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Motion of a gas bubble in fluid under vibration

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Abstract This paper is concerned with the analysis of motion of a gas bubble in a uniformly oscillating incompressible fluid. A theoretical model explaining the effect of sinking of gas bubbles in the absence of a standing pressure wave is validated experimentally. The conditions under which this effect occurs are determined, and a simple formula is derived for the average velocity of a gas bubble in the fluid.

Keywords Compressible bubble · Sinking conditions · Incompressible viscous fluid · Uniform oscillations · Average velocity

Introduction

The motion of a gas bubble in an oscillating fluid is of essential interest for flotation theory and for a whole number of other technological processes.

In papers [1–5] it was shown that gas bubbles can sink and heavy particles can rise in the standing wave field in a fluid. An overview of these papers and a detailed description of the authors' own results are given in the monograph [5]. The solution of the problem for nondeformable particle using the concept of vibrational mechanics and the method of direct separation of motions was provided in paper [4]. According to this work, a necessary condition for the arising of the effects is the presence of a standing wave, which for corresponding relatively low excitation frequencies is possible only in a gas-saturated fluid. The effect of gas bubbles sinking in such a fluid was experimentally observed and described in papers [4, 6].

Papers [7–10] are devoted to the analysis of motion of a solid and deformable particle in a uniformly oscillating fluid. It was shown that the average velocity of "heavy" particle sinking and gas bubble rising can significantly reduce due to the nonlinear character of the resistance force at large Reynolds numbers. This fact is confirmed by experiments [11, 12].

The motion of small ("before resonant") bubbles in an oscillating incompressible fluid was examined by Bleich [13], who obtained an expression for the critical depth, which does not allow for fluid viscosity, and experimentally confirmed the possibility of gas bubble sinking starting from this depth. However, Bleich did not solve the rather complicated differential equation of gas bubble motion considering resistance forces, and so he did not obtain an expression for its average velocity; the experimental part of the research is also rather partial.

The present study appreciably supplements results of the papers [13] and [4]. The supposition that the effect of gas bubble sinking in a vertically oscillating fluid-filled volume can occur in the absence of the standing wave, which is conditional on the gas–fluid medium compressibility, i.e. in a fluid without

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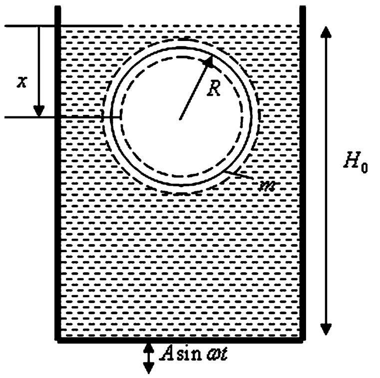

Fig. 1 Model of a bubble in a fluid

other bubbles, is confirmed by theoretical and experimental research. In this case sinking takes place due to the deformability of the bubble under the action of fast-varying pressure caused by oscillations and proportional to the mass of the fluid column placed above the bubble. An approximate expression for the average velocity of gas bubble motion in a fluid, which considerably depends on the depth of its submergence and on vibration parameters, is also derived in this paper.

Thereby, at present two different mechanisms, two explanations of the effect of gas bubbles sinking in a vertically oscillating fluid-filled volume were proposed: "wave" and "nonwave," "vibrational." For the first explanation a key factor is the gradient of the wave amplitude, while for the second it is the compressibility of the bubble. In both cases gas bubble sinking is possible only starting from some critical depth X_0; the presence of gas bubbles at this depth can be caused by emergence of the turbulent fluid-gas layer [14] near the fluid surface. The relatively complicated joint consideration of these mechanisms is the subject of future analysis.

1 Model system

The motion of a bubble in a harmonically oscillating with amplitude A and frequency ω fluid-filled volume is analyzed (Fig. 1). The fluid is considered to be incompressible and oscillating uniformly. The height of the fluid column is designated as H_0.

In our analysis we take into account bubble volume pulsations caused by the influence of the varying pressure, which is exerted on the bubble due to the fluid oscillations. These pulsations are considered to be isothermal and quasistatic; i.e., the following condition is fulfilled:

$$P_t V_b = P_e V_{b0} = \text{const} \tag{1}$$

where V_b is the current volume of the bubble, V_{b0} is the volume of the bubble near the free surface of the fluid, P_e is the external pressure, which in the simplest case is equal to the atmospheric pressure P_0,

$$P_t = P_e + \rho x\left(g + A\omega^2 \sin\omega t\right) \tag{2}$$

is the current value of the fluid pressure exerted on the bubble, x is the coordinate of the center of the bubble, counted off from the free surface, ρ is the density of the fluid, and g is gravitational acceleration. For the fulfillment of the assumption that bubble volume pulsations are quasistatic, it is necessary for the frequency ω to be sufficiently smaller (practically three times smaller) than the frequency λ of its free radial oscillations. This supposition is fulfilled for bubbles with radius $R < 2$ cm at frequencies $\omega < 280$ 1/c [4] (see Sect. 5).

From equalities (1) and (2) we obtain

$$V_b = \frac{P_e V_{b0}}{P_e + \rho x(g + A\omega^2 \sin\omega t)} \tag{3}$$

The following inequality is assumed to hold:

$$\left(\frac{\rho H_0(g + A\omega^2)}{P_e}\right)^2 \ll 1 \tag{4}$$

i.e., the value of the external pressure is much greater than the sum of the hydrostatic pressure and the pressure caused by inertial forces at the bottom of the volume $x = H_0$. For example, for $P_e = P_0 = 10^5$ Pa and $A\omega^2 = 14$ g, $H_0 = 0.2$ m we obtain $\rho H_0(g + A\omega^2) = 0.2 \cdot 15 \cdot 10^4 = 3 \cdot 10^4$ Pa, and $(\rho H_0(g + A\omega^2)/P_e)^2 = 0.09$. We note that assumption (4) is optional, though it simplifies the analytical research significantly; in numerical experiments it can be omitted.

