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COMMENT

Comments on "Control of a class of fractional-order chaotic systems via sliding mode"

[Nonlinear Dyn. (2011). doi:10.1007/s11071-011-0002-x]

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Abstract In this note, some comments on the paper [Chen et al. in Nonlinear Dyn. (2011). doi:10.1007/s11071-011-0002-x] are made.

Keywords Fractional-order calculus · Sliding mode dynamics · Fractional Lyapunov theorem

1 Introduction

In [1], the authors have investigated chaos control of a class of three-dimensional fractional-order chaotic systems via sliding mode control theory. They have proposed two similar fractional-order sliding mode controllers to guarantee the asymptotical stability of the following class of chaotic systems with/without uncertainties $\Delta g(x, y, z) + \xi(t)$ in the second state equation of the system (see (2) and (13) in [1]):

$$\begin{aligned} D^{q_1}x &= y.f(x, y, z) + z.\Phi(x, y, z) - \alpha x \\ D^{q_2}y &= g(x, y, z) - \beta x + \Delta g(x, y, z) \\ &\quad + \xi(t) + u(t) \\ D^{q_3}z &= y.h(x, y, z) - x.\Phi(x, y, z) - \gamma z \end{aligned} \tag{1}$$

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Chen et al. [1] have proposed a fractional-order sliding surface and have claimed that the closed-loop system is globally asymptotically stable via a sliding mode control. They have performed some numerical simulations to show the efficiency of the proposed control scheme. Furthermore, using the Lyapunov stability theorem, they have performed a stability analysis to ensure the existence of the sliding motion. However, as one knows, the stability of a sliding mode controller is guaranteed if (1) a suitable sliding surface is selected to result in a stable sliding mode dynamics and (2) a sliding mode control law is designed to force the system trajectories to reach the prescribed stable sliding surface in finite time [4, 5]. Moreover, since the main results of conventional Lyapunov stability theory are not applicable for showing the stability of the fractional-order systems, it is more appropriate to analysis the stability of the fractional-order systems using the following fractional-order Lyapunov stability theorems [2, 3].

Theorem 1 [2] *Let $x = 0$ be an equilibrium point for the non-autonomous fractional-order system*

$$D^{\alpha}x = f(x, t) \tag{2}$$

where $f(x, t)$ satisfies the Lipschitz condition with Lipschitz constant $l > 0$ and $\alpha \in (0, 1)$. Let $V(t, x(t))$ be a continuous differentiable function and locally Lipschitz with respect to x such that

$$\alpha_1\|x\|^a \le V(t, x) \le \alpha_2\|x\|^{ab}, \tag{3}$$

$$D^{\beta} V(t,x) \le -\alpha_3 \|x\|^{ab} \tag{4}$$

where α_1, α_2, α_3, a and b are arbitrary positive constants and $\beta \in (0,1)$. Then the equilibrium point of the system (2) is Miattag–Leffler stable.

Theorem 2 [2] *Let $x = 0$ be an equilibrium point for the non-autonomous fractional-order system (2). Assume that there exists a Lyapunov function $V(t, x(t))$ satisfying*

$$\alpha_1 \|x\|^{a} \le V(t,x) \le \alpha_2 \|x\|, \tag{5}$$

$$\dot{V}(t,x) \le \alpha_3 \|x\| \tag{6}$$

where $\alpha_1, \alpha_2, \alpha_3$ and a are positive constants. Then the equilibrium point of the system (2) is Miattag–Leffler stable.

Theorem 3 [2] *Let $x = 0$ be an equilibrium point for the non-autonomous fractional-order system (2). Assume that there exist a Lyapunov function $V(t, x(t))$ and class-K functions $\alpha_1, \alpha_2, \alpha_3$ satisfying*

$$\alpha_1(\|x\|) \le V(t,x) \le \alpha_2(\|x\|), \tag{7}$$

$$D^{\beta} V(t,x) \le -\alpha_3(\|x\|) \tag{8}$$

where $\beta \in (0,1)$. Then the equilibrium point of the system (2) is Miattag–Leffler stable.

