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Robust observer for discrete-time Markovian jumping neural networks with mixed mode-dependent delays

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Abstract The robust observer problem is considered in this paper for a class of discrete-time neural networks with Markovian jumping parameters and mode-dependent time delays which are in both discrete-time form and finite distributed form. The neural network switches from one mode to another controlled by a Markov chain with known transition probability. Time-delays considered in this paper are mode-dependent which may reflect a more realistic version of the neural network. By using the Lyapunov functional method and the techniques of linear matrix inequalities (LMIs), sufficient conditions are established in terms of LMIs that ensure the existence of the robust observer. The obtained conditions are easy to be verified via the LMI toolbox. An example is presented to show the effectiveness of the obtained results.

Keywords Robust observer · Discrete-time neural network · Markovian jumping parameters · Mixed mode-dependent delays · Linear matrix inequality

1 Introduction

It is well known that a large amount of neural networks are successfully used in various areas in the last few decades, such as image processing, associative memory, pattern recognition, and so on. In large-scale neural networks, however, it is often that only partial information about the neuron states is available in the form of the network outputs. Therefore, to make use of the neural networks in practice, it becomes necessary to estimate the neuron states. For the discrete delay system, Trinh and Aldeen obtained a memoryless state observer by the state augmentation approach [1]. In [2], a general form of linear observers was given for the continuous delay systems by using the factorization approach. Recently, the state estimation problem for neural networks has drawn research interests (see [3–11]). For example, in [3], the neural state estimation problem was described by an effective LMI approach which developed to verify the stability of the estimation error dynamics. In [12], an adaptive state estimator was addressed by using techniques of optimization theory.

In practice, because of the limited speed of signals traveling through the links, time delays often occur in neural networks [13–15]. It is known that time delays can cause complex dynamics such as periodic or quasi-periodic motions, higher-dimensional chaos [16]. It is worth mentioning that the time delays can be categorized as discrete ones and distributed ones generally. The distributed delays were obtained particular attention since a network usually has a spatial nature due to the presence of an amount of parallel pathways of a variety of axon sizes and lengths [17–19].

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On the other hand, complex networks may be subject to network mode switching. For example, in [20, 21], it revealed that a neural network had finite modes which switched from one to another sometimes, and the switching can be controlled by a Markov chain. In [22], the bufferless packet switching of leveled networks was illustrated to be achievable. In [20], authors investigated the exponential stability of delayed recurrent neural networks with Markovian jumping parameters. The exponential synchronization problem was studied for a class of continuous-time complex networks with Markovian jump and mixed delays [6, 23–28]. In [29], due to the uncomplicated form and size of switching elements, it showed that practical interconnection systems can be designed by the shuffle-exchange network. In [30], the associative memory of a stochastic Hopfield neural system was described to switch between pattern attractors. It is worth mentioning that the control problems for dynamical systems with Markovian jumping parameters were already widely studied; see, e.g., [31], and the references therein. Moreover, discrete-time neural networks were more suitable to model digitally signals in a dynamical way. Note that discrete-time networks have already been applied in a wide range of areas, such as image processing, time series analysis, quadratic optimization problems, and system identification, so it is worth studying.

In this paper, we deal with the robust state estimation problem for a class of discrete-time neural networks with Markovian jumping parameters and mode-dependent mixed time delays. By using the Lyapunov functional method, we get several sufficient conditions under which the estimation error dynamics is asymptotically stable. The obtained criteria are in the form of LMIs whose solution could be easily calculated by utilizing the LMI toolbox.

Notations The standard notations will be used in this paper. Throughout this paper, for real symmetric matrices X and Y, the notation $X \leq Y$ (respectively, $X < Y$) means that the matrix $X - Y$ is negative semidefinite (respectively, negative definite). The superscript "T" represents the transpose. diag$\{\cdots\}$ stands for a block-diagonal matrix. I and 0 is the identity matrix and zero matrix with compatible dimension, respectively. $|\cdot|$ refers to the Euclidean vector norm. $\mathbb{N}$ denotes the natural number set, i.e., $\mathbb{N} = \{0, 1, \ldots\}$. $\mathbb{R}^n$ and $\mathbb{R}^{n\times m}$ denote, respectively, the n-dimensional Euclidean space and the set of all $n \times m$ real matrices. In symmetric block matrices, we use an asterisk "$*$" to represent a term that is induced by symmetry. $\mathbb{E}[x]$ and $\mathbb{E}[x|y]$ mean, respectively, the expectation of x and the expectation of x conditional on y. Matrices, if they are not explicitly, are assumed to have compatible dimensions.

2 Model formulation and preliminaries

In this section, the problem discussed in this paper is formulated and some lemmas are introduced which will play an important role in the proof of the main result in Sect. 3.

In the following, we consider a discrete-time uncertain nonlinear system:

$$
\begin{aligned}
&x(k+1)\\
&\quad = \big(D\big(r(k)\big) + \Delta D\big(k, r(k)\big)\big)x(k)\\
&\qquad + \big(A\big(r(k)\big) + \Delta A\big(k, r(k)\big)\big)F\big(x(k)\big)\\
&\qquad + \big(B\big(r(k)\big) + \Delta B\big(k, r(k)\big)\big)G\big(x(k-\tau_{1,r(k)})\big)\\
&\qquad + \big(C\big(r(k)\big) + \Delta C\big(k, r(k)\big)\big)\sum_{v=1}^{\tau_{2,r(k)}} H\big(x(k-v)\big),
\end{aligned}
\tag{1}
$$

where $\{r(k)|k \geq 0\}$ is a Markov chain taking values in a finite state space $\mathcal{S} = \{1, 2, 3, \ldots, N\}$, and

$$\mathcal{P}\big\{r(k+1) = j|r(k) = i\big\} = \pi_{ij}, \quad \forall i, j \in \mathcal{S},$$

where $\pi_{ij} \geq 0$ $(i, j \in \mathcal{S})$ is the transition rate from i to j and $\sum_{j=1}^{N} \pi_{ij} = 1$, $\forall i \in \mathcal{S}$; $x(k) = (x_1(k), x_2(k), \ldots, x_n(k))^T$ is the neural state vector; $D(r(k)) = \text{diag}\{d_1(r(k)), d_2(r(k)), \ldots, d_n(r(k))\}$ describes the rate with which each neuron will reset its potential to the resting state in isolation when disconnected from the exogenous inputs and networks; $A(r(k)) = [a_{ij}(r(k))]_{n\times n}$, $B(r(k)) = [b_{ij}(r(k))]_{n\times n}$ and $C(r(k)) = [c_{ij}(r(k))]_{n\times n}$ are the connection weight matrix, the discretely delayed connection weight matrix and the distributively delayed connection weight matrix, respectively; $\Delta D(k, r(k))$, $\Delta A(k, r(k))$, $\Delta B(k, r(k))$ and $\Delta C(k, r(k))$ are unknown matrices which representing the mode parameter uncertainties; $F(x(k)) = (f_1(x_1(k)), f_2(x_2(k)), \ldots, f_n(x_n(k)))^T$, $G(x(k - \tau_{1,r(k)})) =$

$(g_1(x_1(k - \tau_{1,r(k)})), g_2(x_2(k - \tau_{1,r(k)})), \dots, g_n(x_n(k - \tau_{1,r(k)})))^T$ and $H(x(k)) = (h_1(x_1(k)), h_2(x_2(k)), \dots, h_n(x_n(k)))^T$ are the nonlinear activation functions; $\tau_{1,r(k)}$ stands for the discrete time delay while $\tau_{2,r(k)}$ denotes the distributed time delay, and both kinds of time delays are dependent on the system mode $r(k)$. Note that $D(r(k))$, $A(r(k))$, $B(r(k))$, and $C(r(k))$ are the constant matrices for any $r(k) \in \mathcal{S}$.

The initial condition of system (1) is assumed to be

$$x(k) = \phi(k), \quad k = -\tau, \dots, 0,$$

where $\phi(k)$ is a known function, and $\tau = \max_{i\in\mathcal{S}}\{\bar{\tau}_1, \bar{\tau}_2\}$, $\bar{\tau}_1 = \max_{i\in\mathcal{S}}\{\tau_{1,i}\}$, $\bar{\tau}_2 = \max_{i\in\mathcal{S}}\{\tau_{2,i}\}$.

Throughout this paper, we make the following assumption on the nonlinear functions in system (1).

Assumption 1 For $i \in \{1, 2, \dots, n\}$, the neuron activation functions in (1) satisfy

$$\begin{aligned} &\lambda_i^- \le \frac{f_i(s_1) - f_i(s_2)}{s_1 - s_2} \le \lambda_i^+, \qquad f_i(0) = 0; \\ &\sigma_i^- \le \frac{g_i(s_1) - g_i(s_2)}{s_1 - s_2} \le \sigma_i^+, \qquad g_i(0) = 0; \\ &\upsilon_i^- \le \frac{h_i(s_1) - h_i(s_2)}{s_1 - s_2} \le \upsilon_i^+, \\ &h_i(0) = 0; \quad \forall s_1, s_2 \in \mathbb{R},\ s_1 \ne s_2 \end{aligned} \tag{2}$$

where λ_i^-, λ_i^+, σ_i^-, σ_i^+, υ_i^- and υ_i^+ are constants.

