Nonlinear Dyn (2012) 69:977–982
DOI 10.1007/s11071-011-0319-5

ORIGINAL PAPER

Fractional Euler–Lagrange equations revisited

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Received: 28 September 2011 / Accepted: 25 December 2011 / Published online: 18 January 2012

Abstract This paper presents the necessary and sufficient optimality conditions for the Euler–Lagrange fractional equations of fractional variational problems with determining in which spaces the functional must exist where the functional contains right and left fractional derivatives in the Riemann–Liouville sense and the upper bound of integration less than the upper bound of the interval of the fractional derivative. In order to illustrate our results, one example is presented.

Keywords Fractional integral · Fractional derivative · Fractional calculus of variations · Lipschitz spaces

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1 Introduction

Fractional calculus is one of the generalizations of the classical calculus. Several fields of application of fractional differentiation and fractional integration are already well established, some others have just started. Many applications of fractional calculus can be found in turbulence and fluid dynamics, stochastic dynamical system, plasma physics and controlled thermonuclear fusion, nonlinear control theory, image processing, nonlinear biological systems, astrophysics, etc. (for more details, see [12, 13, 15, 16, 20–22, 24–27, 33] and the references therein).

Real integer variational calculus plays a significant role in many areas of science, engineering and applied mathematics. Having this in mind, in recent years, there has been a growing interest in the area of fractional variational calculus and its applications which include classical and quantum mechanics, field theory, and optimal control (see [1–17, 19, 20, 23–32, 34–36]). However, many problems are still open in this field, and in the future, therefore, further research is necessary in this direction.

For example, in the previous work the authors were determining the Euler–Lagrange equations without determining the space of the Lagrangian function. In [8, 9], the authors determine the conditions on the spaces of the Lagrangian, but in [9] the conditions are not sufficient to satisfy the conditions required by the fractional integration by parts (see [37]), and in [8] the conditions are very strong.

In this manuscript, we develop the theory of fractional variational calculus further by proving the necessary and sufficient optimality conditions with determining the space in which the Lagrangian must exist which will be seen as a refinement of what is given in [8, 9].

We consider for $\alpha, \beta \in (0,1)$ and $-\infty < a < b < \infty$, the following functional

$$J(y) = \int_a^B L\big(t, y(t), {}_a^R D_t^\alpha y(t), {}_t^R D_b^\beta y(t)\big)\, dt, \quad a < B < b. \tag{1}$$

We find the necessary and sufficient optimality conditions.

This paper is organized as follows:

In Sect. 2, we give the principal definitions used in this paper. In Sect. 3, the necessary optimality conditions are proved for the problem (1) for any arbitrary $B < b$. Sufficient conditions are given in Sect. 4 and some examples are given in Sect. 5 to illustrate our main results. Finally, our conclusions are given in Sect. 6.

2 Preliminaries

Here we give the standard definitions of left and right Riemann–Liouville fractional integral and Riemann–Liouville fractional derivatives (see [22, 27, 33, 37]).

Definition 2.1 If $f(t) \in L^1(a,b)$, the set of all integrable functions, and $\alpha > 0$ then the left and right Riemann–Liouville fractional integrals of order α, denoted respectively by ${}_aI_t^\alpha$ and ${}_tI_b^\alpha$, are defined by

$$ {}_aI_t^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} f(\tau)\, d\tau, \tag{2}$$

$$ {}_tI_b^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_t^b (\tau-t)^{\alpha-1} f(\tau)\, d\tau. \tag{3}$$

Definition 2.2 For $\alpha > 0$ the left and right Riemann–Liouville fractional derivatives of order α, denoted respectively by ${}_a^R D_t^\alpha$ and ${}_t^R D_b^\alpha$, are defined by

$$ {}_a^R D_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} D^n \int_a^t (t-\tau)^{n-\alpha-1} f(\tau)\, d\tau, \tag{4}$$

$$ {}_t^R D_b^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} (-D)^n \int_t^b (\tau-t)^{n-\alpha-1} f(\tau)\, d\tau, \tag{5}$$

where n is such that $n-1 < \alpha < n$ and $D = \frac{d}{dt}$. If α is an integer, these derivatives are defined in the usual sense ${}_a^R D_t^\alpha := D^\alpha$, ${}_t^R D_b^\alpha := (-D)^\alpha$, $\alpha = 1, 2, 3, \ldots$.