Remark 1 [3] Mittag–Leffler stability implies asymptotic stability.

More details about the fractional Lyapunov stability theory can be found in [2, 3].

In Sect. 3 of [1], the authors have defined a fractional sliding surface as (see (3) in [1]):

$$s(t) = D^{q_2 - 1} y(t) + D^{-1} \psi(t) \tag{9}$$

where $\psi(t) = x.f(x,y,z) + z.h(x,y,z) + \beta y$.

To obtain the sliding mode dynamics, we use the well-known property of the sliding mode as follows (see (6) in [1]):

$$\dot{s}(t) = 0 \to D^{q_2} y(t) + \psi(t) = 0 \to D^{q_2} y(t) = -x.f(x,y,z) - z.h(x,y,z) - \beta y \tag{10}$$

Although the authors of [1] have proved the convergence of the closed-loop system state trajectories to the sliding surface (9), but they have not obtained the sliding mode dynamics and, therefore, have not presented any stability analysis for it. However, in order to ensure the global stability of the closed-loop system in [1], the sliding mode dynamics in [1] should be stable. In this regard, to obtain the sliding mode dynamics, inserting $D^{q_2} y(t)$ from (10) into (1), we have

$$\begin{aligned} D^{q_1} x &= y.f(x,y,z) + z.\Phi(x,y,z) - \alpha x \\ D^{q_2} y &= -x.f(x,y,z) - z.h(x,y,z) - \beta y \\ D^{q_3} z &= y.h(x,y,z) - x.\Phi(x,y,z) - \gamma z \end{aligned} \tag{11}$$

Equation (11) describes the dynamics of the closed-loop system on the sliding surface (9). In other word, (11) indicates that when the state trajectories of the system (1) reach to the sliding surface (9), the dynamics of the closed-loop system is represented by (11) (sliding mode dynamics) and the closed-loop system is then insensitive to the system uncertainties. Therefore, the sliding mode dynamics (11) should be stable to guarantee the stability of the closed-loop system. However, Chen et al. [1] have not presented any stability discussion for the stability of the sliding mode dynamics (11). This means that there is no guarantee that the sliding mode dynamics (11) is stable or not and it cannot be ensured that the state trajectories of the chaotic system (1) can converge to zero, when they attain to the sliding surface (3) in [1]. Consequently, the global stability of the proposed fractional sliding mode controller in [1] cannot be guaranteed.

On the other hand, in Sect. 3 of [1] the traditional Lyapunov stability theorem has been used to show the convergence of the system (1) trajectories to the sliding surface (9). The authors of [1] have selected a Lyapunov function as $V(t) = 1/2s^2$ (see (11) in [1]) and have proved that the time derivative of the selected Lyapunov function is negative along the trajectories of the fractional-order system (1). Thus, they have claimed that the closed-loop system is globally asymptotically stable via the sliding mode control. However, one can easily see that the dynamics of the system (1) and sliding surface (9) involves fractional-order terms (see (2), (3) and (13) in [1]). Therefore, it is more appropriate to prove the existence of sliding motion based on the fractional-order Lyapunov stability theorems [2, 3]. Besides, it is easy to check that the selected Lyapunov function $V(t) = 1/2s^2$ in [1] does not satisfy the conditions of the fractional Lyapunov stability theorems (see Theorems 1–3).

