle = \langle\eta(t)\rangle = 0, \\ &\langle\xi(t)\xi(t')\rangle = 2D\delta(t-t'), \\ &\langle\eta(t)\eta(t')\rangle = 2Q\delta(t-t'), \\ &\langle\xi(t)\eta(t')\rangle = \langle\eta(t)\xi(t')\rangle = 0. \end{aligned} \tag{2}$$

Here, D and Q describe the intensity of multiplicative and additive noise.

The periodic rectangular signal $G(t)$ with period T is given by

$$G(t+T) = G(t) = \begin{cases} 1, & 0 < t \le T/2, \\ -1, & T/2 < t \le T. \end{cases} \tag{3}$$

According to (1) and (2), the corresponding Fokker–Plank equation of the monostable system, (1) can be written as

$$\begin{aligned} \frac{\partial\rho(x,t)}{\partial t} &= -\frac{\partial}{\partial t}\big[F(x,t)\rho(x,t)\big] \\ &\quad + \frac{\partial^2}{\partial x^2}\big[B(x)\rho(x,t)\big], \end{aligned} \tag{4}$$

where

$$\begin{aligned} &F(x,t) = Dx - ax^3 + AG(t) + r, \\ &B(x) = Dx^2 + Q. \end{aligned} \tag{5}$$

We assume that external force is so small that there is enough time to reach the local equilibrium during the period of external force, i.e., we make the assumption that the system satisfies the adiabatic approximation condition [16]. The asymptotic long-time distribution function can be derived from (3) and (4) in the adiabatic limit, i.e.,

$$\rho(x,t) = \frac{N}{[B(x)]^{\frac{1}{2}}}\exp\left[-\frac{V(x)}{D}\right], \tag{6}$$

where N is normalization constant, and $V(x)$ is the rectified potential function, which has the form

$$\begin{aligned} V(x) &= \int_{-\infty}^{x}\frac{D(-U'(x) + AG(t) + r)}{B(x)}\,dx \\ &= -\frac{a}{2}x^2 + \frac{D^2 + aQ}{2D}\ln\big(Dx^2 + Q\big) \\ &\quad + \sqrt{D/Q}\big(r + AG(t)\big)\arctan\big(\sqrt{D/Q}\big), \end{aligned} \tag{7}$$

with $U'(x) = \frac{dU}{dx} = ax^3 - Dx$.

From (6) and (7), one can see that, for the case of $D \neq 0$, i.e., in the presence of multiplicative noise, the monostable system (1) can thus be regarded as an equivalent bistable system, i.e., corresponding to the so-called two-state model [16], with $x_u = 0$ and $x_\pm = \pm\sqrt{D/a}$ being the unstable and stable states of the equivalent bistable system. Under the adiabatic limit condition, the transition rates out of $x_\pm$ can be obtained by

$$\begin{aligned} W_\pm(t) &= \frac{|[U''(x_u)U''(x_\pm)]|^{\frac{1}{2}}}{2\pi} \\ &\quad \times \exp\left[-\frac{V(x_u) - V(x_\pm)}{D}\right] \\ &= N_0\exp\big[\mp p \mp qG(t)\big], \end{aligned} \tag{8}$$

where $N_{\pm 0}$ denotes the characteristic switching frequency of the equivalent bistable system when it is only driven by multiplicative and additive noise, which is given

$$N_0 = \frac{D}{\sqrt{2\pi}} \exp\left[-\frac{(D+a/R)\ln(DR/a+1)-D}{2D}\right], \tag{9}$$

with

$$\begin{aligned} p &= \frac{r\sqrt{R}\arctan(\sqrt{DR/a})}{D}, \\ q &= \frac{A\sqrt{R}\arctan(\sqrt{DR/a})}{D}, \\ R &= \frac{D}{Q}. \end{aligned} \tag{10}$$

The occupation probabilities $n_\pm$ of the equivalent bistable system satisfy the following master equation:

$$\begin{aligned} \frac{dn_+}{dt} &= W_+ p_+ + W_- p_-, \\ \frac{dp_-}{dt} &= W_+ p_+ - W_- p_-. \end{aligned} \tag{11}$$

Under the adiabatic limit, the timescale for the transients between the two stable states is much longer than the timescale of intrawall relaxation, then the solution of (11) is given as

$$\begin{aligned} n_+(t) &= \frac{W_-(t)}{W_-(t)+W_+(t)}, \\ n_-(t) &= \frac{W_+(t)}{W_-(t)+W_+(t)}. \end{aligned} \tag{12}$$