We use in our proofs the fractional integration by parts which is satisfied for Lipschitz spaces H_p^λ and $\tilde{H}_p^\lambda$ which are defined as follows (see [37]):

Definition 2.3 We say that $f(x) \in H_p^\lambda = H_p^\lambda([a,b])$, where $0 < \lambda \le 1$, if $f(x) \in L_p(a,b)$ and $w_p(f,\delta) \le c\delta^\lambda$ where $w_p(f,\delta)$ is given by

$$ w_p(f,\delta) = \sup_{|t|<\delta} \left\{ \int_a^b \big|f(x) - f(x-t)\big|^p dx \right\}^{1/p} \tag{6}$$

and it is assumed that $f(x)$ is continued by zero beyond the interval $[a,b]$.

Definition 2.4 We define the space $\tilde{H}_p^\lambda$ as H_p^λ without the zero continuation of a function $f(x)$ beyond $[a,b]$, i.e.,

$$ \tilde{H}_p^\lambda = \left\{ f(x) : f(x) \in L_p(a,b), \int_a^{b-\delta} \big|f(x+\delta) - f(x)\big|^p dx \le c\delta^{\lambda^p}, \delta > 0 \right\}. \tag{7}$$

From Definition 2.3 and Definition 2.4, we note that $H_p^\lambda \subset \tilde{H}_p^\lambda$. H_p^λ and $\tilde{H}_p^\lambda$ are Banach spaces under the following norms:

$$ \|f\|_{\tilde{H}_p^\lambda} = \|f\|_p + \sup_{0<\delta<b-a} \delta^{-\lambda} \left\{ \int_a^{b-\delta} \big|f(x+\delta) - f(x)\big|^p dx \right\}^{1/p} \tag{8}$$

and

$$ \|f\|_{H_p^\lambda} = \|f\|_{\tilde{H}_p^\lambda} + \sup_{0<\delta<b-a} \delta^{-\lambda} \left\{ \left(\int_a^{a+\delta} + \int_{b-\delta}^b \right) \times \big|f(x)\big|^p dx \right\}^{1/p}. \tag{9}$$

Lemma 2.5 *The fractional integration by parts in Riemann–Liouville derivatives*

$$\int_a^b f(x)\left({}_a^R D_x^\alpha g\right)(x)dx = \int_a^b g(x)\left({}_x^R D_b^\alpha f\right)(x)dx, \quad 0<\alpha<1, \tag{10}$$

is valid for the functions

$$f(x) \in \tilde{H}_p^\lambda, \quad g(x) \in H_q^\lambda, \quad \lambda > \alpha, \quad 1/p + 1/q \le 1+\alpha. \tag{11}$$

3 Necessary optimality conditions

To develop the necessary conditions for the extremum of (1), assume $y^*(t)$ is the desired function which makes the value of the given functional a minimum or maximum (most commonly a minimum), let $\epsilon \in R$, and define a family of curves $y(t) = y^*(t) + \epsilon\eta(t)$. Since ${}_a^R D_t^\alpha$ and ${}_t^R D_b^\beta$ are linear operators, we get (1) in the form

$$J(\epsilon) = \int_a^B L\big(t, y(t)+\epsilon\eta(t), {}_a^R D_t^\alpha y + \epsilon {}_a^R D_t^\alpha \eta, {}_t^R D_b^\beta y + \epsilon {}_t^R D_b^\beta \eta\big)\,dt. \tag{12}$$

By differentiating both sides with respect to ϵ and setting $\frac{dJ}{d\epsilon} = 0$, we get

$$\int_a^B \left[\frac{\partial L}{\partial y}\eta + \frac{\partial L}{\partial {}_a^R D_t^\alpha y}{}_a^R D_t^\alpha \eta + \frac{\partial L}{\partial {}_t^R D_b^\beta y}{}_t^R D_b^\beta \eta\right] dt = 0. \tag{13}$$