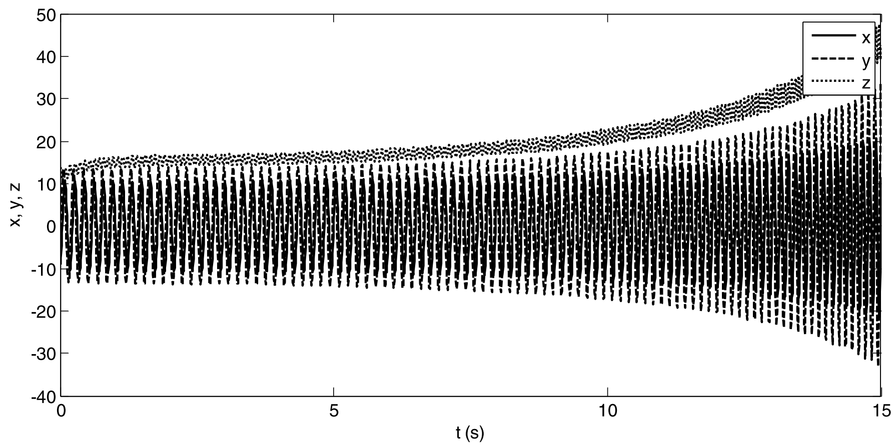


Fig. 1 State trajectories of the Chen system with the controller (15) and (16) in [1]

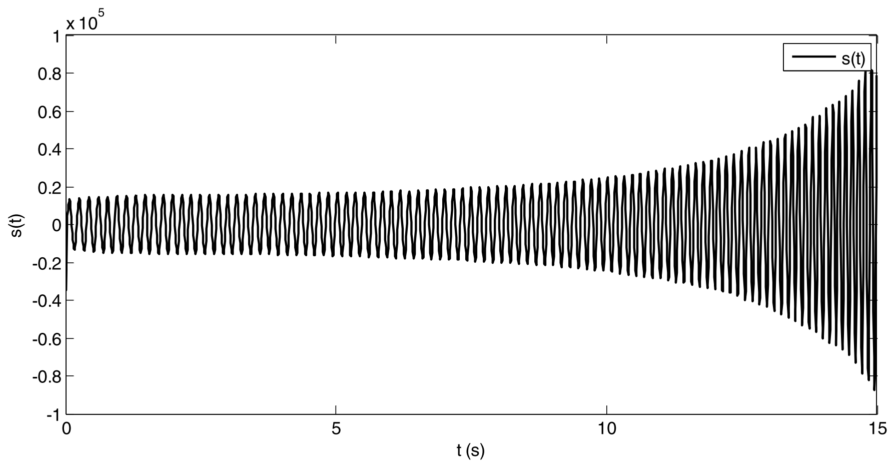


Fig. 2 Sliding surface (15) in [1] applied for the Chen system

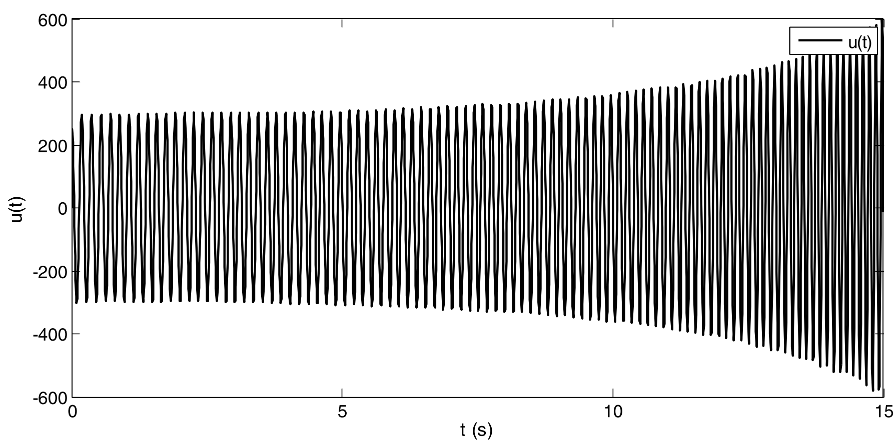


Fig. 3 Control input (16) in [1] applied for the Chen system

Remark 2 It is worth noticing that for the integer-order system (i.e. when $q_1 = q_2 = q_3 = 1$) the sliding mode dynamics (11) is stable for $\alpha > 0$, $\beta > 0$ and $\gamma > 0$, where the system asymptotic stability can be verified using a Lyapunov function candidate such as $V(t) = x^2 + y^2 + z^2$. However, it is known that the stability of an integer-order nonlinear system cannot generally guarantee the stability of the correspond-