Remark 1 Just as pointed out in [19], the constants $\lambda_i^-, \lambda_i^+, \sigma_i^-, \sigma_i^+, \upsilon_i^-, \upsilon_i^+$ in Assumption 1 are allowed to be positive, negative, or zero. So, the activation functions are more general than the usual sigmoid functions because they could be nonmonotonic. This description is very accurate in quantifying the lower and upper bounds of the activation functions, and is very helpful for using the LMI approach.

It is worth noticing that in neural networks, either biological or artificial, it is usually difficult to get the complete information of their states and, therefore, it is necessary to estimate the states of neural networks based only on the output. In this paper, we assume that the output from the system (1) is of the following form:

$$y(k) = \big(E\big(r(k)\big) + \Delta E\big(k, r(k)\big)\big)x(k) + S\big(k, x(k)\big), \tag{3}$$

where $y(k) = (y_1(k), y_2(k), \dots, y_m(k))^T$ stands for the measurement output of the neural network; and we assume $m < n$ because only partial information of the system states could be accessed from output measurements. $E(r(k)) \in \mathbb{R}^{m\times n}$ is a constant matrix and $\Delta E(k, r(k))$ is an unknown matrix. $S(k, x(k)) \in \mathbb{R}^m$ is a nonlinear disturbance dependent on neural states satisfying the following Lipschitz condition:

$$\begin{aligned} &\big|S(k, \varsigma_1) - S(k, \varsigma_2)\big| \le \big|W(\varsigma_1 - \varsigma_2)\big|, \\ &\quad \forall k \in \mathbb{N} \text{ and } \varsigma_1, \varsigma_2 \in \mathbb{R}^n, \end{aligned} \tag{4}$$

where W is a known real constant matrix of appropriate dimension.

Assumption 2 The uncertain matrices $\Delta D(k, r(k))$, $\Delta A(k, r(k))$, $\Delta B(k, r(k))$, $\Delta C(k, r(k))$, and $\Delta E(k, r(k))$ are assumed to be of the following form:

$$\begin{aligned} &\big(\Delta D(k, r(k))\ \Delta A(k, r(k))\ \Delta B(k, r(k))\ \Delta C(k, r(k))\big) \\ &\quad = M_1\big(r(k)\big)\Phi\big(k, r(k)\big) \\ &\qquad \times \big(N_1(r(k))\ N_2(r(k))\ N_3(r(k))\ N_4(r(k))\big), \\ &\Delta E\big(k, r(k)\big) = M_2\big(r(k)\big)\Phi\big(k, r(k)\big)N_5\big(r(k)\big); \end{aligned} \tag{5}$$

where $M_1(r(k))$, $M_2(r(k))$, $N_1(r(k))$, $N_2(r(k))$, $N_3(r(k))$, $N_4(r(k))$, and $N_5(r(k))$ are known real constant matrices and $\Phi(k, r(k))$ is an unknown real-valued matrix satisfying

$$\Phi^T\big(k, r(k)\big)\Phi\big(k, r(k)\big) \le I, \quad \forall k \in \mathbb{N},\ r(k) \in \mathcal{S}. \tag{6}$$

Remark 2 The structure of the parameter uncertainties in (5)–(6) has been widely used for the robust control and robust observer problems for both the discrete-time and the continuous-time systems, which can represent many practical situations [32, 33].

To estimate the state of system (1), we construct a full-order state estimator as follows:

$$\begin{aligned} \hat{x}(k+1) = {} & D\big(r(k)\big)\hat{x}(k) + A\big(r(k)\big)F\big(\hat{x}(k)\big) \\ & + B\big(r(k)\big)G\big(\hat{x}(k - \tau_{1,r(k)})\big) \\ & + C\big(r(k)\big)\sum_{v=1}^{\tau_{2,r(k)}} H\big(\hat{x}(k - v)\big) \end{aligned}$$

$$
\begin{aligned}
&- K\big(r(k)\big)\big[y(k) - E\big(r(k)\big)\hat{x}(k) \\
&- S\big(k, \hat{x}(k)\big)\big],
\end{aligned} \tag{7}
$$

where $\hat{x}(k)$ is the estimation of the neuron state, and $K(r(k))$ $(r(k) \in \mathcal{S})$ are the estimate gain matrices to be designed.

Definition 1 System (7) is said to be an robustly asymptotic state estimator of system (1) with measurement output (3) if the state estimation error $\tilde{x}(k) = (\tilde{x}_1(k), \ldots, \tilde{x}_n(k))^T \triangleq x(k) - \hat{x}(k)$ satisfies

$$
\lim_{k \to +\infty} \mathbb{E}\big[\big|\tilde{x}(k)\big|^2\big] = 0. \tag{8}
$$

Lemma 1 (See [34]) *Let $\mathcal{B} \in \mathbb{R}^{n\times n}$ is a positive semi-definite matrix, $z_i \in \mathbb{R}^n$ is a vector and $a_i \geq 0$ is a scalar $(i = 1, 2, \ldots)$. The following inequality holds*

$$
\left(\sum_{i=1}^{+\infty} a_i z_i\right)^T \mathcal{B}\left(\sum_{i=1}^{+\infty} a_i z_i\right) \leq \left(\sum_{i=1}^{+\infty} a_i\right)\left(\sum_{i=1}^{+\infty} a_i z_i^T \mathcal{B} z_i\right) \tag{9}
$$

if the series are convergent.

Lemma 2 (See [6]) *Let $\mathcal{D} = \mathrm{diag}\{\gamma_1, \gamma_2, \ldots, \gamma_n\} \geq 0$, $z = (z_1, z_2, \ldots, z_n)^T \in \mathbb{R}^n$, $\mathcal{H}(z) = (\hbar_1(z_1), \hbar_2(z_2), \ldots, \hbar_n(z_n))^T$ is a continuous nonlinear function and*

$$
l_i^- \leq \frac{\hbar_i(s)}{s} \leq l_i^+; \quad s \neq 0,\ s \in \mathbb{R},\ i = 1, 2, \ldots, n,
$$

where l_i^- and l_i^+ are constant scalars. Then

$$
\begin{pmatrix} z \\ \mathcal{H}(z) \end{pmatrix}^T \begin{pmatrix} \mathcal{D}L_1 & -\mathcal{D}L_2 \\ -\mathcal{D}L_2 & \mathcal{D} \end{pmatrix} \begin{pmatrix} z \\ \mathcal{H}(z) \end{pmatrix} \leq 0,
$$

or

$$
z^T \mathcal{D} L_1 z - 2z^T \mathcal{D} L_2 \mathcal{H}(z) + \mathcal{H}(z)^T \mathcal{D} \mathcal{H}(z) \leq 0,
$$

where $L_1 = \mathrm{diag}\{l_1^- l_1^+, l_2^- l_2^+, \ldots, l_n^- l_n^+\}$ *and* $L_2 = \mathrm{diag}\{\frac{l_1^- + l_1^+}{2}, \frac{l_2^- + l_2^+}{2}, \ldots, \frac{l_n^- + l_n^+}{2}\}$.

Lemma 3 (See [4]) *Let A, D, E, F, and P be real matrices of appropriate dimensions with $P > 0$ and F satisfying $F^T F \leq I$. Then for any scalar $\epsilon > 0$ satisfying $P^{-1} - \epsilon^{-1} D D^T > 0$, we have*

$$
\begin{aligned}
&(A + DFE)^T P (A + DFE) \\
&\quad \leq A^T \big(P^{-1} - \epsilon^{-1} D D^T\big)^{-1} A + \epsilon E^T E.
\end{aligned} \tag{10}
$$

For presentation convenience, in the following, we denote

$$
\begin{aligned}
&\underline{\tau}_1 = \min_{i \in \mathcal{S}}\{\tau_{1,i}\}, \qquad \underline{\tau}_2 = \min_{i \in \mathcal{S}}\{\tau_{2,i}\}, \\
&\underline{\pi} = \min_{i \in \mathcal{S}}\{\pi_{ii}\}; \\
&\Lambda_1 = \mathrm{diag}\big\{\lambda_1^- \lambda_1^+, \lambda_2^- \lambda_2^+, \ldots, \lambda_n^- \lambda_n^+\big\}, \\
&\Lambda_2 = \mathrm{diag}\left\{\frac{\lambda_1^- + \lambda_1^+}{2}, \frac{\lambda_2^- + \lambda_2^+}{2}, \ldots, \frac{\lambda_n^- + \lambda_n^+}{2}\right\}; \\
&\Sigma_1 = \mathrm{diag}\big\{\sigma_1^- \sigma_1^+, \sigma_2^- \sigma_2^+, \ldots, \sigma_n^- \sigma_n^+\big\}, \\
&\Sigma_2 = \mathrm{diag}\left\{\frac{\sigma_1^- + \sigma_1^+}{2}, \frac{\sigma_2^- + \sigma_2^+}{2}, \ldots, \frac{\sigma_n^- + \sigma_n^+}{2}\right\}; \\
&\Upsilon_1 = \mathrm{diag}\big\{\upsilon_1^- \upsilon_1^+, \upsilon_2^- \upsilon_2^+, \ldots, \upsilon_n^- \upsilon_n^+\big\}, \\
&\Upsilon_2 = \mathrm{diag}\left\{\frac{\upsilon_1^- + \upsilon_1^+}{2}, \frac{\upsilon_2^- + \upsilon_2^+}{2}, \ldots, \frac{\upsilon_n^- + \upsilon_n^+}{2}\right\}.
\end{aligned}
$$

3 Main result

In this section, the robust observer is designed for system (1) by resorting to the Lyapunov functional and the LMI approach.