The following formula is valid for the arbitrary function f,

$$\begin{aligned} f\left[\alpha+\beta G(t)\right] = \frac{1}{2}\big[& f(\alpha+\beta)+f(\alpha-\beta) \\ & + G(t)\big(f(\alpha+\beta)-f(\alpha-\beta)\big)\big]. \end{aligned} \tag{13}$$

By using formula (13) and combing with the method in [17, 18], the expression of the correlation function has the following form:

$$K(t) = B_2(p,q)\langle G(t)G(0)\rangle + C(p,q)\delta(t), \tag{14}$$

where

$$\begin{aligned} \langle G(t)G(0)\rangle &= \frac{4}{\pi^2}\sum_{k=0}^{\infty}(2k+1)^{-2} \\ &\quad \times \exp\left[-i(2k+1)\Omega t\right], \\ B_2(P,Q) &= \frac{D}{4a}\big(\tanh(p+q)-\tanh(p-q)\big)^2, \\ C(p,q) = S_1(0) &= \frac{D}{2aN_0}\left(\frac{1}{\cosh^3(p+q)}\right. \\ &\quad \left. + \frac{1}{\cosh^3(p-q)}\right), \\ \Omega &= \frac{2\pi}{T}. \end{aligned} \tag{15}$$

By using the Fourier transform of the autocorrelation function, we can get the expression of the power spectrum

$$S(\omega) = S_1(0) + S_2(\omega), \tag{16}$$

where

$$\begin{aligned} S_1(0) &= C(p,q), \\ S_2(\omega) &= B_2(p,q)\frac{8}{\pi}\sum_{k=0}^{\infty}(2k+1)^{-2} \\ &\quad \times \delta\left[\omega-(2k+1)\Omega\right]. \end{aligned} \tag{17}$$

So the SNR can be defined as

$$SNR = \frac{8}{\pi}\frac{B_2(p,q)}{C(p,q)}. \tag{18}$$

The formula (18) is derived in the adiabatic limit under the condition $q \ll 1$ is satisfied, so the $B_2(P,Q)$ and $C(p,q)$ can be approximated as follows:

$$\begin{aligned} B_2(p,q) &= \frac{D}{4a}\left(\frac{\sinh(p+q)}{\cosh(p+q)} - \frac{\sinh(p-q)}{\cosh(p-q)}\right)^2 \\ &= \frac{D}{4a}\left(\frac{\sinh(2q)}{\cosh(p+q)\cosh(p-q)}\right)^2 \\ &\approx \frac{D}{4a}\left(\frac{2q}{\cosh^2(p)}\right)^2, \end{aligned} \tag{19}$$

$$C(p,q) \approx \frac{D}{aN_0}\frac{1}{\cosh^3(p+q)}. \tag{20}$$

Then the SNR can be simplified as

$$SNR = \frac{8}{\pi}\frac{N_0 q^2}{\cosh(p)}, \tag{21}$$

where the p and q have been defined earlier.

3 Discussion and conclusion

We discuss the influences of the multiplicative noise intensity D, additive Gaussian white noise intensity Q, noise intensity ratio R, and static asymmetry r on the SNR using (18). For simplicity, we plot the curves in Figs. 1–4 in the case of $r > 0$.

The effects of the multiplicative noise intensity D on the SNR with different values of r are illustrated in Fig. 1. There is a single peak in the curves and SR appears. The peak becomes lower and the position of the peak shifts to the left with increase of the static asymmetry r.

In Fig. 2, the curve of SNR versus the additive Gaussian white noise intensity Q with different values of r exhibits a maximum and SR exists for this

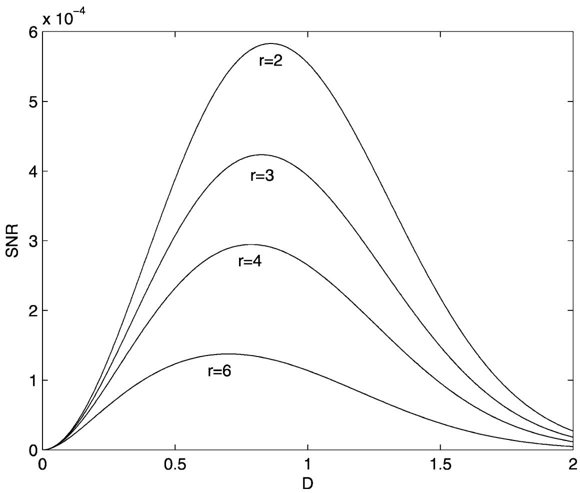

Fig. 1 Signal-to-noise ratio SNR as a function of multiplicative noise intensity D with different values of the parameter r for $a = 1$, $A = 0.1$, $Q = 2$