Now for $\frac{\partial L}{\partial {}_a^R D_t^\alpha y} \in \tilde{H}_p^\lambda([a,B])$, $\frac{\partial L}{\partial {}_t^R D_b^\beta y} \in H_p^\lambda([a,b]) \cap H_p^\lambda([B,b])$, with $\lambda > \max\{\alpha,\beta\}$ and $p > \max\{\frac{1}{1+\alpha}, \frac{1}{1+\beta}\}$. For any $\eta \in H_q^\lambda([a,B]) \cap \tilde{H}_q^\lambda([a,b]) \cap \tilde{H}_q^\lambda([B,b])$ with $1/p+1/q \le \min\{1+\alpha, 1+\beta\}$, we get by fractional integration by parts that

$$\int_a^B \frac{\partial L}{\partial {}_a^R D_t^\alpha y}{}_a^R D_t^\alpha \eta\, dt = \int_a^B \eta {}_t^R D_B^\alpha \left(\frac{\partial L}{\partial {}_a^R D_t^\alpha y}\right) dt \tag{14}$$

and

$$\begin{aligned}\int_a^B \frac{\partial L}{\partial {}_t^R D_b^\beta y}{}_t^R D_b^\beta \eta\, dt &= \int_a^b \frac{\partial L}{\partial {}_t^R D_b^\beta y}{}_t^R D_b^\beta \eta\, dt - \int_B^b \frac{\partial L}{\partial {}_t^R D_b^\beta y}{}_t^R D_b^\beta \eta\, dt \\ &= \int_a^b \eta {}_a^R D_t^\beta \left(\frac{\partial L}{\partial {}_t^R D_b^\beta y}\right) dt - \int_B^b \eta {}_B^R D_t^\beta \left(\frac{\partial L}{\partial {}_t^R D_b^\beta y}\right) dt.\end{aligned} \tag{15}$$

Thus, we get

$$\begin{aligned}0 = {}&\int_a^B \eta\left(\frac{\partial L}{\partial y} + {}_t^R D_B^\alpha\left(\frac{\partial L}{\partial {}_a^R D_t^\alpha y}\right) + {}_a^R D_t^\beta\left(\frac{\partial L}{\partial {}_t^R D_b^\beta y}\right)\right) dt \\ &+ \int_B^b \eta\left({}_a^R D_t^\beta\left(\frac{\partial L}{\partial {}_t^R D_b^\beta y}\right) - {}_B^R D_t^\beta\left(\frac{\partial L}{\partial {}_t^R D_b^\beta y}\right)\right) dt.\end{aligned} \tag{16}$$

And where η is arbitrary, taking $\eta(t) = 0$ for $t \in [B,b]$, we get

$$\frac{\partial L}{\partial y} + {}_t^R D_B^\alpha\left(\frac{\partial L}{\partial {}_a^R D_t^\alpha y}\right) + {}_a^R D_t^\beta\left(\frac{\partial L}{\partial {}_t^R D_b^\beta y}\right) = 0, \quad t \in [a,B]. \tag{17}$$

Taking $\eta(t) = 0$ for $t \in [a,B]$, we get

$${}_a^R D_t^\beta\left(\frac{\partial L}{\partial {}_t^R D_b^\beta y}\right) - {}_B^R D_t^\beta\left(\frac{\partial L}{\partial {}_t^R D_b^\beta y}\right) = 0, \quad t \in [B,b]. \tag{18}$$

Thus we have the fractional Euler equations for our problem in the forms (17) and (18).

Note that by a simple calculation we can write (18) in the form

$$\frac{d}{dt}\left(\int_a^B \frac{(t-s)^{-\beta}}{\Gamma(1-\beta)} \frac{\partial L}{\partial {}^R D_{b-}^\beta y}\, dt\right) = 0, \quad t \in [B,b]. \tag{19}$$

Thus, we prove

Theorem 3.1 *Let $y^* \in AC([a,b])$ be an extremal of the functional $J(y)$ in (1), whose Lagrangian L satisfies for some $\lambda > \max\{\alpha,\beta\}$ and $p > \max\{\frac{1}{1+\alpha}, \frac{1}{1+\beta}\}$ the conditions:*

- $L \in C^1((a,b) \times R \times R \times R)$;
- $\frac{\partial L}{\partial {}_a^R D_t^\alpha y} \in \tilde{H}_p^\lambda([a,B])$;
- $\frac{\partial L}{\partial {}_t^R D_b^\beta y} \in H_p^\lambda([a,b]) \cap H_p^\lambda([B,b])$.