Theorem 1 *Let $K(i)$ $(i \in \mathcal{S})$ be known constant matrices; suppose Assumptions 1–2 and condition (4) hold. Then system (7) is an robustly asymptotic state estimator of system (1) with measurement output (3) if there exist scalar constants $\varepsilon_i > 0$, $\vartheta_i > 0$, matrices $P_i > 0$, $Q > 0$ and $R > 0$, and diagonal matrices $\Omega_{1i} > 0$, $\Omega_{2i} > 0$, $\Theta_{1i} > 0$, $\Theta_{2i} > 0$, $\Delta_{1i} > 0$, $\Delta_{2i} > 0$ $(i \in \mathcal{S})$ satisfying*:

$$
\Psi_i \triangleq \begin{pmatrix} \Pi_i & Z_o^T(i)\bar{P}_i & 0 \\ * & -\bar{P}_i & \bar{P}_i \tilde{M}(i) \\ * & * & -\varepsilon_i I \end{pmatrix} < 0, \quad i \in \mathcal{S} \tag{11}
$$

where

$$Z_o(i)=\big(D_o(i)\ A_o(i)\ 0\ B_o(i)\ 0\ C_o(i)\ K_o(i)\big),$$

$$D_o(i)=\begin{pmatrix} D(i) & 0\\ 0 & D(i)+K(i)E(i)\end{pmatrix},$$

$$A_o(i)=\operatorname{diag}\big\{A(i),A(i)\big\},\qquad B_o(i)=\operatorname{diag}\big\{B(i),B(i)\big\},$$

$$C_o(i)=\operatorname{diag}\big\{C(i),C(i)\big\},\qquad K_o(i)=\operatorname{diag}\big\{0,K(i)\big\};\qquad \bar{P}_i=\sum_{j=1}^{N}\pi_{ij}P_j;$$

$$\tilde{M}(i)=\big(\tilde{M}_1(i)\ \tilde{M}_2(i)\big),\qquad \tilde{M}_1(i)=\begin{pmatrix}\bar{M}_1(i)\\ \bar{M}_1(i)\end{pmatrix},\qquad \tilde{M}_2(i)=\begin{pmatrix}0\\ K(i)\bar{M}_2(i)\end{pmatrix},$$

$$\bar{M}_1(i)=\big(M_1(i)\ 0\big),\qquad \bar{M}_2(i)=\big(0\ M_2(i)\big);\qquad \Pi_i=\varphi_i+\varepsilon_i\tilde{N}^T(i)\tilde{N}(i),\qquad \tilde{N}(i)=\begin{pmatrix}\hat{N}_1(i)\\ \hat{N}_2(i)\end{pmatrix},$$

$$\varphi_i=\begin{pmatrix}
\Xi_i & \Omega_i\bar{\Lambda}_2 & \Theta_i\bar{\Sigma}_2 & 0 & \Delta_i\bar{\Upsilon}_2 & 0 & 0\\
* & -\Omega_i & 0 & 0 & 0 & 0 & 0\\
* & * & \kappa_0 Q-\Theta_i & 0 & 0 & 0 & 0\\
* & * & * & -Q & 0 & 0 & 0\\
* & * & * & * & \kappa_i R-\Delta_i & 0 & 0\\
* & * & * & * & * & -\frac{1}{\tau_{2,i}}R & 0\\
* & * & * & * & * & * & -\vartheta_i I
\end{pmatrix}$$

---

and

$$\hat{N}_1(i)=\big(\tilde{N}_1(i)\ \tilde{N}_2(i)\ 0\ \tilde{N}_3(i)\ 0\ \tilde{N}_4(i)\ 0\big),$$

$$\hat{N}_2(i)=\big(\tilde{N}_1(i)\ 0\ 0\ 0\ 0\ 0\ 0\big),$$

$$\tilde{N}_1(i)=\big(\bar{N}_1(i)\ 0\big),\qquad \tilde{N}_2(i)=\big(\bar{N}_2(i)\ 0\big),$$

$$\tilde{N}_3(i)=\big(\bar{N}_3(i)\ 0\big),\qquad \tilde{N}_4(i)=\big(\bar{N}_4(i)\ 0\big),$$

$$\bar{N}_1(i)=\begin{pmatrix}N_1\\ N_5\end{pmatrix},\qquad \bar{N}_2(i)=\begin{pmatrix}N_2\\ 0\end{pmatrix},$$

$$\bar{N}_3(i)=\begin{pmatrix}N_3\\ 0\end{pmatrix},\qquad \bar{N}_4(i)=\begin{pmatrix}N_4\\ 0\end{pmatrix};$$

$$\Xi_i=-P_i-\Omega_i\bar{\Lambda}_1-\Theta_i\bar{\Sigma}_1-\Delta_i\bar{\Upsilon}_1+\vartheta_i W_o^T W_o,$$

$$W_o=\operatorname{diag}\{0,W\},\qquad \Omega_i=\operatorname{diag}\{\Omega_{1i},\Omega_{2i}\},$$

$$\Theta_i=\operatorname{diag}\{\Theta_{1i},\Theta_{2i}\},\qquad \Delta_i=\operatorname{diag}\{\Delta_{1i},\Delta_{2i}\},$$

$$\bar{\Lambda}_1=\operatorname{diag}\{\Lambda_1,\Lambda_1\},\qquad \bar{\Sigma}_1=\operatorname{diag}\{\Sigma_1,\Sigma_1\},$$

$$\bar{\Upsilon}_1=\operatorname{diag}\{\Upsilon_1,\Upsilon_1\},\qquad \bar{\Lambda}_2=\operatorname{diag}\{\Lambda_2,\Lambda_2\},$$

$$\bar{\Sigma}_2=\operatorname{diag}\{\Sigma_2,\Sigma_2\},\qquad \bar{\Upsilon}_2=\operatorname{diag}\{\Upsilon_2,\Upsilon_2\},$$

$$\kappa_0=(1-\underline{\pi})(\bar{\tau}_1-\underline{\tau}_1)+1,$$

$$\kappa_i=\tau_{2,i}+(1-\pi_{ii})(\bar{\tau}_2-\underline{\tau}_2)+\frac{1}{2}(1-\underline{\pi})(\bar{\tau}_2-\underline{\tau}_2)(\bar{\tau}_2-\underline{\tau}_2-1).$$

Proof From (1) and (7), we have

$$\begin{aligned}
\tilde{x}(k+1)
&=\big(D\big(r(k)\big)+K\big(r(k)\big)E\big(r(k)\big)\big)\tilde{x}(k)\\
&\quad+A\big(r(k)\big)\big(F\big(x(k)\big)-F\big(\hat{x}(k)\big)\big)\\
&\quad+B\big(r(k)\big)\big(G\big(x(k-\tau_{1,r(k)})\big)\\
&\quad-G\big(\hat{x}(k-\tau_{1,r(k)})\big)\big)\\
&\quad+C\big(r(k)\big)\sum_{v=1}^{\tau_{2,r(k)}}\big(H\big(x(k-v)\big)-H\big(\hat{x}(k-v)\big)\big)\\
&\quad+K\big(r(k)\big)\big(S\big(k,x(k)\big)-S\big(k,\hat{x}(k)\big)\big)\\
&\quad+\big(\Delta D\big(k,r(k)\big)+K\big(r(k)\big)\Delta E\big(k,r(k)\big)\big)x(k)\\
&\quad+\Delta A\big(k,r(k)\big)F\big(x(k)\big)\\
&\quad+\Delta B\big(k,r(k)\big)G\big(x(k-\tau_{1,r(k)})\big)\\
&\quad+\Delta C\big(k,r(k)\big)\sum_{v=1}^{\tau_{2,r(k)}}H\big(x(k-v)\big).
\end{aligned}\tag{12}$$