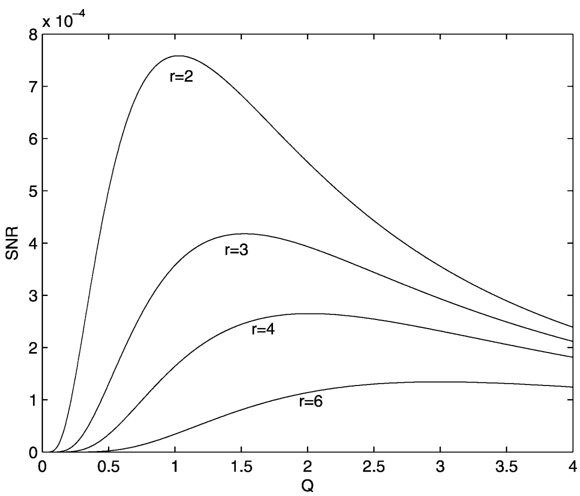

Fig. 2 Signal-to-noise ratio SNR as a function of additive noise intensity Q with different values of the parameter r for $a = 1$, $A = 0.1$, $D = 1$

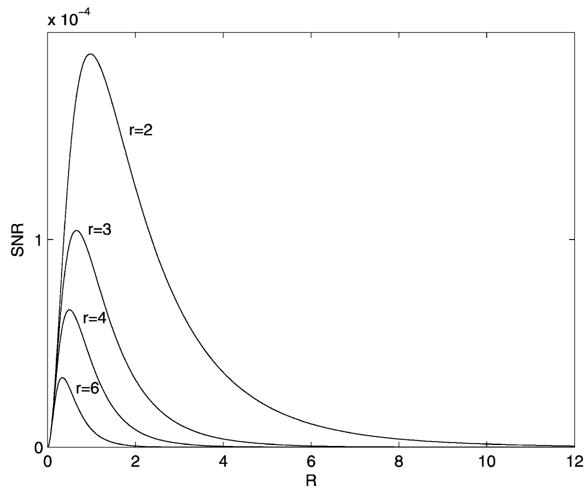

Fig. 3 Signal-to-noise ratio SNR as a function of noise ratio R with different values of the parameter r for $a = 1$, $A = 0.05$, $D = 1$

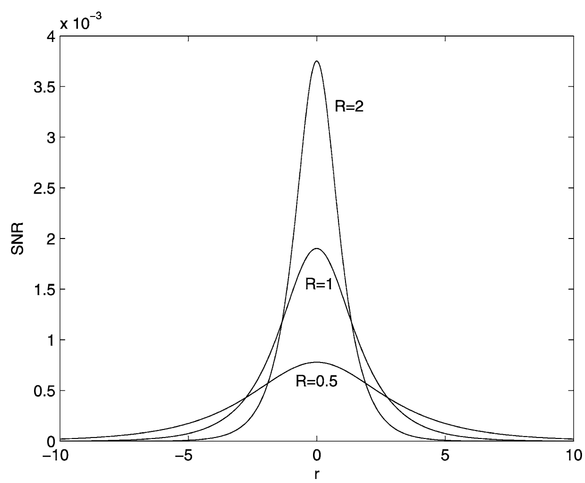

Fig. 4 Signal-to-noise ratio SNR as a function of the static asymmetry r with different values of the parameter R for $a = 1$, $A = 0.1$, $D = 1$

case. The position of the peak shifts to the right with the increase r. From Fig. 1 and Fig. 2, we can see the effects of the multiplicative noise intensity D and the additive noise intensity Q are different on the SR. The SR appears when Q is larger but D is smaller.

In Fig. 3, the SNR is shown as a function of the noise intensity ratio $R = D/Q$ for different values of r. The curve exhibits a pronounced single peak and SR appears on the R-SNR parameter plane. When asymmetry r is increased, the peak becomes lower and the position of the peak moves to the left, which is consistent with the result of [19].

The effects of the static asymmetry on the SNR is given in Fig. 4. We can see the SNR is decreased when $|r|$ is increased. Which is consistent with the result of [18, 19]. Meanwhile, we can see the SNR is increased when the noise intensity ratio $R = D/Q$ is increased.

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