So expression (3) can be written in the form

$$V_b = \frac{V_{b0}}{1 + \frac{\rho x}{P_e}(g + A\omega^2 \sin\omega t)} \approx V_{b0}\left[1 - \frac{\rho x g}{P_e}\left(1 + \frac{A\omega^2}{g}\sin\omega t\right)\right] \tag{5}$$

Introducing the nondimensional small parameter

$$\gamma = \frac{\rho H_0 g}{P_e} \ll 1 \tag{6}$$

and load coefficient

$$w = \frac{A\omega^2}{g} \tag{7}$$

expression (5) for the current volume of the bubble transforms to the form

$$V_\mathrm{b} = V_{\mathrm{b}0}\left(1 - \gamma\frac{x}{H_0} - \gamma\frac{x}{H_0} w \sin\omega t\right) \tag{8}$$

In the present paper the motion of the bubble is analyzed in the presence of the high intensity vibration $w \gg 1$.

Due to the employed assumption (4), the fluid pressure at the bottom of the volume is always positive

$$P_d = P_\mathrm{e} + \rho H_0\big(g + A\omega^2 \sin\omega t\big) > 0 \tag{9}$$

i.e., cavitations cannot occur.

The following expression for instantaneous radius of the bubble can be derived from equality (8), with correlation (6) taken into account:

$$R = R_0\left(1 - \frac{\gamma}{3}\frac{x}{H_0} - \frac{\gamma}{3}\frac{x}{H_0} w \sin\omega t\right) \tag{10}$$

where $R_0 = \sqrt[3]{\frac{3}{4\pi}V_{\mathrm{b}0}}$ is the radius of the bubble near the free surface of the fluid.

2 Governing equation

The equation of the gas bubble motion has the following form:

$$(m + m_0)\ddot{x} + \dot{m}_0\dot{x} = -F(\dot{x}) + (m - \rho V_\mathrm{b})\big(A\omega^2 \sin\omega t + g\big) \tag{11}$$

Here m is the mass of the bubble (particles attached to the bubble), which is usually much smaller than the mass of the fluid in its volume. When this condition is fulfilled, the Archimedes force is much greater than the weight of the bubble (particles attached to the bubble), so it rises in the absence of vibration. m_0 is the added mass of the fluid, defined by the formula $m_0 = \chi V_\mathrm{b}\rho$, where χ is an added mass coefficient, with the magnitude of $\chi = 1/2$.

In (11) the term $\dot{m}_0\dot{x}$ corresponds to the additional force caused by added mass variations [21]. $F(\dot{x})$ designates the resistance force to the gas bubble motion. In the general case this force is defined by the following expression [11, 12, 16]:

$$F(\dot{x}) = 4\rho R^2 \Psi(\mathrm{Re})\dot{x}^2 \operatorname{sgn}\dot{x} \tag{12}$$

Here, as above, ρ is the density of the fluid, R is the radius of the bubble, and $\Psi(\mathrm{Re})$ is the resistance coefficient, whose dependence on the Reynolds number

$$\mathrm{Re} = 2\rho R V/\mu \tag{13}$$

is given by the classical Rayleigh diagram [7, 12, 16]. In expression (13) μ is the dynamic viscosity of the fluid, and V is the velocity of the gas bubble motion. Formulas (12) and (13), strictly speaking, correspond to the case of motion with constant velocity. However in problems similar to that concerned here, the velocity $\dot{x}$ varies significantly during one period of oscillation. So, for such problems questions about the applicability of expression (12), and about the magnitude of the velocity V, which should be used to determine the Reynolds number, are stated (see, for example, [17–19]). As concerns expression (12), it is employed frequently in the case of variable velocity $\dot{x}$ (the stationarity hypothesis), sometimes (in the case of large Reynolds number Re) with the linear term being added [18]. There are no precise recommendations about the value of the velocity V in expression (13). In the present paper the velocity V means the period average velocity of the gas bubble motion. The corresponding Reynolds number is considered to be "large" (Re > 1000), but is less than the Reynolds number corresponding to the crisis of the flow around, $\mathrm{Re} \approx 2 \cdot 10^5$. Under this assumption, according to the Rayleigh diagram, the resistance coefficient can be considered to be approximately constant:

$$\Psi(\mathrm{Re}) \approx 0.2 \equiv \Psi_\infty$$

3 Solution by the method of direct separation of motions

3.1 Equations of "fast" and "slow" motions

Employing the expression (8), the equation of the gas bubble motion (11) can be written in the form

$$\big(D_0(x) + D_1(x)\sin\omega t\big)\ddot{x} + \big(E_0(\dot{x}) + E_1(\dot{x})\sin\omega t + G_1(x)\cos\omega t\big)\dot{x}$$

$$= -F(\dot{x}) + C_0(x) + C_1(x)\sin\omega t + C_2(x)\sin^2\omega t \qquad (14)$$

where

$$C_0(x) = \left[m - \rho V_{b0}\left(1 - \gamma\frac{x}{H_0}\right)\right]g$$

$$C_1(x) = \left[m - \rho V_{b0}\left(1 - 2\gamma\frac{x}{H_0}\right)\right]A\omega^2$$

$$C_2(x) = \rho V_{b0}\gamma\frac{x}{H_0}wA\omega^2$$

$$D_0(x) = m + \chi\rho V_{b0}\left(1 - \gamma\frac{x}{H_0}\right)$$

$$D_1(x) = -\gamma w\chi\rho V_{b0}\frac{x}{H_0}$$

$$E_0(\dot{x}) = -\gamma\chi\rho V_{b0}\frac{\dot{x}}{H_0}$$

$$E_1(\dot{x}) = -\gamma w\chi\rho V_{b0}\frac{\dot{x}}{H_0}$$

$$G_1(x) = -\gamma w\chi\rho V_{b0}\omega\frac{x}{H_0}$$

For the solution of the problem we use the concept of vibrational mechanics and the method of direct separation of motions [20], and we seek solutions to (14) in the form

$$x = X(t) + \psi(t,\tau)$$

where X is "slow" and ψ is "fast", 2π periodic in dimensionless ("fast") time, $\tau = \omega t$ variable, with average zero:

$$\big\langle\psi(t,\tau)\big\rangle = 0,$$

where for any $h = h(t,\tau)$, $T = 2\pi$ the periodic in τ, we define $\langle h(t,\tau)\rangle = \frac{1}{T}\int_0^T h\,d\tau$.

