Nonlinear Dyn (2011) 66:831–837
DOI 10.1007/s11071-011-9944-2

ORIGINAL PAPER

# Adaptive synchronization of fractional-order chaotic systems via a single driving variable

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Received: 27 November 2010 / Accepted: 5 January 2011 / Published online: 18 February 2011


**Abstract** This letter investigates the synchronization of a class of three-dimensional fractional-order chaotic systems. Based on sliding mode variable structure control theory and adaptive control technique, a single-state adaptive-feedback controller containing a novel fractional integral sliding surface is developed to synchronize a class of fractional-order chaotic systems. The present controller, which only contains a single driving variable, is simple both in design and implementation. Simulation results for three fractional-order chaotic systems are provided to illustrate the effectiveness of the proposed scheme.

**Keywords** Sliding mode control · Fractional-order chaotic system · Fractional integral sliding surface · Adaptive synchronization

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## 1 Introduction

Even though fractional calculus is a mathematical topic with more than 300 years history, its application to physics and engineering has attracted lots of attention only in the recent years. It has been found that many systems in interdisciplinary fields can be described by fractional differential equations, such as viscoelastic systems [1], dielectric polarization [2], electrode–electrolyte polarization [3], some finance systems, and electromagnetic wave systems [4]. Moreover, applications of fractional calculus have been reported in many areas such as signal processing [5], image processing [6], automatic control [7], and robotics [8, 9]. These examples and many other similar samples perfectly clarify the importance of consideration and analysis of dynamical systems with fractional-order models.

Recently, studying fractional-order chaotic systems has become an active research field. Synchronization of fractional-order chaotic systems starts to attract increasing attention due to its potential applications in secure communication and control processing. Some approaches have been proposed to achieve chaos synchronization in fractional-order chaotic systems, such as PC control [10, 11], nonlinear state observer method [12, 13], adaptive control [14], unidirectional linear error feedback coupling [15], sliding mode control [16], a scalar transmitted signal method [17], etc. However, in the formulation of the chaos synchronization problem, the proposed controllers in previous works in most cases are too complex both in design and implementation.

In this letter, we design a fractional integral (FI) sliding surface and propose a single driving variable feedback control approach to synchronize a class of three-dimensional fractional-order chaotic systems. The proposed scheme, based on sliding mode variable structure control theory and adaptive control technique is simple, global, and theoretically rigorous. To show a wider applicability of our method, we give illustrations by using three different fractional-order chaotic systems with numerical simulations to verify the effectiveness of the proposed synchronization scheme. The rest of the letter is organized as follows. Section 2 gives the main results. In Sect. 3, three groups of examples are used to verify the effectiveness of the proposed scheme. The letter is concluded in Sect. 4.

## 2 Main results

Let a three-dimensional fractional-order chaotic error system be

$$\begin{aligned} D_t^\alpha x &= f_1(x,z),\\ D_t^\alpha z &= f_2(x,z), \end{aligned} \tag{1}$$

where $D_t^\alpha = d^\alpha/dt^\alpha$, $0<\alpha<1$, $x\in R^2$, $z\in R^1$ are the state error vectors, $f_1$ and $f_2$ are continuous differential nonlinear functions with $f_1(0,0) = f_2(0,0) = 0$.

To describe the new design and analysis, the following assumption is needed:

**Assumption 1** *The function $f_1(x,z)$ is smooth in a neighborhood of $z=0$, and the subsystem $D_t^\alpha x = f_1(x,0)$ is asymptotically stable about the origin $x=0$ for all $x$.*

*Remark 1* System (1) is very general, which contains almost all fractional-order chaotic error systems.