Let

$$\eta(k) \triangleq \begin{pmatrix} x(k) \\ \tilde{x}(k) \end{pmatrix},$$

$$\tilde{F}\big(\eta(k)\big) = \begin{pmatrix} F(x(k)) \\ F(x(k)) - F(\hat{x}(k)) \end{pmatrix},$$

$$\tilde{G}\big(\eta(k)\big) = \begin{pmatrix} G(x(k)) \\ G(x(k)) - G(\hat{x}(k)) \end{pmatrix},$$

$$\tilde{H}\big(\eta(k)\big) = \begin{pmatrix} H(x(k)) \\ H(x(k)) - H(\hat{x}(k)) \end{pmatrix},$$

$$\tilde{S}\big(k, \eta(k)\big) = \begin{pmatrix} 0 \\ S(k, x(k)) - S(k, \hat{x}(k)) \end{pmatrix},$$

$$\begin{aligned} &\Delta D_o\big(k, r(k)\big) \\ &\quad = \begin{pmatrix} \Delta D(k, r(k)) & 0 \\ \Delta D(k, r(k)) + K(r(k))\Delta E(k, r(k)) & 0 \end{pmatrix}, \end{aligned}$$

$$\Delta A_o\big(k, r(k)\big) = \begin{pmatrix} \Delta A(k, r(k)) & 0 \\ \Delta A(k, r(k)) & 0 \end{pmatrix},$$

$$\Delta B_o\big(k, r(k)\big) = \begin{pmatrix} \Delta B(k, r(k)) & 0 \\ \Delta B(k, r(k)) & 0 \end{pmatrix},$$

$$\Delta C_o\big(k, r(k)\big) = \begin{pmatrix} \Delta C(k, r(k)) & 0 \\ \Delta C(k, r(k)) & 0 \end{pmatrix};$$

with the above notations and from (1) and (12), we obtain

$$\begin{aligned} \eta(k+1) = {} & \bar{D}_o\big(k, r(k)\big)\eta(k) + \bar{A}_o\big(k, r(k)\big)\tilde{F}\big(\eta(k)\big) \\ & + \bar{B}_o\big(k, r(k)\big)\tilde{G}\big(\eta(k - \tau_{1,r(k)})\big) \\ & + \bar{C}_o\big(k, r(k)\big) \sum_{v=1}^{\tau_{2,r(k)}} \tilde{H}\big(\eta(k-v)\big) \\ & + K_o\big(r(k)\big)\tilde{S}\big(k, \eta(k)\big), \end{aligned} \tag{13}$$

where

$$\begin{aligned} \bar{D}_o\big(k, r(k)\big) &= D_o\big(r(k)\big) + \Delta D_o\big(k, r(k)\big), \\ \bar{A}_o\big(k, r(k)\big) &= A_o\big(r(k)\big) + \Delta A_o\big(k, r(k)\big); \\ \bar{B}_o\big(k, r(k)\big) &= B_o\big(r(k)\big) + \Delta B_o\big(k, r(k)\big), \\ \bar{C}_o\big(k, r(k)\big) &= C_o\big(r(k)\big) + \Delta C_o\big(k, r(k)\big). \end{aligned}$$

For convenience, we denote

$$\begin{aligned} &\eta_k = \big(\eta^T(k)\ \eta^T(k-1)\ \dots\ \eta^T(k-\tau)\big)^T, \\ &\xi(k,i) = \big(\eta^T(k)\ \tilde{F}^T(\eta(k))\ \tilde{G}^T(\eta(k))\ \tilde{G}^T(\eta(k-\tau_{1,i}))\ \tilde{H}^T(\eta(k))\ \textstyle\sum_{v=1}^{\tau_{2,i}} \tilde{H}^T(\eta(k-v))\ \tilde{S}^T(k,\eta(k))\big)^T, \\ &\Delta Z_o(k,i) = \big(\Delta D_o(k,i)\ \Delta A_o(k,i)\ 0\ \Delta B_o(k,i)\ 0\ \Delta C_o(k,i)\ 0\big), \\ &Z(k,i) = \big(\bar{D}_o(k,i)\ \bar{A}_o(k,i)\ 0\ \bar{B}_o(k,i)\ 0\ \bar{C}_o(k,i)\ K_o(i)\big) = Z_o(i) + \Delta Z_o(k,i) \end{aligned} \tag{14}$$

where $Z_o(i)$ is as defined in (11) and $i \in \mathcal{S}$.

Now, to ensure that system (7) is a robust state estimator of system (1) with measurement output (3), we just need to show that the system (13) is asymptotically stable in the mean square. Construct the following Lyapunov–Krasovskii functional $V(\eta_k, k, r(k))$ as

$$\begin{aligned} &V\big(\eta_k, k, r(k)\big) \\ &\quad = V_1\big(\eta_k, k, r(k)\big) + V_2\big(\eta_k, k, r(k)\big) \\ &\qquad + V_3\big(\eta_k, k, r(k)\big) + V_4\big(\eta_k, k, r(k)\big) \\ &\qquad + V_5\big(\eta_k, k, r(k)\big), \end{aligned} \tag{15}$$

where

$$V_1\big(\eta_k, k, r(k)\big) = \eta^T(k) P_{r(k)} \eta(k),$$

$$V_2\big(\eta_k, k, r(k)\big) = \sum_{v=k-\tau_{1,r(k)}}^{k-1} \tilde{G}^T\big(\eta(v)\big) Q \tilde{G}\big(\eta(v)\big),$$

$$V_3\big(\eta_k, k, r(k)\big) = \sum_{l=\underline{\tau}_1}^{\bar{\tau}_1 - 1} \sum_{v=k-l}^{k-1} \tilde{G}^T\big(\eta(v)\big) \bar{Q} \tilde{G}\big(\eta(v)\big),$$

$$V_4\big(\eta_k, k, r(k)\big) = \sum_{l=1}^{\tau_{2,r(k)}} \sum_{v=k-l}^{k-1} \tilde{H}^T\big(\eta(v)\big) R \tilde{H}\big(\eta(v)\big),$$

$$V_5\big(\eta_k, k, r(k)\big) = \sum_{s=\underline{\tau}_2+1}^{\bar{\tau}_2} \sum_{l=1}^{s-1} \sum_{v=k-l}^{k-1} \tilde{H}^T\big(\eta(v)\big)\bar{R}\tilde{H}\big(\eta(v)\big)$$

and $\bar{Q} = (1-\underline{\pi})Q$, $\bar{R} = (1-\underline{\pi})R$.

Calculate the expectation of the difference of the Lyapunov functional at two consecutive time instants, and one has

$$\begin{aligned}
&\mathbb{E}\big[V_1\big(\eta_{k+1}, k+1, r(k+1)\big)|\eta_k, r(k)=i\big] - V_1(\eta_k, k, i) \\
&\quad = \xi^T(k,i)Z^T(i)\bar{P}_i Z(i)\xi(k,i) - \eta^T(k)P_i\eta(k), \qquad (16)
\end{aligned}$$

$$\begin{aligned}
&\mathbb{E}\big[V_2\big(\eta_{k+1}, k+1, r(k+1)\big)|\eta_k, r(k)=i\big] - V_2(\eta_k, k, i) \\
&\quad = \sum_{j=1}^{N}\pi_{ij}\sum_{v=k-\tau_{1,j}+1}^{k}\tilde{G}^T\big(\eta(v)\big)Q\tilde{G}\big(\eta(v)\big) - \sum_{v=k-\tau_{1,i}}^{k-1}\tilde{G}^T\big(\eta(v)\big)Q\tilde{G}\big(\eta(v)\big) \\
&\quad = \pi_{ii}\Bigg[\sum_{v=k-\tau_{1,i}+1}^{k} - \sum_{v=k-\tau_{1,i}}^{k-1}\Bigg]\tilde{G}^T\big(\eta(v)\big)Q\tilde{G}\big(\eta(v)\big) \\
&\qquad + \sum_{j\neq i}\pi_{ij}\Bigg[\sum_{v=k-\tau_{1,j}+1}^{k} - \sum_{v=k-\tau_{1,i}}^{k-1}\Bigg] \times \tilde{G}^T\big(\eta(v)\big)Q\tilde{G}\big(\eta(v)\big) \\
&\quad \le \pi_{ii}\big[\tilde{G}^T\big(\eta(k)\big)Q\tilde{G}\big(\eta(k)\big) - \tilde{G}^T\big(\eta(k-\tau_{1,i})\big)Q\tilde{G}\big(\eta(k-\tau_{1,i})\big)\big] \\
&\qquad + \sum_{j\neq i}\pi_{ij}\Bigg[\sum_{v=k+1-\bar{\tau}_1}^{k} - \sum_{v=k-\tau_{1,i}}^{k-1}\Bigg] \times \tilde{G}^T\big(\eta(v)\big)Q\tilde{G}\big(\eta(v)\big) \\
&\quad \le \tilde{G}^T\big(\eta(k)\big)Q\tilde{G}\big(\eta(k)\big) - \tilde{G}^T\big(\eta(k-\tau_{1,i})\big)Q\tilde{G}\big(\eta(k-\tau_{1,i})\big) \\
&\qquad + (1-\pi_{ii})\sum_{v=k+1-\bar{\tau}}^{k-\tau_{1,i}}\tilde{G}^T\big(\eta(v)\big)Q\tilde{G}\big(\eta(v)\big) \\
&\quad \le \tilde{G}^T\big(\eta(k)\big)Q\tilde{G}\big(\eta(k)\big) - \tilde{G}^T\big(\eta(k-\tau_{1,i})\big)Q\tilde{G}\big(\eta(k-\tau_{1,i})\big) \\
&\qquad + (1-\underline{\pi})\sum_{v=k-\bar{\tau}_1+1}^{k-\underline{\tau}_1}\tilde{G}^T\big(\eta(v)\big)Q\tilde{G}\big(\eta(v)\big), \qquad (17)
\end{aligned}$$