The following equations for the new variables X and ψ are obtained:

$$\begin{aligned}
&(m+m_{01})\ddot{X} - \gamma\cdot m_{01}\left(\frac{X}{H_0}\ddot{X} + \frac{\langle\psi\ddot{\psi}\rangle}{H_0}\right.\\
&\quad + w\left(\frac{X}{H_0}\langle\ddot{\psi}\sin\omega t\rangle\right.\\
&\quad \left.\left. + \frac{\langle\psi\sin\omega t\rangle}{H_0}\ddot{X} + \frac{\langle\ddot{\psi}\psi\sin\omega t\rangle}{H_0}\right)\right)\\
&\quad - \gamma\cdot m_{01}\left(\frac{\dot{X}^2 + \langle\dot{\psi}^2\rangle}{H_0} + 2\frac{\dot{X}}{H_0}w\langle\dot{\psi}\sin\omega t\rangle\right.\\
&\quad \left. + \frac{X}{H_0}w\omega\langle\dot{\psi}\cos\omega t\rangle + \frac{\dot{X}}{H_0}w\omega\langle\psi\cos\omega t\rangle\right)\\
&= -\big\langle F(\dot{X}+\dot{\psi})\big\rangle + (m - \rho V_{b0})g\\
&\quad + \gamma\cdot\rho V_{b0}\left(\frac{X}{H_0}g + \frac{A\omega^2}{H_0}\langle\psi\sin\omega t\rangle\right.\\
&\quad + w\left(\frac{X}{H_0}\frac{A\omega^2}{2} + \frac{\langle\psi\sin^2\omega t\rangle}{H_0}A\omega^2\right.\\
&\quad \left.\left. + \frac{\langle\psi\sin\omega t\rangle}{H_0}g\right)\right)
\end{aligned} \qquad (15)$$

and

$$\begin{aligned}
&(m+m_{01})\ddot{\psi} - \gamma\cdot m_{01}\left(\frac{X}{H_0}\ddot{\psi} + \frac{\psi}{H_0}\ddot{X}\right.\\
&\quad + \frac{\psi\ddot{\psi} - \langle\psi\ddot{\psi}\rangle}{H_0}\\
&\quad + w\left(\frac{X}{H_0}\big((\ddot{X}+\ddot{\psi})\sin\omega t - \langle\ddot{\psi}\sin\omega t\rangle\big)\right.\\
&\quad + \ddot{X}\left(\frac{\psi\sin\omega t}{H_0} - \frac{\langle\psi\sin\omega t\rangle}{H_0}\right)\\
&\quad \left.\left. + \frac{\psi\ddot{\psi}\sin\omega t}{H_0} - \frac{\langle\psi\ddot{\psi}\sin\omega t\rangle}{H_0}\right)\right)\\
&\quad - \gamma\cdot m_{01}\left(\frac{2\dot{X}}{H_0}\dot{\psi} + \frac{\dot{\psi}^2 - \langle\dot{\psi}^2\rangle}{H_0}\right.\\
&\quad + \frac{(\dot{X}+\dot{\psi})^2}{H_0}w\sin\omega t - 2\frac{\dot{X}}{H_0}w\langle\dot{\psi}\sin\omega t\rangle\\
&\quad + \frac{(X+\psi)}{H_0}(\dot{X}+\dot{\psi})w\omega\cos\omega t\\
&\quad \left. - \frac{X}{H_0}w\omega\langle\dot{\psi}\cos\omega t\rangle - \frac{\dot{X}}{H_0}w\omega\langle\psi\cos\omega t\rangle\right)\\
&= -F(\dot{X}+\dot{\psi}) + \big\langle F(\dot{X}+\dot{\psi})\big\rangle\\
&\quad + (m - \rho V_{b0})A\omega^2\sin\omega t\\
&\quad + \gamma\cdot\rho V_{b0}\left(\frac{X}{H_0}A\omega^2\sin\omega t + \frac{\psi}{H_0}g\right.\\
&\quad + \frac{A\omega^2}{H_0}\big(\psi\sin\omega t - \langle\psi\sin\omega t\rangle\big)\\
&\quad + w\left(\frac{X+\psi}{H_0}\sin\omega t\big(A\omega^2\sin\omega t + g\big)\right.
\end{aligned}$$

$$- \frac{X}{H_0}\frac{A\omega^2}{2} - \frac{\langle \psi \sin^2 \omega t\rangle}{H_0} A\omega^2 - \frac{\langle \psi \sin \omega t\rangle}{H_0} g \Big)\Big) \tag{16}$$

where $m_{01} = \chi \rho V_{b0}$ is the added mass of the fluid near its free surface.

The equation of "slow" motions (15) is simplified using correlation (6), inequalities $w \gg 1$, $\frac{A}{H_0} \ll 1$, $\frac{\psi}{H_0} \ll 1$ and neglecting relatively small terms. So this equation is written in the form

$$\begin{aligned}(m + m_{01})\ddot{X} &- \gamma \cdot m_{01} w \frac{X}{H_0}\big(\langle \ddot{\psi} \sin \omega t\rangle \\ &+ \omega\langle \dot{\psi} \cos \omega t\rangle\big) \\ &= -\big\langle F(\dot{X} + \dot{\psi})\big\rangle + (m - \rho V_{b0})g \\ &+ \gamma \cdot \rho V_{b0} w \frac{X}{H_0}\frac{A\omega^2}{2}\end{aligned} \tag{17}$$

The essential advantage of the method of direct separation of motions is the possibility to solve equations of "fast" motions in an approximate manner, if, as is typical, the equation of "slow" motion is the one of primary interest. In the considered case all terms with the factor $\gamma \cdot w$(which is assumed to be much less than unity) can be neglected, because they are small in comparison with $(m - \rho V_b)A\omega^2 \sin \omega t$. As a result, the equation of "fast" motions (16) is simplified:

$$\begin{aligned}(m + m_{01})\ddot{\psi} &= -F(\dot{X} + \dot{\psi}) + \big\langle F(\dot{X} + \dot{\psi})\big\rangle \\ &+ (m - \rho V_{b0})A\omega^2 \sin \omega t\end{aligned} \tag{18}$$