*Remark 2* The vector function $f_1(x,z)$ being smooth in a neighborhood of $z=0$, i.e., there are two positive constants $\lambda_0$, $\lambda_1$ such that $\|f_1(x,z)-f_1(x,0)\|\le \lambda_0|z|$, and then $\|x\|_1 \le \lambda_1|z|$. Thus, for chaos error system (1), there exist two positive numbers $p_1$, $p_2$ and a large enough positive constant $\lambda_2$ such that

$$\begin{aligned} \|f_2(x,z)\| &\le p_1\|x\|_1 + p_2|z| \le p_1\lambda_1|z| + p_2|z|\\ &\le \lambda_2|z|. \end{aligned}$$

In order to stabilize the error system (1) to its equilibrium point $x=0$, $z=0$, we add the following adaptive controller $u$ to the error system (1), so that the controlled error system (1) is given by

$$\begin{aligned} D_t^\alpha x &= f_1(x,z),\\ D_t^\alpha z &= f_2(x,z)+u, \end{aligned} \tag{2}$$

with

$$u = -k|z|\,\mathrm{sign}(s), \tag{3}$$

where $s$ is the FI sliding surface defined as

$$s = D_t^{\alpha-1}z + \int_0^t cz(\tau)\,d\tau \quad (c>0), \tag{4}$$

and $k$ is adapted according to the following update law:

$$\dot{k} = \theta|z||s| \quad (\theta>0). \tag{5}$$

**Theorem 1** *Starting from any initial values, the controlled error system (3) is asymptotically stable under the FI sliding surface (4) and updating law (5).*

*Proof* The FI sliding surface is defined by (4). For the existence of the sliding mode, it is necessary and sufficient that

$$s = D_t^{\alpha-1}z + \int_0^t cz(\tau)\,d\tau = 0 \tag{6}$$

and

$$\dot{s} = D_t^\alpha z + cz = 0. \tag{7}$$

Therefore, the following sliding mode dynamics can be obtained as

$$D_t^\alpha z = -cz. \tag{8}$$

Obviously, if the design parameter $c>0$, the stability of (8) is surely guaranteed, that is, $\lim_{t\to\infty} z = 0$. According to Assumption 1, we have $\lim_{t\to\infty} x = 0$. Hence,

$$\lim_{t\to\infty} x = 0, \qquad \lim_{t\to\infty} z = 0. \tag{9}$$

In what follows, the proposed adaptive control scheme will be proved to be able to derive the controlled error system (3) onto the sliding surface $s(t)=0$.

Consider the following Lyapunov function candidate:

$$V(t) = \frac{1}{2}s^2 + \frac{1}{2\theta}(k-k^*)^2 \quad (k^* > \lambda_2). \tag{10}$$

Taking the derivative of $V(t)$ with respect to time, one has

$$\begin{aligned}\dot{V}(t) &= s\dot{s} + (k-k^*)\dot{k}/\theta \\ &= s\big(D_t^\alpha z + cz\big) + (k-k^*)\dot{k}/\theta \\ &= s\big(f_2(x,z) - k|z|\operatorname{sign}(s) + cz\big) \\ &\quad + (k-k^*)|z||s| \\ &\le \lambda_2|z||s| - k|z||s| + (k-k^*)|z||s| \\ &= -(k^*-\lambda_2)|z||s| \le 0. \end{aligned}$$

□

According to Lyapunov stability theory, $\lim_{t\to\infty} s = 0$. Furthermore, according to (9), the sliding mode is asymptotically stable. It clearly shows that the controlled error system (3) can be stabilized with the single controller $u = -k|z|\operatorname{sign}(s)$.

## 3 Applications

This section of the paper presents three illustrative examples to demonstrate the effectiveness of the proposed synchronization scheme. MATLAB software is used in this simulation.

*Example 1* Consider the fractional-order Arneodo's system [18]

$$\begin{aligned} D_t^\alpha x_1 &= x_2, \\ D_t^\alpha x_2 &= x_3, \\ D_t^\alpha x_3 &= ax_1 - bx_2 - cx_3 - dx_1^3, \end{aligned} \tag{11}$$

where $a = 5.5$, $b = 3.5$, $c = 1$, $d = 1$, and $x_1, x_2, x_3$ are state variables. Let system (11) be master system. Then, the slave system with variable $y$ is given as

$$\begin{aligned} D_t^\alpha y_1 &= y_2, \\ D_t^\alpha y_2 &= y_3, \\ D_t^\alpha y_3 &= ay_1 - by_2 - cy_3 - dy_1^3 \end{aligned} \tag{12}$$

and the error system ($e = y - x$) is as follows:

$$\begin{aligned} D_t^\alpha e_1 &= e_2, \\ D_t^\alpha e_2 &= e_3, \\ D_t^\alpha e_3 &= ae_1 - be_2 - ce_3 - d\big(x_1^2 + x_1y_1 + y_1^2\big)e_1. \end{aligned} \tag{13}$$