$$\begin{aligned}
&\mathbb{E}\big[V_3\big(\eta_{k+1}, k+1, r(k+1)\big)|\eta_k, r(k)=i\big] - V_3(\eta_k, k, i) \\
&\quad = \sum_{l=\underline{\tau}_1}^{\bar{\tau}_1-1}\big(\tilde{G}^T\big(\eta(k)\big)\bar{Q}\tilde{G}\big(\eta(k)\big) - \tilde{G}^T\big(\eta(k-l)\big)\bar{Q}\tilde{G}\big(\eta(k-l)\big)\big) \\
&\quad = (1-\underline{\pi})(\bar{\tau}_1-\underline{\tau}_1)\tilde{G}^T\big(\eta(k)\big)Q\tilde{G}\big(\eta(k)\big) - (1-\underline{\pi})\sum_{v=k-\bar{\tau}_1+1}^{k-\underline{\tau}_1}\tilde{G}^T\big(\eta(v)\big)Q\tilde{G}\big(\eta(v)\big), \qquad (18)
\end{aligned}$$

and

$$\begin{aligned}
&\mathbb{E}\big[V_4\big(\eta_{k+1}, k+1, r(k+1)\big)|\eta_k, r(k)=i\big] - V_4(\eta_k, k, i) \\
&\quad = \sum_{j=1}^{N}\pi_{ij}\sum_{l=1}^{\tau_{2,j}}\sum_{v=k-l+1}^{k}\tilde{H}^T\big(\eta(v)\big)R\tilde{H}\big(\eta(v)\big) - \sum_{l=1}^{\tau_{2,i}}\sum_{v=k-l}^{k-1}\tilde{H}^T\big(\eta(v)\big)R\tilde{H}\big(\eta(v)\big) \\
&\quad = \pi_{ii}\sum_{l=1}^{\tau_{2,i}}\big[\tilde{H}^T\big(\eta(k)\big)R\tilde{H}\big(\eta(k)\big) - \tilde{H}^T\big(\eta(k-l)\big)R\tilde{H}\big(\eta(k-l)\big)\big] \\
&\qquad + \sum_{j\neq i}\pi_{ij}\Bigg[\sum_{l=1}^{\tau_{2,i}}\sum_{v=k-l+1}^{k} - \sum_{l=1}^{\tau_{2,i}}\sum_{v=k-l}^{k-1}\Bigg] \times \tilde{H}^T\big(\eta(v)\big)R\tilde{H}\big(\eta(v)\big) \\
&\qquad + \sum_{j\neq i}\pi_{ij}\Bigg[\sum_{l=1}^{\tau_{2,j}}\sum_{v=k-l+1}^{k} - \sum_{l=1}^{\tau_{2,i}}\sum_{v=k-l+1}^{k}\Bigg] \times \tilde{H}^T\big(\eta(v)\big)R\tilde{H}\big(\eta(v)\big) \\
&\quad \le \sum_{l=1}^{\tau_{2,i}}\big(\tilde{H}^T\big(\eta(k)\big)R\tilde{H}\big(\eta(k)\big) - \tilde{H}^T\big(\eta(k-l)\big)R\tilde{H}\big(\eta(k-l)\big)\big)
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{j\neq i}\pi_{ij}\Bigg[\sum_{l=\underline{\tau}_2+1}^{\bar{\tau}_2}\sum_{v=k-l+1}^{k}\tilde{H}^T\big(\eta(v)\big)R\tilde{H}\big(\eta(v)\big)\Bigg]\\
&\le\big(\tau_{2,i}+(1-\pi_{ii})(\bar{\tau}_2-\underline{\tau}_2)\big)\tilde{H}^T\big(\eta(k)\big)R\tilde{H}\big(\eta(k)\big)\\
&\quad-\sum_{v=1}^{\tau_{2,i}}\tilde{H}^T\big(\eta(k-v)\big)R\tilde{H}\big(\eta(k-v)\big)\\
&\quad+(1-\underline{\pi})\\
&\quad\times\sum_{l=\underline{\tau}_2+1}^{\bar{\tau}_2}\sum_{v=k-l+1}^{k-1}\tilde{H}^T\big(\eta(v)\big)R\tilde{H}\big(\eta(v)\big),
\end{aligned}\tag{19}
$$

$$
\begin{aligned}
&\mathbb{E}\big[V_5\big(\eta_{k+1},k+1,r(k+1)\big)|\eta_k,r(k)=i\big]\\
&\quad-V_5(\eta_k,k,i)\\
&=\sum_{s=\underline{\tau}_2+1}^{\bar{\tau}_2}\sum_{l=1}^{s-1}\big[\tilde{H}^T\big(\eta(k)\big)\bar{R}\tilde{H}\big(\eta(k)\big)\\
&\quad-\tilde{H}^T\big(\eta(k-l)\big)\bar{R}\tilde{H}\big(\eta(k-l)\big)\big]\\
&=(1-\underline{\pi})\Bigg[\frac{1}{2}(\bar{\tau}_2-\underline{\tau}_2)(\bar{\tau}_2+\underline{\tau}_2-1)\\
&\quad\times\tilde{H}^T\big(\eta(k)\big)R\tilde{H}\big(\eta(k)\big)\\
&\quad-\sum_{l=\underline{\tau}_2+1}^{\bar{\tau}_2}\sum_{v=k-l+1}^{k-1}\tilde{H}^T\big(\eta(v)\big)R\tilde{H}\big(\eta(v)\big)\Bigg].
\end{aligned}\tag{20}
$$

From (15)–(20), we have

$$
\begin{aligned}
&\mathbb{E}\big[V\big(\eta_{k+1},k+1,r(k+1)\big)|\eta_k,r(k)=i\big]\\
&\quad-V(\eta_k,k,i)\\
&\le\xi^T(k,i)Z^T(k,i)\bar{P}_iZ(k,i)\xi(k,i)\\
&\quad-\eta^T(k)P_i\eta(k)+\kappa_0\tilde{G}^T\big(\eta(k)\big)Q\tilde{G}\big(\eta(k)\big)\\
&\quad-\tilde{G}^T\big(\eta(k-\tau_{1,i})\big)Q\tilde{G}\big(\eta(k-\tau_{1,i})\big)\\
&\quad+\kappa_i\tilde{H}^T\big(\eta(k)\big)R\tilde{\eta(k)}\\
&\quad-\sum_{v=1}^{\tau_{2,i}}\tilde{H}^T\big(\eta(k-v)\big)R\tilde{H}\big(\eta(k-v)\big).
\end{aligned}\tag{21}
$$

Lemma 1 ensures that

$$
\begin{aligned}
&-\sum_{v=1}^{\tau_{2,i}}\tilde{H}^T\big(\eta(k-v)\big)R\tilde{H}\big(\eta(k-v)\big)\\
&\le-\frac{1}{\tau_{2,i}}\Bigg[\sum_{v=1}^{\tau_{2,i}}\tilde{H}\big(\eta(k-v)\big)\Bigg]^T\\
&\quad\times R\sum_{v=1}^{\tau_{2,i}}\tilde{H}\big(\eta(k-v)\big).
\end{aligned}\tag{22}
$$

From Assumption 1 and (13), the definition of functions $\tilde{F}(\cdot)$, $\tilde{G}(\cdot)$, and $\tilde{H}(\cdot)$, also considering Lemma 2, it implies easily that the following inequalities hold:

$$
\begin{aligned}
&\eta^T(k)\varOmega_i\bar{\varLambda}_1\eta(k)-2\eta^T(k)\varOmega_i\bar{\varLambda}_2\tilde{F}\big(\eta(k)\big)\\
&\quad+\tilde{F}^T\big(\eta(k)\big)\varOmega_i\tilde{F}\big(\eta(k)\big)\le 0,
\end{aligned}\tag{23}
$$

$$
\begin{aligned}
&\eta^T(k)\varTheta_i\bar{\varSigma}_1\eta(k)-2\eta^T(k)\varTheta_i\bar{\varSigma}_2\tilde{G}\big(\eta(k)\big)\\
&\quad+\tilde{G}^T\big(\eta(k)\big)\varTheta_i\tilde{G}\big(\eta(k)\big)\le 0,
\end{aligned}\tag{24}
$$

$$
\begin{aligned}
&\eta^T(k)\Delta_i\bar{\varUpsilon}_1\eta(k)-2\eta^T(k)\Delta_i\bar{\varUpsilon}_2\tilde{H}\big(\eta(k)\big)\\
&\quad+\tilde{H}^T\big(\eta(k)\big)\Delta_i\tilde{H}\big(\eta(k)\big)\le 0.
\end{aligned}\tag{25}
$$