As was noted above, in this paper we consider the case of high intensity vibration $w \gg 1$. Thus, we assume that the condition $|\dot{\psi}| \gg |\dot{X}|$ holds; i.e., the "fast" component of the velocity is much greater than the "slow" component for almost all values of the "fast" time τ (verification of this assumption is provided in Sect. 5.1. For some values of $\tau \dot{\psi}$ is equal to zero). Hence, $\operatorname{sgn}(\dot{X} + \dot{\psi}) = \operatorname{sgn}\dot{\psi}$; from the expression (12) with correlation (10) taken into account we obtain (all terms of higher order of smallness were neglected)

$$\begin{aligned}-F(\dot{X} + \dot{\psi}) &= -4\rho R_0^2 \Psi_\infty \bigg(\dot{\psi}^2 + 2\dot{X}\dot{\psi} \\ &- \frac{2}{3}\gamma \frac{X}{H_0} w \dot{\psi}^2 \sin \omega t\bigg) \operatorname{sgn}\dot{\psi}\end{aligned} \tag{19}$$

In expression (19) the first term is much greater than the rest ($|\dot{\psi}| \gg |\dot{X}|$ and $\gamma \cdot w \ll 1$). So, using this expression and neglecting relatively small terms, we obtain the equation of "fast" motions in the form

$$\begin{aligned}(m + m_{01})\ddot{\psi} &= -4\rho R_0^2 \Psi_\infty \big(\dot{\psi}^2 \operatorname{sgn}\dot{\psi} - \big\langle \dot{\psi}^2 \operatorname{sgn}\dot{\psi}\big\rangle\big) \\ &+ (m - \rho V_{b0})A\omega^2 \sin \omega t\end{aligned} \tag{20}$$

Note that in the considered problem the resistance force should be taken into account when solving the equation of "fast" motions, because it considerably affects the total results.

3.2 The solution of the equation of "fast" motions

An approximate solution of the equation of "fast" motions (20) is sought by the method of harmonic balance in the form

$$\psi = B \sin(\omega t + \varphi) \tag{21}$$

where B and φ are constants. Hence, we have

$$\big\langle \dot{\psi}^2 \operatorname{sgn}\dot{\psi}\big\rangle = 0$$

Placing expression (21) into (20), we multiply it by $\sin(\omega t + \varphi)$, and then by $\cos(\omega t + \varphi)$. Integrating over period $T = 2\pi/\omega$, we obtain equalities

$$-\frac{(m + m_{01})B}{2} = (m - \rho V_{b0})A\frac{\cos \varphi}{2} \tag{22}$$

$$0 = -\frac{16}{3\pi}\rho R_0^2 \Psi_\infty B^2 - (m - \rho V_{b0})A\frac{\sin \varphi}{2} \tag{23}$$

As a result, the amplitude B of bubble "fast" oscillations can be determined from the equation

$$\begin{aligned}&\left(\frac{16}{3\pi}\right)^2 \rho^2 R_0^4 \Psi_\infty^2 B^4 + \frac{(m + m_{01})^2}{4} B^2 \\ &- \frac{(m - \rho V_{b0})^2}{4} A^2 = 0\end{aligned} \tag{24}$$

the positive solution of which has the form

$$B^2 = \frac{2(m - \rho V_{b0})^2 A^2}{(m + m_{01})^2 + \sqrt{(m + m_{01})^4 + \frac{16^3}{9\pi^2}(m - \rho V_{b0})^2 \rho^2 R_0^4 \Psi_\infty^2 A^2}} \tag{25}$$

Employing the condition $m \ll \rho V_{b0}$, expression (25) can be transformed into the form

$$B^2 = \frac{2A^2}{\chi^2 + \sqrt{\chi^4 + \frac{16^2}{\pi^4}\Psi_\infty^2 \frac{A^2}{R_0^2}}} \tag{26}$$

Taking into account correlation (25), the following expression for $\sin\varphi$ can be derived from (23):

$$\sin\varphi = \frac{64}{3\pi}\frac{\rho R_0^2 \Psi_\infty A}{\rho V_{b0} - m} \cdot \frac{1}{(\frac{m+m_{01}}{\rho V_{b0}-m})^2 + \sqrt{(\frac{m+m_{01}}{\rho V_{b0}-m})^4 + (\frac{64}{3\pi}\frac{\rho R_0^2 \Psi_\infty A}{\rho V_{b0}-m})^2}} \tag{27}$$

which for $m \ll \rho V_{b0}$ is simplified as follows:

$$\sin\varphi = \frac{16}{\pi^2}\frac{\Psi_\infty}{\chi^2}\frac{A}{R_0} \cdot \frac{1}{1 + \sqrt{1 + (\frac{16}{\pi^2}\frac{\Psi_\infty}{\chi^2}\frac{A}{R_0})^2}} \tag{28}$$

To compose the equation of "slow" motions (17) it is necessary to define expression $\langle \ddot{\psi} \sin\omega t\rangle$ and expression $\langle \dot{\psi} \cos\omega t\rangle$. But $\langle \ddot{\psi} \sin\omega t\rangle = -\omega^2 B \frac{\cos\varphi}{2}$, and $\langle \dot{\psi} \cos\omega t\rangle = \omega B \frac{\cos\varphi}{2}$. Hence, we obtain

$$\langle \ddot{\psi} \sin\omega t\rangle + \omega\langle \dot{\psi} \cos\omega t\rangle = 0 \tag{29}$$

3.3 Determination of the effective resistance force to gas bubble motion

Employing correlation (29), we obtain the equation of "slow" motions (17) in the form

$$(m + m_{01})\ddot{X} + \langle F(\dot{X} + \dot{\psi})\rangle = \gamma \cdot \rho V_{b0} w \frac{X}{H_0}\frac{A\omega^2}{2} - (\rho V_{b0} - m) g \tag{30}$$

To define the range of parameters at which the gas bubble will sink in the fluid, it is necessary to determine the period $T = 2\pi/\omega$ average value of the resistance force $\langle F(\dot{X} + \dot{\psi})\rangle$. Using correlation (19) and taking into account that $\langle \dot{\psi}^2 \mathrm{sgn}\dot{\psi}\rangle = 0$, the following expression for this value is obtained:

$$\langle F(\dot{X} + \dot{\psi})\rangle = 4\rho\Psi_\infty R_0^2 \Big(2\dot{X}\langle \dot{\psi}\,\mathrm{sgn}\dot{\psi}\rangle - \frac{2}{3}\gamma\frac{X}{H_0} w \langle \dot{\psi}^2 \mathrm{sgn}\dot{\psi}\,\sin\omega t\rangle\Big) \tag{31}$$