Then y^* satisfies the Euler–Lagrange equations given by (17) and (18).

4 Sufficient conditions

The study of sufficient conditions of optimality for fractional variational problems was started by R. Almeida and D. Torres in [5], then many papers carried the research further (see [6–8, 18, 30]). In this section, we prove the sufficient conditions that ensure the existence of a minimum (maximum) of our fractional variational problem. Some conditions of convexity (concavity) are in order.

Definition 4.1 Given a function $L = L(t, y, z, u)$, we say that L is jointly convex (concave) in (y, z, u) if $\frac{\partial L}{\partial y}, \frac{\partial L}{\partial z}, \frac{\partial L}{\partial u}$ exist, are continuous, and verify the following condition:

$$\begin{aligned} &L(t, y + y_1, z + z_1, u + u_1) - L(t, y, z, u) \\ &\quad \geq (\leq) \frac{\partial L}{\partial y} y_1 + \frac{\partial L}{\partial z} z_1 + \frac{\partial L}{\partial u} u_1 \end{aligned} \tag{20}$$

for all $(t, y, z, u), (t, y + y_1, z + z_1, u + u_1) \in [a, b] \times R^3$.

Theorem 4.2 *Suppose that the function $L(t, y, z, u)$ is jointly convex in (y, z, u). Then every solution y_0 of the fractional Euler–Lagrange equations (17)–(18) provides an extremal of the functional $J(y)$ given by (1).*

Proof Since L is jointly convex in (y, z, u), for any admissible function $y_0 + h$ (where $y_0(t)$ is a solution of the fractional Euler–Lagrange equations (17)–(18) and $h(t)$ satisfies the conditions satisfied by $\eta(t)$ used in proving Theorem 3.1), using integration by parts, we get

$$\begin{aligned} &J(y_0 + h) - J(y_0) \\ &\quad = \int_a^B \Big[L\big(t, y_0(t) + h(t), {}_a^R D_t^\alpha y_0 + {}_a^R D_t^\alpha h, \\ &\qquad {}_t^R D_b^\beta y_0 + {}_t^R D_b^\beta h\big) \\ &\qquad - L(t, y_0(t), {}_a^R D_t^\alpha y_0, {}_t^R D_b^\beta y_0 \Big] dt \\ &\quad \geq \int_a^B \left[\frac{\partial L}{\partial y_0} h + \frac{\partial L}{{}_a^R D_t^\alpha y_0} {}_a^R D_t^\alpha h \right. \\ &\qquad \left. + \frac{\partial L}{\partial_t^R D_b^\beta y_0} {}_t^R D_b^\beta h \right] dt \\ &\quad = \int_a^B h \left(\frac{\partial L}{\partial y_0} + {}_t^R D_B^\alpha \left(\frac{\partial L}{\partial_a^R D_t^\alpha y_0} \right) \right. \\ &\qquad \left. + {}_a^R D_t^\beta \left(\frac{\partial L}{\partial_t^R D_b^\beta y_0} \right) \right) dt \\ &\qquad + \int_B^b h \left({}_a^R D_t^\beta \left(\frac{\partial L}{\partial_t^R D_b^\beta y_0} \right) \right. \\ &\qquad \left. - {}_B^R D_t^\beta \left(\frac{\partial L}{\partial_t^R D_b^\beta y_0} \right) \right) dt \\ &\quad = 0 \end{aligned} \tag{21}$$

which completes the proof. □

5 Example

We shall provide in this section an example in order to illustrate our main results. Consider the following problem:

$$\begin{aligned} \min J(y) = \frac{1}{2} \int_0^B &\big[y^2(t) + \big({}_0^R D_t^\alpha y(t)\big)^2 \\ &+ \delta \big({}_t^R D_1^\beta y(t)\big)^2 \big], \quad t \in [0, 1], \delta \geq 0, \\ &y(0) = y_0. \end{aligned} \tag{22}$$