It is easy to see that if $e_1 = 0$, the following two-dimensional subsystem of system (13),

$$\begin{aligned} D_t^\alpha e_2 &= e_3, \\ D_t^\alpha e_3 &= -be_2 - ce_3 \end{aligned}$$

is asymptotically stable about the origin $e_2 = 0, e_3 = 0$ for all $e_2, e_3$. Therefore, if we let $x = (e_2, e_3)^T$, $z = e_1$, system (13) satisfies Assumption 1, and thus, the controlled error system (13) is

$$\begin{aligned} D_t^\alpha e_1 &= e_2 - k|e_1|\operatorname{sign}(s), \\ D_t^\alpha e_2 &= e_3, \\ D_t^\alpha e_3 &= ae_1 - be_2 - ce_3 - d(x_1^2 + x_1y_1 + y_1^2)e_1 \end{aligned} \tag{14}$$

i.e., the controlled slave system (12) is

$$\begin{aligned} D_t^\alpha y_1 &= y_2 - k|e_1|\operatorname{sign}(s), \\ D_t^\alpha y_2 &= y_3, \\ D_t^\alpha y_3 &= ay_1 - by_2 - cy_3 - dy_1^3, \end{aligned} \tag{15}$$

where $s = D_t^{\alpha-1}e_1 + \int_0^t ce_1(\tau)\,d\tau = 0$, and $\dot{k} = \theta|e_1||s|$ $(\theta > 0)$.

According to Theorem 1, the controlled error system (14) can be stabilized, i.e., the master system (11) can synchronize the slave system (15) with a single controller.

To confirm the validity of the above conclusion, we give numerical simulations with the following choices of the initial conditions: $x(0) = (1, -1, 1)^T$, $y(0) = (-1, 1, -1)^T$, and $k(0) = 0$, $c = 3$, $\alpha = 0.9$, $\theta = 1$. The numerical results are illustrated in Figs. 1 and 2. Figure 1 shows that the error system $e = y - x$ is stabilized to origin asymptotically as $t \to \infty$, that is to say, the master system (11) and the salve system (12) are synchronized by the above controller. Figure 2 displays the feedback gain $k$ tends to a positive constant, while FI sliding surface $s$ converges to zero.