Employing solution (21) of the equation of "fast" motions, we get

$$\langle \dot{\psi}\,\mathrm{sgn}\dot{\psi}\rangle = \frac{2}{\pi}B\omega$$
$$\langle \dot{\psi}^2 \mathrm{sgn}\dot{\psi}\,\sin\omega t\rangle = -\frac{4\sin\varphi}{3\pi}B^2\omega^2 \tag{32}$$

Thereby, expression (31) transforms into the form

$$\langle F(\dot{X} + \dot{\psi})\rangle = \frac{16}{\pi}\rho\Psi_\infty R_0^2 \dot{X} B\omega + \frac{32\Psi_\infty}{9\pi}\rho R_0^2 \gamma w \frac{X}{H_0} B^2\omega^2 \sin\varphi \tag{33}$$

The presence of the second term in this expression is conditioned on the compressibility of the bubble; i.e., if we did not take compressibility into account, then we would obtain the following expression for the average value of the resistance force:

$$\langle F(\dot{X} + \dot{\psi})\rangle_n = \frac{16}{\pi}\rho\Psi_\infty R_0^2 \dot{X} B\omega \tag{34}$$

The second term in expression (33) can be transformed using correlation (23):

$$\langle F(\dot{X} + \dot{\psi})\rangle = \langle F(\dot{X} + \dot{\psi})\rangle_n + \gamma w^2 (\rho V_{b0} - m) g \frac{X}{H_0}\frac{\sin^2\varphi}{3} \tag{35}$$

where $\sin\varphi$ is defined by formula (27).

3.4 The condition of gas bubble sinking

Employing expression (35) for the period average value of the resistance force, the equation of "slow" motions (30) can be written in the form

$$(m + m_{01})\ddot{X} + \langle F(\dot{X} + \dot{\psi})\rangle_n = \gamma \cdot w^2 \frac{\rho V_{b0} g}{2}\frac{X}{H_0}\Big(1 - \frac{2}{3}\Big(1 - \frac{m}{\rho V_{b0}}\Big)\sin^2\varphi\Big) - (\rho V_{b0} - m) g \tag{36}$$

We transform this equation, using expression (28) and taking into account that $m \ll \rho V_{b0}$:

$$m_{01}\ddot{X} + \langle F(\dot{X} + \dot{\psi})\rangle_n = \gamma \cdot w^2 \frac{X}{H_0}\frac{\rho V_{b0} g}{2} \times \left(1 - \frac{2}{3}\frac{\theta\frac{A^2}{R_0^2}}{2(1 + \sqrt{1 + \theta\frac{A^2}{R_0^2}}) + \theta\frac{A^2}{R_0^2}}\right) - \rho V_{b0} g \tag{37}$$

Here the coefficient $\theta = \frac{16^2}{\pi^4}\frac{\Psi_\infty^2}{\chi^4} \equiv 1.68$. (We recall that $\chi = 1/2$, and $\Psi_\infty = 0.2$.)

The range of parameters at which the gas bubble sinks (or rises) in the fluid can be determined from the form of (37). Indeed, if the expression written on the right side of this equation has a positive value, then the bubble sinks after a certain period of time, which depends on its initial velocity; if it is negative, then the bubble rises. So the condition of gas bubble sinking can be written in the form

$$\gamma \cdot w^2 \frac{X}{H_0} \frac{1}{2}\left(1 - \frac{2}{3} \frac{\theta \frac{A^2}{R_0^2}}{2(1+\sqrt{1+\theta \frac{A^2}{R_0^2}}) + \theta \frac{A^2}{R_0^2}}\right) > 1 \tag{38}$$

The "slow" variable X is presented on the left side of inequality (38), so the fulfillment of the condition of gas bubble sinking depends on its current position in the fluid. In other words, if the bubble is situated in the position with coordinate $X > X_0$, then it will sink; if it is in the position $X < X_0$, then it will rise. That is, the gas bubble sinking in fluid occurs only from a certain depth X_0. The value of X_0 is determined from the expression

$$X_0 = \frac{2H_0}{\gamma \cdot w^2} \cdot \frac{2(1+\sqrt{1+\theta \frac{A^2}{R_0^2}}) + \theta \frac{A^2}{R_0^2}}{2(1+\sqrt{1+\theta \frac{A^2}{R_0^2}}) + \frac{\theta}{3} \frac{A^2}{R_0^2}} \tag{39}$$

from which we can see that for any magnitudes of parameters X_0 is positive. If $X_0 > H_0$, then gas bubbles rise throughout the whole volume. The case when $X_0 < H_0$ seems to be the most interesting—bubbles situated lower than a certain level sink; those situated higher rise.

For example, when the external pressure P_e is equal to the atmospheric pressure $P_e = P_0 = 10^5$ Pa and the height of the fluid column in the volume is equal to $H_0 = 0.16$ m, the small parameter γ is equal to $\gamma = 0.016$. The volume of the bubble is considered to be equal to $V_{b0} = 0.1 \cdot 10^{-6}$ m^3 (radius $R_0 = 2.9$ mm). The amplitude of the external force is assumed to be equal to $A = 6.5$ mm, and the frequency $\omega = 170$ 1/s. Then the load coefficient is equal to $w = 19$. Using expression (39) for X_0 we obtain $X_0 = 0.54H_0$; i.e., bubbles situated in the positions with coordinates $H_0 > X > 0.54H_0$ will sink in the fluid. We note that in this case condition (4) holds, because $(\rho H_0(g + A\omega^2)/P_e)^2 = 0.1$. For the frequency of the external excitation $\omega = 195$ 1/s and the same values of the rest parameters from expression (39), we obtain $X_0 = 0.31H_0$ (in this case $(\rho H_0(g + A\omega^2)/P_e)^2 = 0.17$).