Condition (4) easily guarantees that the following inequality holds:

$$
\big|\tilde{S}\big(k,\eta(k)\big)\big|\le\big|W\tilde{x}(k)\big|=\big|W_o\eta(k)\big|,\quad\forall\eta(k)\in\mathbb{R}^{2n}.\tag{26}
$$

Consider (22)–(26) with (21), we have

$$
\begin{aligned}
&\mathbb{E}\big[V\big(\eta_{k+1},k+1,r(k+1)\big)|\eta_k,r(k)=i\big]\\
&\quad-V(\eta_k,k,i)\\
&\le\xi^T(k,i)Z^T(k,i)\bar{P}_iZ(k,i)\xi(k,i)\\
&\quad-\eta^T(k)P_i\eta(k)+\kappa_0\tilde{G}^T\big(\eta(k)\big)Q\tilde{G}\big(\eta(k)\big)\\
&\quad-\tilde{G}^T\big(\eta(k-\tau_{1,i})\big)Q\tilde{G}\big(\eta(k-\tau_{1,i})\big)\\
&\quad+\kappa_i\tilde{H}^T\big(\eta(k)\big)R\tilde{H}\big(\eta(k)\big)\\
&\quad-\frac{1}{\tau_{2,i}}\Bigg[\sum_{v=1}^{\tau_{2,i}}\tilde{H}\big(\eta(k-v)\big)\Bigg]R\sum_{v=1}^{\tau_{2,i}}\tilde{H}\big(\eta(k-v)\big)\\
&\quad-\big(\eta^T(k)\varOmega_i\bar{\varLambda}_1\eta(k)-2\eta^T(k)\varOmega_i\bar{\varLambda}_2\tilde{F}\big(\eta(k)\big)\\
&\quad+\tilde{F}^T\big(\eta(k)\big)\varOmega_i\tilde{F}\big(\eta(k)\big)\big)\\
&\quad-\big(\eta^T(k)\varTheta_i\bar{\varSigma}_1\eta(k)-2\eta^T(k)\varTheta_i\bar{\varSigma}_2\tilde{G}\big(\eta(k)\big)\\
&\quad+\tilde{G}^T\big(\eta(k)\big)\varTheta_i\tilde{G}\big(\eta(k)\big)\big)\\
&\quad-\big(\eta^T(k)\Delta_i\bar{\varUpsilon}_1\eta(k)-2\eta^T(k)\Delta_i\bar{\varUpsilon}_2\tilde{H}\big(\eta(k)\big)\\
&\quad+\tilde{H}^T\big(\eta(k)\big)\Delta_i\tilde{H}\big(\eta(k)\big)\big)
\end{aligned}
$$

$$- \big(\vartheta_i \tilde{S}^T\big(k,\eta(k)\big)\tilde{S}\big(k,\eta(k)\big) - \vartheta_i \eta^T(k) W_o^T W_o \eta(k)\big) = \xi^T(k,i)\tilde{\Psi}_i(k)\xi(k,i), \tag{27}$$

where $\tilde{\Psi}_i(k) = \varphi_i + Z^T(k,i)\bar{P}_i Z(k,i)$.

Letting

$$\tilde{\Phi}(k,i) = \begin{pmatrix} \Phi(k,i) & 0 & 0 & 0 \\ 0 & \Phi(k,i) & 0 & 0 \\ 0 & 0 & \Phi(k,i) & 0 \\ 0 & 0 & 0 & \Phi(k,i) \end{pmatrix},$$

from Assumption 2, we have

$$\begin{aligned} &\Delta Z_o(k,i) \\ &\quad = \tilde{M}_1(i)\begin{pmatrix} \Phi(k,i) & 0 \\ 0 & \Phi(k,i) \end{pmatrix}\hat{N}_1(i) \\ &\qquad + \tilde{M}_2(i)\begin{pmatrix} \Phi(k,i) & 0 \\ 0 & \Phi(k,i) \end{pmatrix}\hat{N}_2(i) \\ &\quad = \tilde{M}(i)\tilde{\Phi}(k,i)\tilde{N}(i). \end{aligned}$$

Noting (11), it can be seen that

$$\bar{P}_i - \varepsilon_i^{-1}\bar{P}_i\tilde{M}(i)\tilde{M}^T(i)\bar{P}_i > 0, \tag{28}$$

which easily implies that

$$\bar{P}_i^{-1} - \varepsilon_i^{-1}\tilde{M}(i)\tilde{M}^T(i) > 0. \tag{29}$$

By Lemma 3, we can get

$$\begin{aligned} &Z^T(k,i)\bar{P}_i Z(i) \\ &\quad = \big(Z_o(i) + \Delta Z_o(k,i)\big)^T \bar{P}_i\big(Z_o(i) + \Delta Z_o(k,i)\big) \\ &\quad = \big(Z_o(i) + \tilde{M}(i)\tilde{\Phi}(k,i)\tilde{N}(i)\big)^T \\ &\qquad \times \bar{P}_i\big(Z_o(i) + \tilde{M}(i)\tilde{\Phi}(k,i)\tilde{N}(i)\big) \\ &\quad \le Z_o^T(i)\big(\bar{P}_i^{-1} - \varepsilon_i^{-1}\tilde{M}(i)\tilde{M}^T(i)\big)^{-1}Z_o(i) \\ &\qquad + \varepsilon_i\tilde{N}^T(i)\tilde{N}(i), \end{aligned} \tag{30}$$

then one obtains

$$\tilde{\Psi}_i \le \varphi_i + Z_o^T(i)\big(\bar{P}_i^{-1} - \varepsilon_i^{-1}\tilde{M}(i)\tilde{M}^T(i)\big)^{-1}Z_o(i) + \varepsilon_i\tilde{N}^T(i)\tilde{N}(i) < 0; \tag{31}$$

the last inequality is assured from condition (11) by the well-known Schur lemma. Therefore, we have that system (13) is robustly asymptotically stable. This completes the proof. □

In the following, it is proved that if a set of matrix inequalities are feasible, the desired estimator gain matrices can be obtained.

Theorem 2 *Suppose Assumptions 1–2 and condition (4) hold. Then system (7) is an robustly asymptotic state estimator of system (1) with measurement output (3) if there exist scalar constants $\varepsilon_i > 0$, $\vartheta_i > 0$, matrices $P_{1i} > 0$, $P_{2i} > 0$, $Q > 0$ and $R > 0$, diagonal matrices $\Omega_{1i} > 0$, $\Omega_{2i} > 0$, $\Theta_{1i} > 0$, $\Theta_{2i} > 0$, $\Delta_{1i} > 0$, $\Delta_{2i} > 0$, and Y_i $(i \in \mathcal{S})$ satisfying:*

$$\Psi_i \triangleq \begin{pmatrix} \Pi_i & U_{1,i}^T & 0 \\ * & -\bar{P}_i & U_{2,i}^T \\ * & * & -\varepsilon_i I \end{pmatrix} < 0, \quad i \in \mathcal{S} \tag{32}$$

where

$$\bar{P}_i = \mathrm{diag}\{\bar{P}_{1i}, \bar{P}_{2i}\}, \qquad \bar{P}_{1i} = \sum_{j=1}^{N}\pi_{ij}P_{1j}, \qquad \bar{P}_{2i} = \sum_{j=1}^{N}\pi_{ij}P_{2j};$$

$$U_{1,i} = \big(\bar{P}_i\tilde{D}_o(i) + \bar{Y}(i)E_o(i)\ \bar{P}_iA_o(i)\ 0\ \bar{P}_iB_o(i)\ 0\ \bar{P}_iC_o(i)\ \bar{Y}(i)\big),$$

$$\tilde{D}_o(i) = \mathrm{diag}\big\{D(i), D(i)\big\}, \qquad E_o(i) = \begin{pmatrix} 0 & 0 \\ 0 & E(i) \end{pmatrix}, \qquad \bar{Y}(i) = \begin{pmatrix} 0 & 0 \\ 0 & Y(i) \end{pmatrix},$$

$$U_{2,i} = \begin{pmatrix} M_1^T(i)\bar{P}_{1i} & M_1^T(i)\bar{P}_{2i} \\ 0 & M_2^T(i)Y_i^T \end{pmatrix}$$

and the other notations are the same as defined in Theorem 1. Moreover, the robust observers in (7) are designed to be as

$$K(i) = \bar{P}_{2i}^{-1} Y_i, \quad i \in \mathcal{S}. \tag{33}$$

Proof In the proof of Theorem 1, let $P_i = \mathrm{diag}\{P_{1i}, P_{2i}\}$, then $\bar{P}_i = \mathrm{diag}\{\bar{P}_{1i}, \bar{P}_{2i}\}$. Also by noting (33) that $Y_i = \bar{P}_{2i} K(i)$, it implies easily that $\bar{P}_i D_o(i) = \bar{P}_i \tilde{D}_o + \bar{Y}(i) E_o(i)$ and $\bar{P}_i \tilde{M}(i) = U_{2,i}^T$. From Theorem 1, it is easily shown that Theorem 2 holds, and this completes the proof. □

Remark 3 In Theorem 2, the obtained conditions are LMIs, so that one can verify them easily by using LMI toolbox. Notice that in the two theorems, the conditions $\tau_{i,r(k)} \geq 1$ and $\bar{\tau}_i > \underline{\tau}_i$ $(i = 1, 2; r(k) \in \mathcal{S})$ are utilized. If they turn to be constant delays, we can also get the similar conclusions.