For this problem, we get the fractional Euler–Lagrange equations in the form:

$$\begin{aligned} &{}_t^R D_B^\alpha \big({}_0^R D_t^\alpha y(t)\big) + \delta {}_0^R D_t^\beta \big({}_t^R D_1^\beta y(t)\big) + y = 0, \\ &\quad t \in [0, B] \end{aligned} \tag{23}$$

and

$$\begin{aligned} &{}_a^R D_t^\beta \big({}_t^R D_1^\beta y(t)\big) - {}_B^R D_t^\beta \big({}_t^R D_1^\beta y(t)\big) = 0, \\ &\quad t \in [B, 1] = 0, \end{aligned} \tag{24}$$

and where $L(y, z, u) = \frac{1}{2}(y^2 + z^2 + \delta u^2)$ is jointly convex. Then the solution of (23)–(24) is a global minimizer to problem (22). Note that it is difficult to solve the above fractional equations, for $0 < \alpha < 1$, $0 < \beta < 1$; a numerical method should be used to get an approximation to the solution.

6 Conclusion

The fractional Euler–Lagrange equations involved both the left and the right derivative although the Lagrangian contains only one kind of derivatives. This fact has attracted many researchers because the obtained equations are new from the mathematical point of view and the classical Euler–Lagrange equations are recovered as a particular case. However, the problem of the surface terms remains one of the hot topics in this area, and it was attacked from various points of view. On the other hand, finding the appropriate necessary and sufficient conditions for this problem is still an open problem, especially when the main idea is to find appropriate conditions and not very strong ones. On this line of thought, we gave the space in which the Lagrangian is contained with an arbitrary upper limit of the functional, and we presented the necessary and sufficient conditions for the existence of the optimizer. Finally, we presented an example in order to illustrate our results.