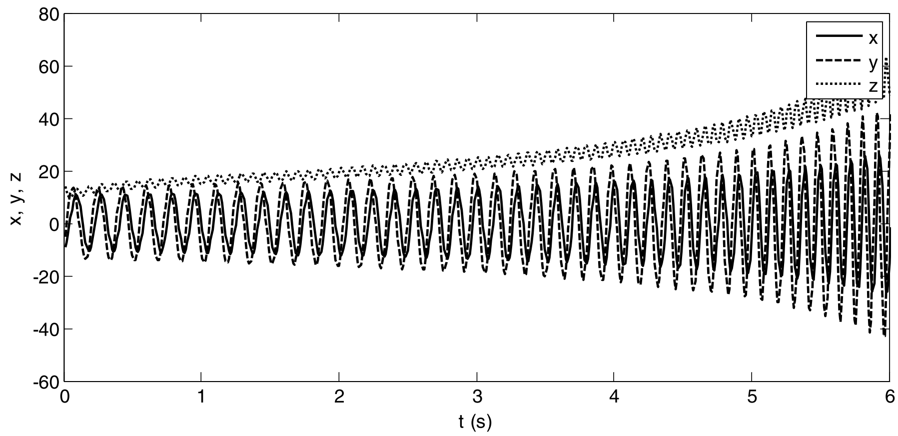


Fig. 4 State trajectories of the uncertain Chen system with the controller (15) and (16) in [1]

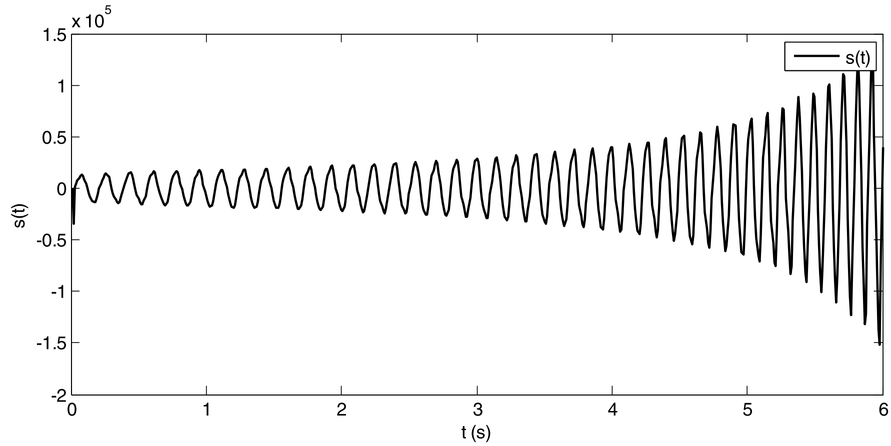


Fig. 5 Sliding surface (15) in [1] applied for the uncertain Chen system

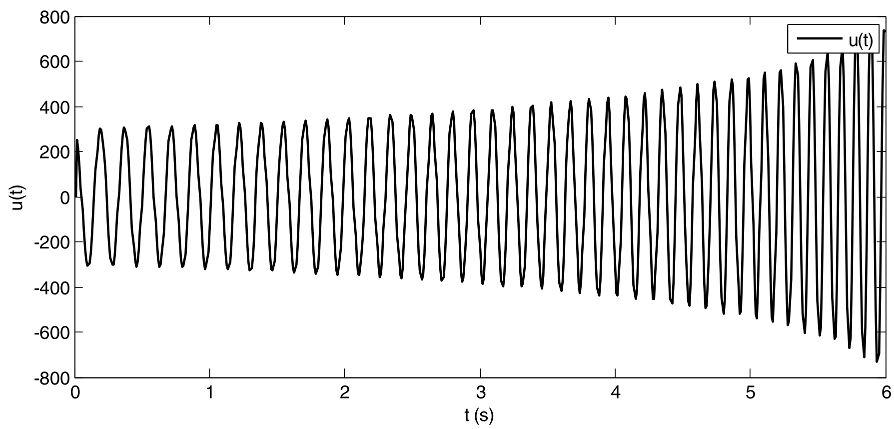


Fig. 6 Control input (16) in [1] applied for the uncertain Chen system

Fig. 7 State trajectories of the Lorenz system with the controller (18) and (19) in [1]