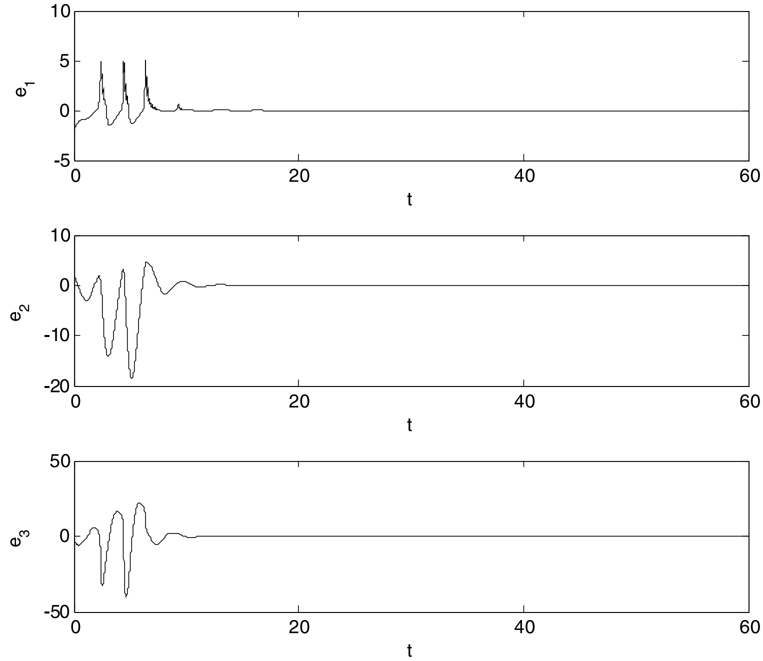



**Fig. 1** The time evolution of synchronization errors between two fractional-order Arneodo chaotic systems

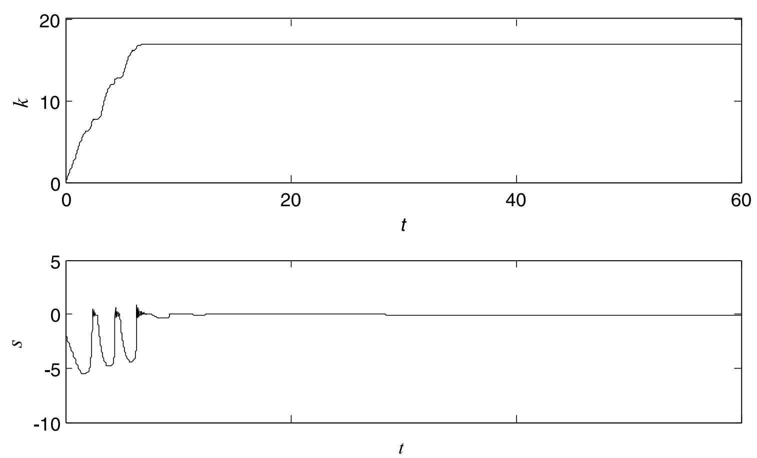



**Fig. 2** The time evolution of the feedback gain $k$ and FI sliding surface $s$

*Example 2* The fractional-order unified chaotic system [19]

$$\begin{aligned} D_t^\alpha x_1 &= (25a+10)(x_2-x_1), \\ D_t^\alpha x_2 &= (28-35a)x_1+(29a-1)x_2-x_1x_3, \\ D_t^\alpha x_3 &= x_1x_2-(a+8)x_3/3, \end{aligned} \tag{16}$$

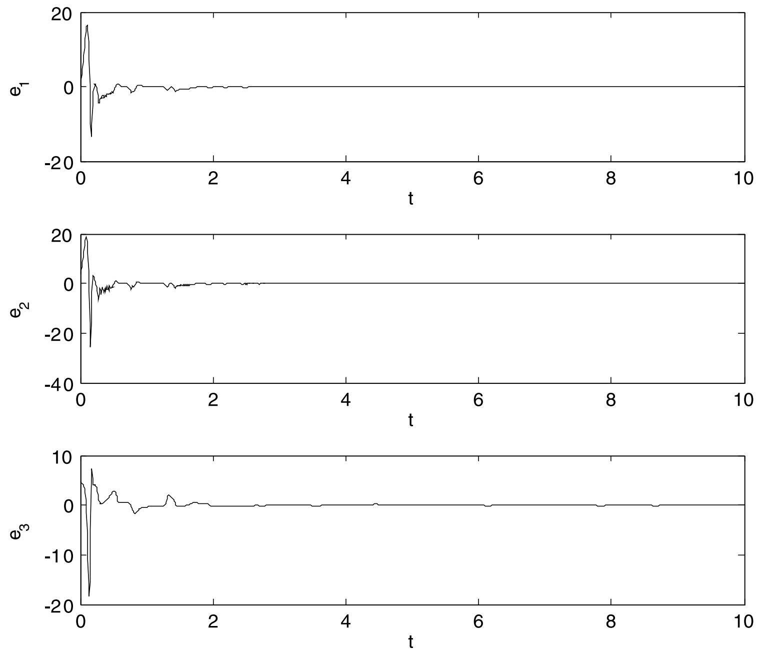
where $a \in [0,1]$. When $a = 0.8$, system (16) becomes the fractional-order Lü system; when $a = 1$, it is the fractional-order Chen system. Let system (16) be the master system, then, the slave system with variable $y$ is given as

$$\begin{aligned} D_t^\alpha y_1 &= (25a+10)(y_2-y_1), \\ D_t^\alpha y_2 &= (28-35a)y_1+(29a-1)y_2-y_1y_3, \\ D_t^\alpha y_3 &= y_1y_2-(a+8)y_3/3 \end{aligned} \tag{17}$$

and the error system ($e = y - x$) is as follows:

$$\begin{aligned} D_t^\alpha e_1 &= (25a+10)(e_2-e_1), \\ D_t^\alpha e_2 &= (28-35a)e_1+(29a-1)e_2 \\ &\quad -x_1e_3-y_3e_1, \\ D_t^\alpha e_3 &= x_1e_2+y_2e_1-(a+8)e_3/3. \end{aligned} \tag{18}$$