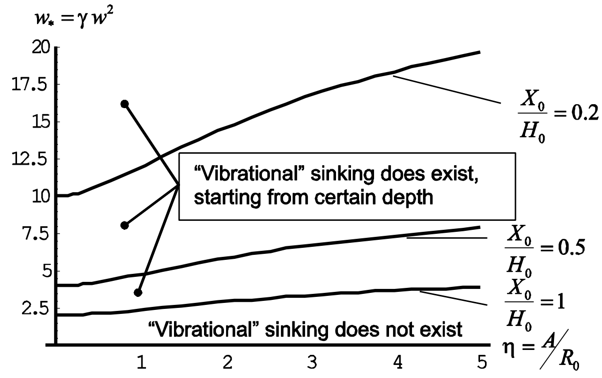

Fig. 2 Dependences of w_* on η for several magnitudes of the threshold depth

According to (39) the value of the ratio $\frac{X_0}{H_0}$ depends on two parameters: $w_* = \gamma \cdot w^2$, the characteristic load parameter, and $\eta = \frac{A}{R_0}$, the ratio of the excitation amplitude to the radius of the bubble. The dependences of w_* on η for several magnitudes of the ratio $\frac{X_0}{H_0}$, i.e., the threshold depths starting from which gas bubble sinking occurs, are shown in Fig. 2.

For the magnitudes of the parameters w_* and η from the area situated below the curve corresponding to $\frac{X_0}{H_0} = 1$, gas bubble sinking does not occur, and below the curve corresponding, for example, to $\frac{X_0}{H_0} = 0.5$ sinking occurs only for $X > 0.5H_0$, i.e., starting from half of the volume depth.

According to the analytical research, gas bubble sinking takes place due to its volume pulsations caused by the influence of the varying pressure, i.e., due to its deformability. We note that pressure pulsations, and therefore bubble volume pulsations, depend on the depth of its submergence x; i.e., the deeper the bubble, the greater the amplitudes of these pulsations. That is why gas bubble sinking occurs only from some certain depth, starting from which its volume pulsations become sufficient for the emergence of the effect.

4 The solution of the equation of "slow" motion

In this section we will find solutions of the equation of "slow" motion (37). Employing expression (34), (37) can be transformed into the form

$$m_{01}\ddot{X}+\frac{16}{\pi}\rho\Psi_\infty R_0^2\dot{X}B\omega$$
$$=\gamma\cdot w^2\frac{X}{H_0}\frac{\rho V_{b0}g}{2}$$
$$\times\left(1-\frac{2}{3}\frac{\theta\frac{A^2}{R_0^2}}{2(1+\sqrt{1+\theta\frac{A^2}{R_0^2}})+\theta\frac{A^2}{R_0^2}}\right)-\rho V_{b0}g \tag{40}$$

Assuming that the "slow" gas bubble acceleration $\ddot{X}$ is small, for the velocity of its "slow" motion we determine the following approximate expression:

$$\dot{X}\approx v\left(\frac{X}{X_0}-1\right) \tag{41}$$

where $v=\frac{\pi^2}{12\cdot\Psi_\infty}\frac{R_0}{B}\frac{g}{\omega}$, and X_0 is defined by expression (39). Actually, v is the absolute value of the "slow" levitation velocity of a rigid bubble; i.e., if we had not take into account pulsations of gas bubble volume, then we would obtain the value v for the "slow" velocity of its levitation. It may be concluded that the compressibility influence on the velocity of "slow" gas bubble motion is approximately reduced to the presence of the term $v\frac{X}{X_0}$ in expression (41) for this velocity.

As is seen from expression (41), the velocity of gas bubble sinking (or rising) depends on the coordinate X; for the critical value $X=X_0$ this velocity is equal to zero.

5 Discussion of the analytical results

5.1 Verification of the assumptions used in the analysis

Here we discuss the main assumptions employed in our analysis.

The condition that gas bubble volume pulsations are quasistatic, i.e., the condition of infinitesimality of the frequency ω in comparison with the frequency λ of its free radial vibrations, has the following form [15]:

$$\omega\ll\frac{1}{R_0}\sqrt{\frac{3P_e}{\rho}} \tag{42}$$

We consider the frequencies ratio $\frac{\omega}{\lambda}<\frac{1}{3}$ to be sufficient; then via characteristic parameters w_* and η condition (42) can be written in the form

$$\sqrt[4]{w_*}\cdot\sqrt{A}<K_0\cdot\eta, \tag{43}$$

where $K_0=\sqrt{\frac{P_e}{3\rho g}}\cdot\sqrt[4]{\gamma}$.

The assumption about the quasistatic nature is fulfilled, because even for bubbles with diameter 4 cm situated in water at external pressure equal to the atmospheric pressure, the value of the frequency of radial free vibrations λ exceeds 850 1/s. At the same time the considered excitation frequencies are less than 250 1/s, and the gas bubble diameters are less than 3 cm.

Solving the equation of gas bubble motion by the method of direct separation of motions, we have assumed that $|\dot{X}|$ is small compared with the amplitude of the velocity of gas bubble "fast" oscillations $B\omega$. Here we verify this assumption. The "slow" velocity of gas bubble motion $\dot{X}$ is defined by formula (41). So the ratio of the amplitude $B\omega$ to the absolute value of the "slow" velocity of the bubble is equal to

$$\xi=\frac{X_0B\omega}{v|X-X_0|}, \tag{44}$$

but $\frac{B\omega}{v}\gg 1$, therefore $\xi\gg 1$ for all magnitudes of X; i.e., the considered assumption really holds. For example, for the frequency $\omega=195$ 1/s and the same magnitudes of the rest of the parameters as used in Sect. 3, we obtain $\frac{B\omega}{v}=26.7$.

5.2 On the influence of gas bubble compressibility

As determined in Sect. 4, the influence of the compressibility of the bubble on the velocity of its "slow" motion is reduced to the term $v\frac{X}{X_0}$, where v is the absolute value of the velocity of rigid bubble levitation. When inequality $\frac{H_0}{X_0}\ll 1$ holds true, term $v\frac{X}{X_0}$ in expression (41) can be neglected, because in that case $v\frac{X}{X_0}\ll v$. Hence, one may conclude that the influence of gas bubble compressibility on its motion is weak when condition $\frac{H_0}{X_0}\ll 1$ holds true; as a matter of fact, in that case the bubble can be considered as nondeformable. We assume the value $\frac{H_0}{X_0}=0.1$ to be sufficient. Then the condition of gas bubble nondeformability, with expression (39) for X_0 taken into account,