4 Numerical example

In this section, a 3-node network with Markovian switching between two modes is considered to illustrate the effectiveness of the criteria obtained in the above section. In the nonlinear delay system model (1), let $\tau_{1,1} = 1$, $\tau_{1,2} = 3$, $\tau_{2,1} = 2$, $\tau_{2,2} = 4$, $D(1) = \mathrm{diag}\{-0.4, -0.5, -0.6\}$, $D(2) = \mathrm{diag}\{-0.3, -0.5, -0.7\}$;

$$A(1) = \begin{pmatrix} 0.2 & 0.5 & 0.1 \\ 0.2 & -0.4 & 0 \\ 0 & -0.1 & 0.2 \end{pmatrix},$$

$$A(2) = \begin{pmatrix} 0.6 & -0.2 & 0.1 \\ 0.1 & -0.1 & 0.2 \\ 0.1 & 0 & 0.2 \end{pmatrix};$$

$$B(1) = \begin{pmatrix} 0.2 & 0.2 & 0.2 \\ 0.2 & -0.2 & 0 \\ 0.2 & -0.1 & -0.1 \end{pmatrix},$$

$$B(2) = \begin{pmatrix} 0.2 & 0.2 & 0.1 \\ 0.2 & -0.2 & 0 \\ 0.3 & -0.1 & -0.1 \end{pmatrix};$$

$$C(1) = \begin{pmatrix} 0.2 & 0.2 & -0.1 \\ 0 & 0.4 & 0.3 \\ -0.3 & 0 & 0.2 \end{pmatrix},$$

$$C(2) = \begin{pmatrix} 0.2 & -0.2 & 0.1 \\ 0.1 & 0.2 & 0.3 \\ 0.8 & 0 & 0.2 \end{pmatrix};$$

$$E(1) = E(2) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix};$$

$$\Pi = \begin{pmatrix} 0.6 & 0.4 \\ 0.5 & 0.5 \end{pmatrix};$$

$$M_1(1) = M_1(2) = \begin{pmatrix} 0.3 & 0 & 0 & 0 \\ 0.1 & 0.4 & 0 & 0 \\ 0 & 0.2 & 0 & 0 \end{pmatrix},$$

$$M_2(1) = M_2(2) = \begin{pmatrix} 0 & 0 & 0.1 & 0 \\ 0 & 0 & 0 & 0.1 \end{pmatrix},$$

$$N_1(1) = N_1(2) = \begin{pmatrix} 0.2 & 0.2 & 0.1 & 0 \\ 0.1 & 0.1 & 0 & 0.1 \\ 0.1 & 0.2 & 0 & 0 \end{pmatrix}^T,$$

$$N_2(1) = N_3(1) = N_4(1) = N_2(2) = N_3(2) = N_4(2) = \begin{pmatrix} 0.1 & 0 & 0 & 0 \\ 0.3 & 0.1 & 0 & 0 \\ 0.1 & 0.2 & 0 & 0 \end{pmatrix}^T.$$

The activation functions are taken as follows:

$$f_1(s) = g_1(s) = h_1(s) = \tanh(0.1s),$$
$$f_2(s) = g_2(s) = h_2(s) = \tanh(0.14s),$$
$$f_3(s) = g_3(s) = h_3(s) = 0.1\tanh(s), \quad s \in \mathbb{R}$$

and the nonlinear function $S(k, x) = (0.2\sin x_1, 0.2\cos x_2)^T$. It can be readily verified that

$$\Lambda_1 = \Sigma_1 = \Upsilon_1 = 0,$$
$$\Lambda_2 = \Sigma_2 = \Upsilon_2 = \mathrm{diag}\{0.05, 0.07, 0.05\},$$
$$W = \begin{pmatrix} 0.2 & 0 & 0 \\ 0 & 0.2 & 0 \end{pmatrix}.$$

With the above parameters, by using the Matlab toolbox, inequalities (32) have feasible solutions. Here, only some of the solution matrices are given for space consideration:

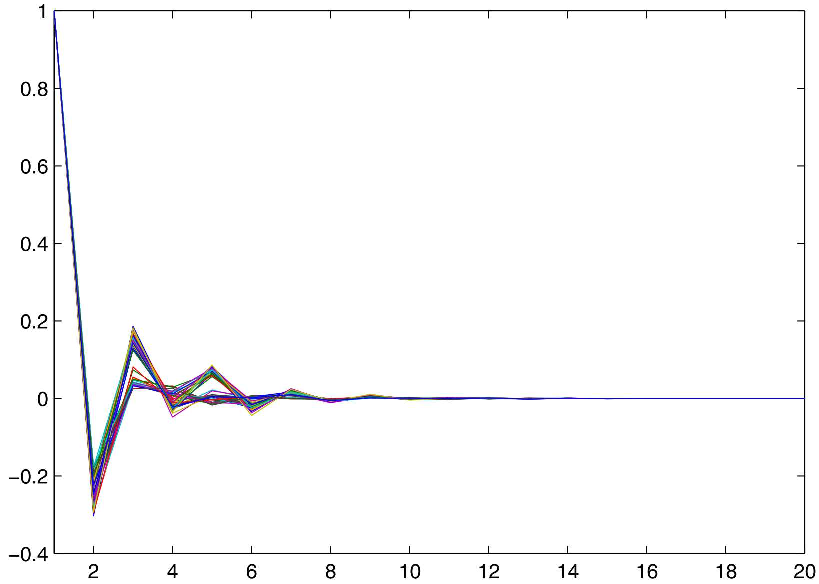

Fig. 1 State x_1 in system (1)

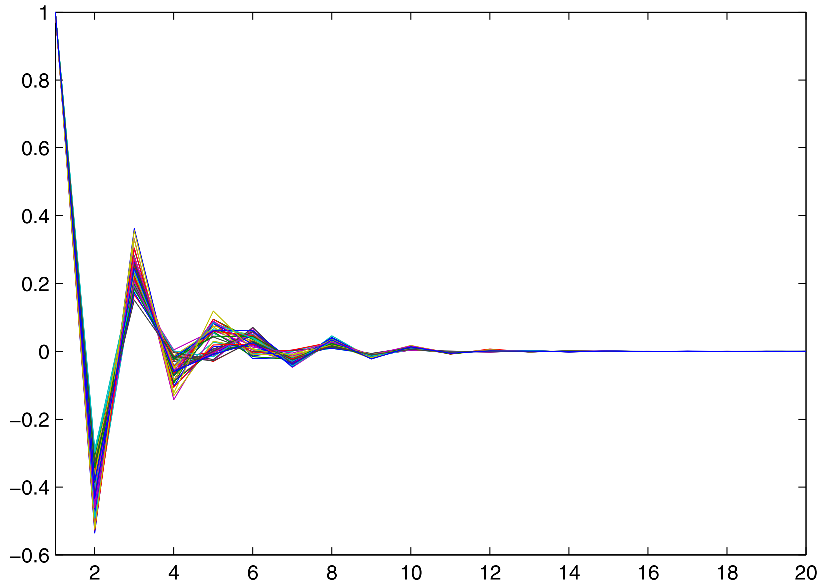

Fig. 2 State x_2 in system (1)

$$P_{11} = \begin{pmatrix} 0.1217 & -0.0043 & 0.0148 \\ -0.0043 & 0.0440 & 0.0049 \\ 0.0148 & 0.0049 & 0.0451 \end{pmatrix}, \qquad P_{12} = \begin{pmatrix} 0.0021 & -0.0002 & -0.0000 \\ -0.0002 & 0.0010 & -0.0009 \\ -0.0000 & -0.0009 & 0.0021 \end{pmatrix},$$

$$P_{21} = \begin{pmatrix} 0.1526 & -0.0067 & 0.0262 \\ -0.0067 & 0.0468 & 0.0048 \\ 0.0262 & 0.0048 & 0.0734 \end{pmatrix}, \qquad P_{22} = \begin{pmatrix} 0.0027 & -0.0004 & 0.0002 \\ -0.0004 & 0.0017 & -0.0017 \\ 0.0002 & -0.0017 & 0.0038 \end{pmatrix},$$