**Fig. 3** The time evolution of synchronization errors between two fractional-order unified chaotic systems ($a = 1$)




It is easy to see that if $e_2 = 0$, the following two-dimensional subsystem of system (18),

$$\begin{aligned} D_t^\alpha e_1 &= -(25a + 10)e_1, \\ D_t^\alpha e_3 &= y_2 e_1 - (a + 8)e_3/3 \end{aligned} \tag{19}$$

is asymptotically stable about the origin $e_1 = 0, e_3 = 0$ for all $e_1, e_3$. Therefore, if we let $x = (e_1, e_3)^T$, $z = e_2$, system (18) satisfies Assumption 1, and thus, the controlled error system (18) is

$$\begin{aligned} D_t^\alpha e_1 &= (25a + 10)(e_2 - e_1), \\ D_t^\alpha e_2 &= (28 - 35a)e_1 + (29a - 1)e_2 - x_1 e_3 \\ &\quad - y_3 e_1 - k|e_2| \operatorname{sign}(s), \\ D_t^\alpha e_3 &= x_1 e_2 + y_2 e_1 - (a + 8)e_3/3, \end{aligned} \tag{20}$$

i.e., the controlled slave system (17) is

$$\begin{aligned} D_t^\alpha y_1 &= (25a + 10)(y_2 - y_1), \\ D_t^\alpha y_2 &= (28 - 35a)y_1 + (29a - 1)y_2 - y_1 y_3 \\ &\quad - k|e_2| \operatorname{sign}(s), \\ D_t^\alpha y_3 &= y_1 y_2 - (a + 8)y_3/3, \end{aligned} \tag{21}$$

where $s = D_t^{\alpha-1} e_2 + \int_0^t c e_2(\tau)\, d\tau = 0$, and $\dot{k} = \theta|e_2||s| (\theta > 0)$.

According to Theorem 1, the controlled error system (20) can be stabilized, i.e., the master system (16) can synchronize the slave system (21) with a single controller.

For this simulation, we employed the initial conditions $x(0) = (1, -1, 10)^T$, $y(0) = (-1, 1, 15)^T$, and $k(0) = 0$, $c = 3$, $\alpha = 0.88$, $\theta = 1$. Figure 3 shows that the error system $e = y - x$ is stabilized to origin asymptotically as $t \to \infty$, i.e., the master system (16) and the controlled salve system (26) are synchronized by the above controller. Here and for the remaining examples, we omit the feedback gain $k$ and FI sliding surface $s$ for brevity.

*Example 3* Consider the fractional-order Chua's system [20]

$$\begin{aligned} D_t^\alpha x_1 &= a\left(x_2 + \frac{x_1 - 2x_1^3}{7}\right), \\ D_t^\alpha x_2 &= x_1 - x_2 + x_3, \\ D_t^\alpha x_3 &= -\frac{100}{7} x_2. \end{aligned} \tag{22}$$

When $\alpha = 0.95$ and $a = 12$, the fractional-order Chua's system is chaotic. Let system (22) be the master system. Then, the slave system with variable $y$ is given as

$$D_t^\alpha y_1 = a\left(y_2 + \frac{y_1 - 2y_1^3}{7}\right),$$
$$D_t^\alpha y_2 = y_1 - y_2 + y_3, \qquad (23)$$
$$D_t^\alpha y_3 = -\frac{100}{7} y_2$$

and the error system ($e = y - x$) is as follows:

$$D_t^\alpha e_1 = a\left(e_2 + \frac{1 - 2(x_1^2 + x_1 y_1 + y_1^2)}{7} e_1\right),$$
$$D_t^\alpha e_2 = e_1 - e_2 + e_3, \qquad (24)$$
$$D_t^\alpha e_3 = -\frac{100}{7} e_2.$$