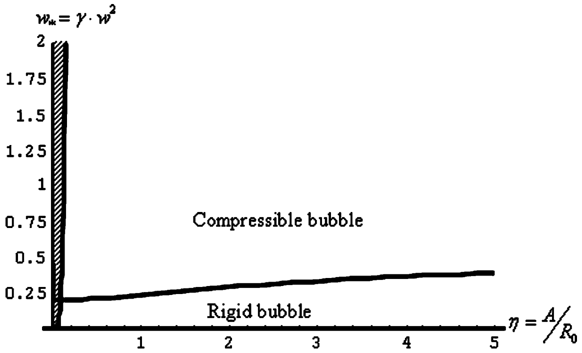

Fig. 3 Dependence of w_* on η corresponding to the condition of gas bubble nondeformability; *shaded area* is the area of non-fulfillment of quasistatic condition for $A = 6.5$ mm, $\gamma = 0.02$, and $P_e = P_0 = 10^5$ Pa

takes the form

$$\frac{2}{\gamma \cdot w^2} \cdot \frac{2(1 + \sqrt{1 + \theta \frac{A^2}{R_0^2}}) + \theta \frac{A^2}{R_0^2}}{2(1 + \sqrt{1 + \theta \frac{A^2}{R_0^2}}) + \frac{\theta}{3} \frac{A^2}{R_0^2}} > 10. \qquad (45)$$

The influence of the surface tension also can be a cause of gas bubble nondeformability. However, this influence is substantial only for very small bubbles; for frequencies $\omega < 200$ 1/s it can be considered as small for bubbles with radius greater than $2 \cdot 10^{-6}$ m [4].

As is seen, condition (45) depends on two characteristic parameters, w_* and η. A plot corresponding to this condition is shown in Fig. 3; the shaded area in the figure corresponds to the area for which condition (43) for the quasistatic character of gas bubble volume pulsations is not fulfilled for $A = 6.5$ mm, $\gamma = 0.02$, and $P_e = P_0 = 10^5$ Pa.

Thereby, for small magnitudes of the characteristic load parameter $w_* < 0.2$ the influence of gas bubble compressibility on its motion turn out to be insignificant.

As seen from Fig. 3, the magnitude of the characteristic load parameter w_*, which corresponds to the condition of gas bubble nondeformability, increases when parameter η is increased. Moreover, from expression (39) it follows that value X_0 at $\eta = \frac{A}{R_0} \to \infty$ is three times greater than value X_0 at $\eta \to 0$. Thereby, it can be concluded that the compressibility effect is more significant for large bubbles than for small ones.

Thus, for very small bubbles, for which $\eta = \frac{A}{R_0} \to \infty$, the compressibility effect can be considered as negligibly small for values of the characteristic load parameter $w_* < 0.6$, and for large bubbles ($\eta \to 0$), for $w_* < 0.2$.

For an amplitude of the external excitation equal to $A = 6.5$ mm, bubbles of radius $R_0 = 2.9$ mm can be considered as nondeformable for characteristic load parameter $w_* < 0.3$.

6 Comparison with the results of direct numerical integration

To verify the analytically obtained results, a numerical experiment was conducted. The equation of gas bubble motion (11) in view of expression (3) for its volume and of expression (12) for the resistance force was integrated directly by means of Mathematica 7, and the obtained results were compared with the analytical solution.

We examine the motion of the system with the following parameters: $P_e = P_0 = 10^5$ Pa, $H_0 = 0.16$ m, $V_{b0} = 0.1 \cdot 10^{-6}$ m^3 ($R_0 = 2.9 \cdot 10^{-3}$ m), $A = 6.5$ mm, $\omega = 195$ 1/s.

For these values of parameters using expression (39) for X_0, it is obtained that $X_0 = 0.31 H_0$ (in this case $(\rho H_0(g + A\omega^2)/P_e)^2 = 0.17$). It was derived analytically that bubbles situated in the positions $X > X_0$ sink in the fluid, and bubbles situated in the positions $X < X_0$ rise. Two numerical experiments were conducted to verify these suppositions. In the first experiment the motion of a bubble with initial position $X(0) = 0.25 H_0$ was examined; in the second experiment $X(0) = 0.5 H_0$. The corresponding dependences of the coordinate of the center of the bubble on time are shown in Fig. 4. The dashed line corresponds to the "slow" component of the analytical solution (expression (41)), and the solid line to the numerical solution.

In the numerical experiments the initial conditions were determined with the obtained analytical results taken into account, namely, expression (21) for the "fast" variable ψ, and correlations $x = X + \psi$ and $x(0) = X(0) + \psi(0, 0)$. Thus, the following initial conditions were employed when numerically integrating the equation of gas bubble motion (11):

$$\begin{aligned} x(0) &= X(0) + B \sin\varphi \\ \dot{x}(0) &= B\omega \cos\varphi \end{aligned} \qquad (46)$$

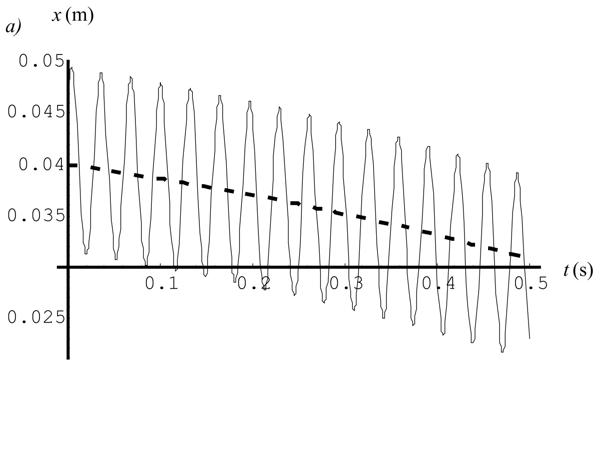

Fig. 4 The dependences of the coordinate of the center of the bubble on time: **a** initial position of the bubble $X(0) = 0.25H_0$, **b** initial position of the bubble $X(0) = 0.5H_0$

in which parameters B and φ were determined according to (26) and (28).

As is seen from Fig. 4, the analytically obtained results are in good agreement with the results of direct numerical integration: bubbles situated in the positions $X > X_0 = 0.31H_0$ sink, and those in the positions $X < X_0$ rise. The analytical solution comprising both "slow" and "fast" components virtually coincides with the results of the numerical integration shown in Fig. 4.

7 Comparison with the results of a natural experiment

A series of experiments has been conducted on the universal vibrating stand of the Joint Laboratory of Vibrational Mechanics IPME RAS (Fig. 5) to verify the analytically obtained results.