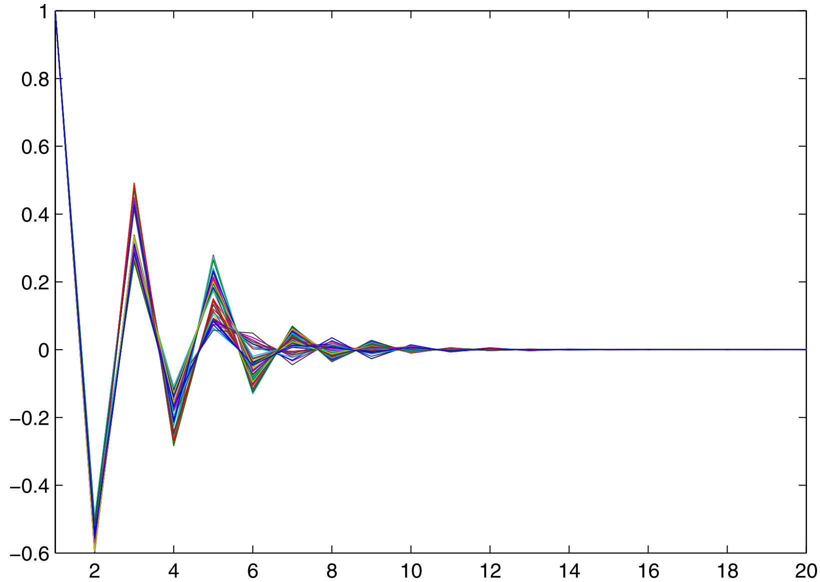

Fig. 3 State x_3 in system (1)

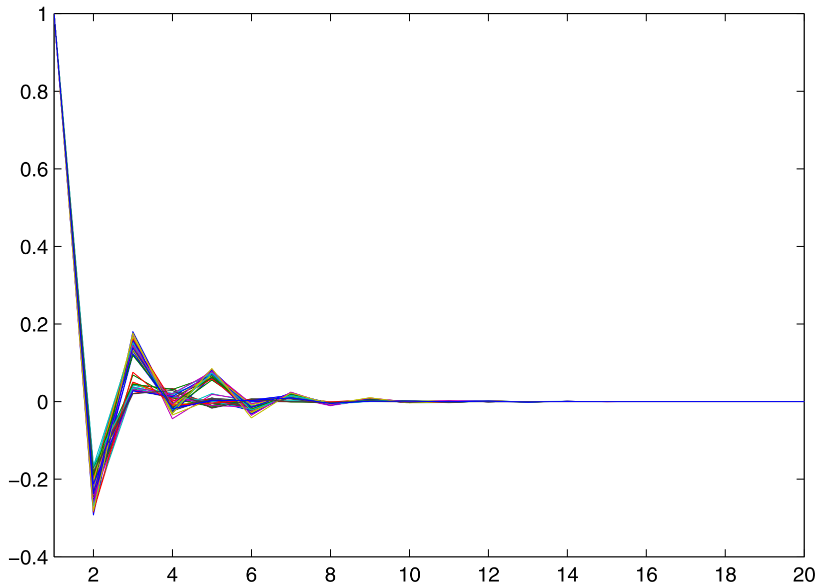

Fig. 4 Estimation error $\tilde{x}_1$ in system (12)

$$Q = \begin{pmatrix} 0.5850 & -0.0304 & -0.0355 & 0.0005 & 0.0003 & 0.0007 \\ -0.0304 & 0.0878 & -0.0057 & 0.0002 & 0.0001 & -0.0000 \\ -0.0355 & -0.0057 & 0.0814 & -0.0004 & 0.0000 & 0.0000 \\ 0.0005 & 0.0002 & -0.0004 & 0.0198 & -0.0018 & -0.0021 \\ 0.0003 & 0.0001 & 0.0000 & -0.0018 & 0.0026 & -0.0018 \\ 0.0007 & -0.0000 & 0.0000 & -0.0021 & -0.0018 & 0.0111 \end{pmatrix},$$

$$\Omega_{11} = \mathrm{diag}\{0.7735, 0.4766, 0.1932\}, \qquad \Theta_{12} = \mathrm{diag}\{2.4745, 0.3452, 0.2948\},$$

$$\Delta_{12} = \mathrm{diag}\{16.3915, 1.4253, 2.9396\}, \qquad \Delta_{21} = \mathrm{diag}\{0.2139, 0.0254, 0.0683\},$$

$$\vartheta_1 = 0.0043, \qquad \vartheta_2 = 0.0052, \qquad \varepsilon_1 = 0.0634, \qquad \varepsilon_2 = 0.0650.$$

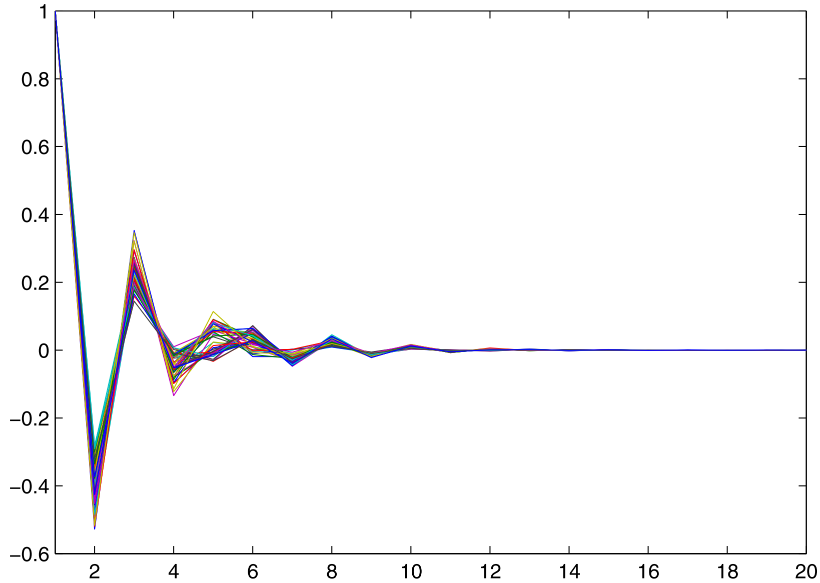

Fig. 5 Estimation error $\tilde{x}_2$ in system (12)

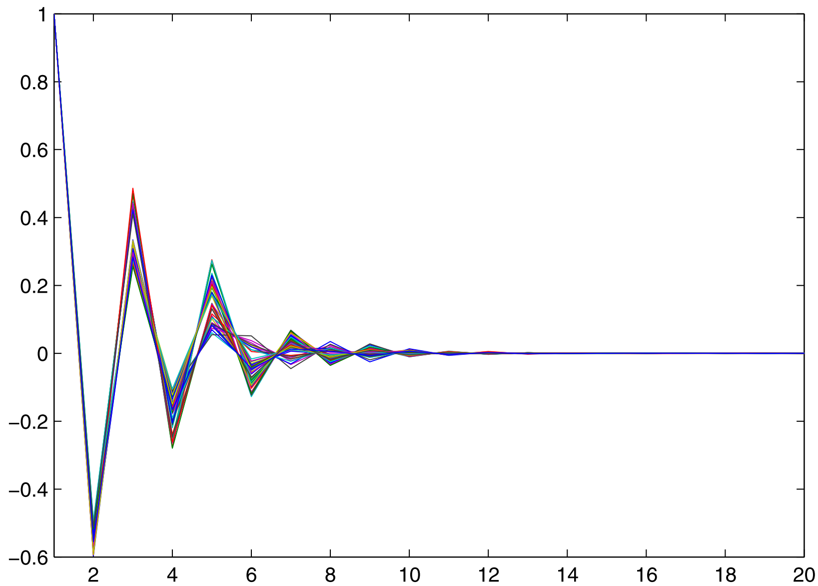

Fig. 6 Estimation error $\tilde{x}_3$ in system (12)

Therefore, from Theorem 2, the robust observers are designed to be as:

$$K(1) = \begin{pmatrix} 0.0112 & -0.0026 \\ -0.0026 & 0.0118 \\ -0.0027 & 0.0057 \end{pmatrix},$$

$$K(2) = \begin{pmatrix} 0.0119 & -0.0022 \\ -0.0006 & 0.0097 \\ -0.0074 & 0.0119 \end{pmatrix}.$$

For simulation, let $\Phi(k,i)$ $(i = 1, 2)$ take values randomly in interval $[-1, 1]$; the initial conditions are chosen to be $x(-4) = x(-3) = x(-2) = x(-1) = [0\ 0\ 0]^T$, $x(0) = [1\ 1\ 1]^T$; and $\hat{x}(-4) = \hat{x}(-3) = \hat{x}(-2) = \hat{x}(-1) = \hat{x}(0) = [0\ 0\ 0]^T$. From Figs. 1, 2 and 3, one can get the original system is stable; and from Figs. 4, 5 and 6, it is shown that with the robust state estimator given above, the full-order state estimator (7) approaches globally robustly to the original system.

5 Conclusions

In this paper, we have discussed the robust state estimation for a class of discrete-time neural networks with Markovian parameters and mode-dependent mixed time-delays. An robust observer is designed to estimate the neuron states by using available output measurements. By using the Lyapunov–Krasovskii functional, we have obtained the sufficient conditions in LMI form ensuring the existence of the robust state estimators. A numerical example has been given to demonstrate the usefulness of the derived LMI-based conditions. Inspired from the excellent work in [9, 28], in the near future, the randomly varying sensor delays and missing measurement phenomena should be taken into account for the system discussed in our paper.

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