It is easy to see that if $e_1 = 0$, the following two-dimensional subsystem of the system (24),

$$D_t^\alpha e_2 = -e_2 + e_3,$$
$$D_t^\alpha e_3 = -\frac{100}{7} e_2$$

is uniformly exponentially stable about the origin $e_2 = 0$, $e_3 = 0$ for all $e_2$, $e_3$. Therefore, if we let $x = (e_2, e_3)^T$, $z = e_1$, system (24) satisfies Assumption 1, and thus, the controlled error system (24) is

$$D_t^\alpha e_1 = a\left(e_2 + \frac{1 - 2(x_1^2 + x_1 y_1 + y_1^2)}{7} e_1\right) - k|e_1| \operatorname{sign}(s),$$
$$D_t^\alpha e_2 = e_1 - e_2 + e_3, \qquad (25)$$
$$D_t^\alpha e_3 = -\frac{100}{7} e_2,$$

i.e., the controlled slave system (23) is

$$D_t^\alpha y_1 = a\left(y_2 + \frac{y_1 - 2y_1^3}{7}\right) - k|e_1| \operatorname{sign}(s),$$
$$D_t^\alpha y_2 = y_1 - y_2 + y_3, \qquad (26)$$
$$D_t^\alpha y_3 = -\frac{100}{7} y_2,$$

where $s = D_t^{\alpha-1} e_1 + \int_0^t c e_1(\tau)\, d\tau = 0$, and $\dot{k} = \theta |e_1||s|$ $(\theta > 0)$.

According to Theorem 1, the controlled error system (25) can be stabilized, i.e., the master system (22) can synchronize the slave system (26) with a single controller.

In the simulation, we choose the initial conditions $x(0) = (0.2, -0.2, 0.3)^T$, $y(0) = (-0.1, 0.1, -1)^T$, and $\alpha = 0.95$, $c = 1$, $\theta = 1$. The numerical results are illustrated in Fig. 4. It displays that the error system $e = y - x$ is stabilized to origin asymptotically

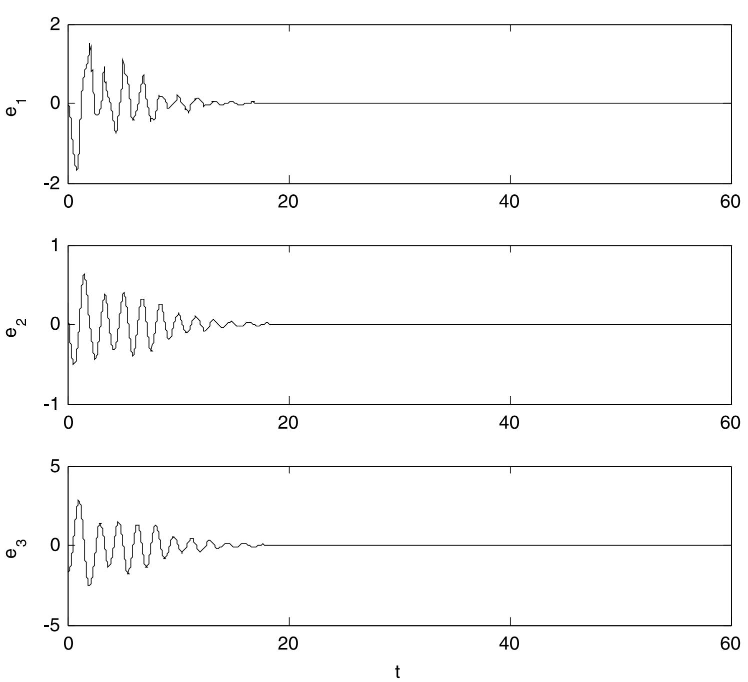



**Fig. 4** The time evolution of synchronization errors between two fractional-order Chua chaotic systems

as $t \to \infty$, that is to say, the master system (22) and the salve system (23) are synchronized by the above controller.

## 4 Conclusions

In conclusion, we have investigated synchronization of 3-d fractional-order chaotic systems. We have designed a novel fractional integral (FI) sliding surface and have proposed a single adaptive controller for fractional-order chaos synchronization. The present controller, which only contains a single driving variable, is easy both in design and implementation. Numerical simulations of three fractional-order chaotic systems verify the effectiveness of the proposed synchronization scheme.

**Acknowledgement** The present work is supported by Natural Science Foundation of Hebei Province under Grant No. 2010000343.

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