References

1. Agrawal, O.P.: Formulation of Euler–Lagrange equations for fractional variational problems. J. Math. Anal. Appl. **272**, 368–379 (2002)
2. Agrawal, O.P.: Fractional variational calculus and the transversality conditions. J. Phys. A **39**(33), 10375–10384 (2006)
3. Agrawal, O.P.: Generalized Euler–Lagrange equations and transversality conditions for FVPs in terms of the Caputo derivative. J. Vib. Control **13**(9–10), 1217–1237 (2007)
4. Agrawal, O.P., Baleanu, D.: Hamiltonian formulation and a direct numerical scheme for Fractional Optimal Control Problems. J. Vib. Control **13**, 1217–1237 (2007)
5. Almeida, R., Torres, D.F.M.: Calculus of variations with fractional derivative and fractional integrals. Appl. Math. Lett. **22**(12), 1816–1820 (2009)
6. Almeida, R., Malinowska, A.B., Torres, D.F.M.: A fractional calculus of variations for multiple integrals with application to vibrating string. J. Math. Phys. **51**(4), 033503 (2010)
7. Almeida, R., Torres, D.F.M.: Leitmann's direct method for fractional optimization problems. Appl. Math. Comput. **217**, 956–962 (2010)
8. Almeida, R., Torres, D.F.M.: Necessary and sufficient conditions for the fractional calculus of variations with Caputo derivatives. Commun. Nonlinear Sci. Numer. Simul. **16**, 1490–1500 (2011)
9. Atanackovic, T.M., Konjik, S., Pilipovic, S.: Variational problems with fractional derivatives: Euler–Lagrange equations. arxiv:1101.2961v1 [math.FA] 15 Jan (2011)
10. Baleanu, D., Agrawal, O.P.: Fractional Hamilton formalism within Caputo's derivative. Czechoslov. J. Phys. **56**(10–11), 1087–1092 (2006)
11. Baleanu, D., Muslih, S.I., Rabei, E.M.: On fractional Euler–Lagrange and Hamilton equations and the fractional generalization of total time derivative. Nonlinear Dyn. **53**(1–2), 67–74 (2008)
12. Chen, Y.Q., Vinagre, B.M.: A new IIR-type digital fractional order differentiator. Signal Process. **83**(11), 2359–2365 (2003)
13. Debnath, L.: Recent applications of fractional calculus to science and engineering. Int. J. Math. Math. Sci. **54**, 3413–3442 (2003)
14. El-Nabulsi, R.A.: A fractional approach of nonconservative Lagrangian dynamics. Fizika A **14**(4), 289–298 (2005)
15. Fonseca Ferreira, N.M., Duarte, F.B., Lima, M.F.M., Marcos, M.G., Tenreiro, Machado J.A.: Application of fractional calculus in the dynamical analysis and control of mechanical manipulators. Fract. Calc. Appl. Anal. **11**(1), 91–113 (2008)
16. Frederico, G.S.F., Torres, D.F.M.: Fractional conservation laws in optimal control theory. Nonlinear Dyn. **53**(3), 215–222 (2008)
17. Herzallah, M.A.E., Baleanu, D.: Fractional-order Euler–Lagrange equations and formulation of Hamiltonian equations. Nonlinear Dyn. **58**, 385–391 (2009)
18. Herzallah, M.A.E., Baleanu, D.: Fractional-order variational calculus with generalized boundary conditions. Adv. Differ. Equ. (2011). doi:10.1155/2011/357580
19. Herzallah, M.A.E., Muslih, S.I., Baleanu, D., Rabei, E.M.: Hamilton–Jacobi and fractional like action with time scaling. Nonlinear Dyn. (2011). doi:10.1007/s11071-010-9933-x
20. Hilfer, R.: Applications of Fractional Calculus in Physics. World Sci., River Edge (2000)
21. Jesus, I.S., Machado, J.A.T.: Fractional control of heat diffusion systems. Nonlinear Dyn. **54**(3), 263–282 (2008)
22. Kilbas, A.A., Srivastava, H.M., Trujillo, J.J.: Theory and Applications of Fractional Differential Equations. North-Holland Mathematical Studies, vol. 204. Elsevier, Amsterdam (2006)
23. Klimek, M.: Fractional sequential mechanics–models with symmetric fractional derivative. Czechoslov. J. Phys. **51**, 1348–1354 (2001)
24. Kulish, V.V., Lage, J.L.: Application of fractional calculus to fluid mechanics. J. Fluids Eng. **124**(3), 803–806 (2002)
25. Lorenzo, C.F., Hartley, T.T.: Fractional trigonometry and the spiral functions. Nonlinear Dyn. **38**(1–4), 23–60 (2004)
26. Magin, R.: Fractional calculus in bioengineering. Part 1–3. Crit. Rev. Bioeng. **32** (2004)
27. Miller, K.S., Ross, B.: An Introduction to the Fractional Calculus and Fractional Differential Equations. Wiley, New York (1993)
28. Muslih, S.I., Baleanu, D.: Hamiltonian formulation of systems with linear velocities within Riemann–Liouville fractional derivatives. J. Math. Anal. Appl. **304**, 599–606 (2005)
29. Muslih, S., Baleanu, D.: Formulation of Hamiltonian equations for fractional variational problems. Czechoslov. J. Phys. **55**(6), 633–642 (2005)
30. Malinowska, A.B., Torres, D.F.M.: Generalized natural boundary conditions for fractional variational problems in terms of the Caputo derivative. Comput. Math. Appl. **59**, 3110–3116 (2010)
31. Odzijewicz, T., Torres, D.F.M.: Calculus of variations with fractional and classical derivatives. In: Podlubny, I., Vinagre Jara, B.M., Chen, Y.Q., Feliu Batlle, V., Tejado Balsera, I. (eds.) Proceedings of FDA'10, The 4th IFAC Workshop on Fractional Differentiation and its Applications, Badajoz, Spain, October 18–20, 2010. ISBN 9788055304878. Article no. FDA10-076, 5 pp

32. Muslih, S.I., Baleanu, D.: Fractional Euler–Lagrange equations of motion in fractional space. J. Vib. Control **13**(9–10), 1209–1216 (2007)
33. Podlubny, I.: Fractional Differential Equations. Acad. Press, San Diego, New York, London (1999)
34. Rabei, E.M., Altarazi, I.A., Muslih, S.I., Baleanu, D.: Fractional WKB approximation. Nonlinear Dyn. **57**(1–2), 171–175 (2009)
35. Rabei, E.M., Ababenh, B.S.: Hamilton–Jacobi fractional mechanics. J. Math. Anal. Appl. **344**, 799–805 (2008)
36. Rabei, E.M., Nawafleh, K.I., Hijjawi, R.S., Muslih, S.I., Baleanu, D.: The Hamilton formalism with fractional derivatives. J. Math. Anal. Appl. **327**, 891–897 (2007)
37. Samko, S.G., Kilbas, A.A., Marichev, O.I.: Fractional Integrals and Derivatives: Theory and Applications. Gordon and Breach, New York (1993)