A vessel, which was almost full to the brim with water (the height of the air column was 1–2 mm) and closed by a plastic cover, was fastened on the stand. The external pressure on the free surface of the water was equal to the atmospheric pressure, and generation of a large quantity of bubbles due to the emergence of the turbulent coating surface [14] was prevented by the small amount of air in the vessel. A deformable rubber ball, filled with air, was attached to the bottom of the vessel by a thread (Fig. 6a). The mass of the rubber was negligibly small compared with the mass of the water in the volume of the bubble; the influence of the rubber tensile force on the pressure inside the ball can also be neglected. Thus, such a ball can be considered as the simplest gas bubble model.

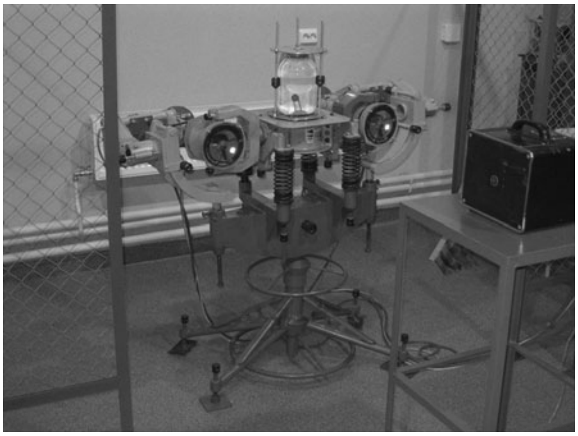

Fig. 5 The experimental setup

The height of the water column in the vessel was equal to 214 mm. The experiment was conducted for balls with different diameters (from 12 mm to 30 mm) and for different lengths of the thread, i.e., initial distances from the bottom of the vessel (from 33 mm to 170 mm). The amplitude of the external excitation was fixed and equal to 4.3 mm. The frequency of the external excitation was increased from 0 to 200 rad/s. In every experiment it was observed that, as soon as the frequency of excitation reaches a certain value ω_*, the ball falls to the bottom of the vessel (Fig. 6b). Further increasing of the frequency did not influence the ball's

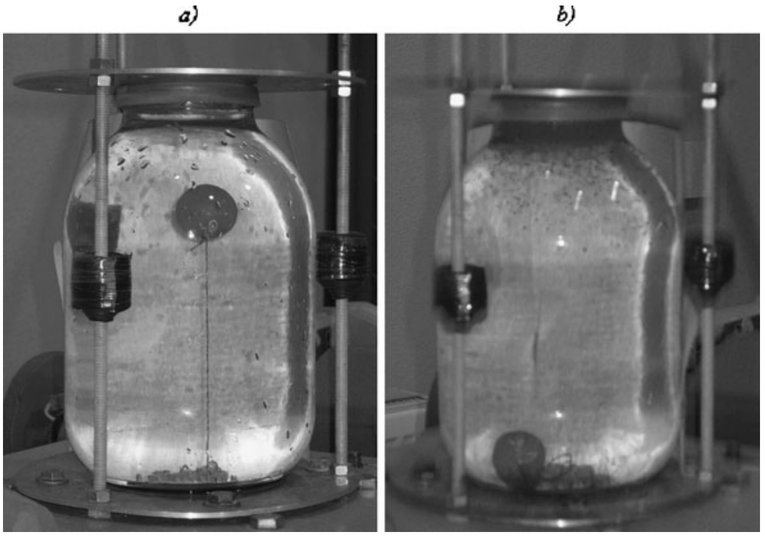

Fig. 6 Sinking of the gas bubble model (deformable rubber ball) in the vibrating vessel with water: **a** the ball in the still vessel, **b** the ball in the vibrating vessel after the threshold frequency ω_* is exceeded

position—it remained at the bottom of the vessel. Decreasing of the frequency led to rising of the ball; i.e., it returned to the initial position. Thereby the theoretical conclusion that the effect of gas bubble sinking can occur in a uniformly oscillating fluid was entirely confirmed by the experiment. We note that earlier this effect was observed by S.S. Grigoryan and Y.L. Yakimov in their experiments [22].

It was found that decreasing the initial distance from the bottom of the vessel to the ball (the length of the thread) as well as increasing the size of the ball led to a decrease of the frequency ω_*. These results are in good agreement with analytical predictions. The magnitudes of the overloads necessary for the occurrence of the effect of rubber balls sinking in the water were slightly smaller than the magnitudes of the overloads necessary for the occurrence of the effect of gas bubbles sinking in gas-saturated fluid [4, 6]. This fact conforms to the analytically revealed dependence of the condition of sinking on the size of the bubble, because the diameters of the balls used in the described experiment considerably exceeded the diameters of the real bubbles.

To verify the assumption that the effect of gas bubble sinking is conditional on its volume variations due to vibration, an experiment with a nondeformable rigid ball, whose density was much smaller than the density of the water, was conducted. The effect of sinking was not registered in this case.

As an illustrative example, the results of one of the experiments are presented. For a rubber ball with diameter $d = 14$ mm and total mass (the mass of the rubber + the mass of the thread) $m = 30$ mg, whose initial distance from the bottom of the vessel is equal to 81 mm, the experimentally obtained value of the threshold frequency was equal to 168 $1/\mathrm{s}$. This value is in good agreement with the analytical result of 166 $1/\mathrm{s}$. For all conducted experiments the same qualitative result was obtained; i.e., the theoretical values of the threshold frequencies almost coincided with the experimental values.

8 Conclusion

In this paper, the effect of sinking of a free or an "equipped" (carrying rigid particles) gas bubble in a uniformly oscillating incompressible fluid has been theoretically established and experimentally confirmed. The conditions at which this effect occurs are formulated. An expression for the average velocity of the gas bubble's sinking or rising, which strongly depends on the depth of its submergence and vibration parameters, is derived. The reported results are applicable for control and optimization of relevant technological processes.

The results of the paper supplement and enhance previous studies, which have been concerned with the "wave" mechanism of a gas bubble sinking. In contrast to this mechanism, the theoretical model analyzed in this paper is "non-gradient" and can be called vibrational. It is shown that the effect of gas bubble sinking is controlled by two dimensionless parameters: w_*, the characteristic load parameter, and the ratio of the excitation amplitude to the radius of the bubble, η. These parameters define the bubble's compressibility in the course of its motion.

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