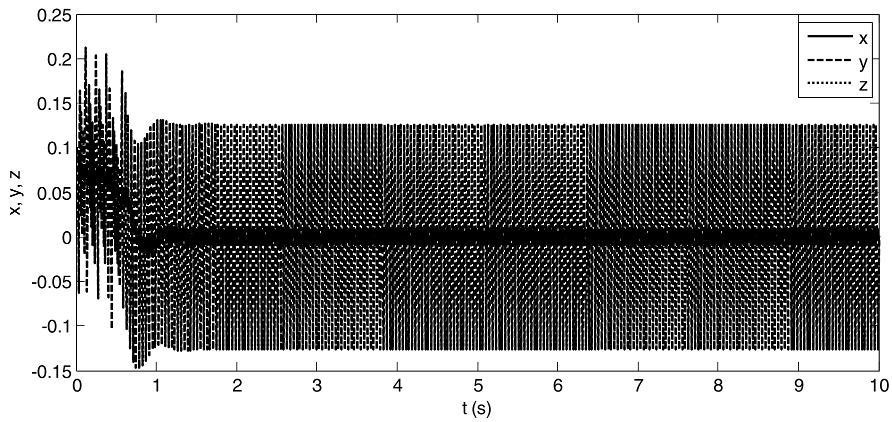


Fig. 8 Sliding surface (18) in [1] applied for the Lorenz system

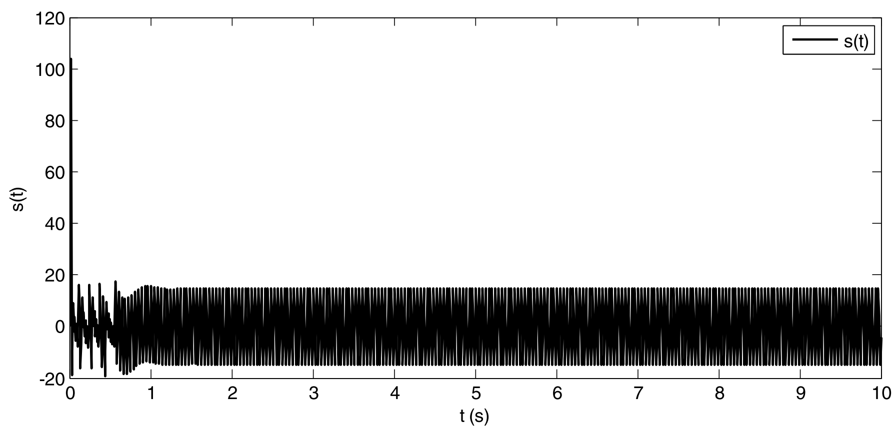
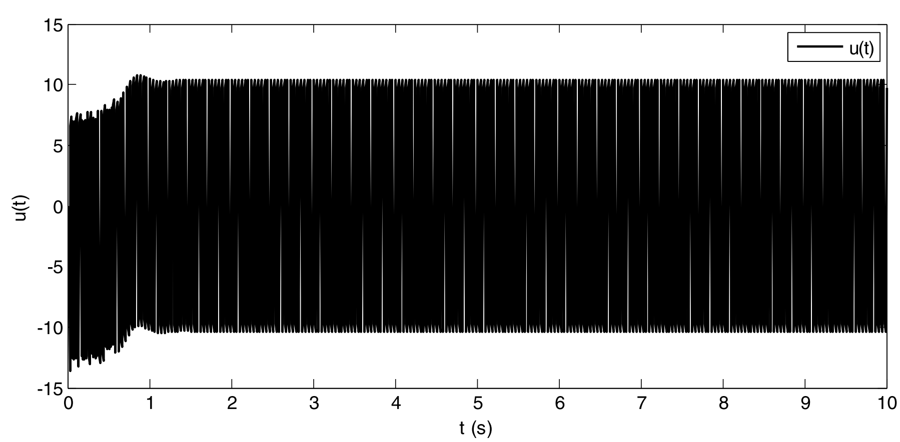


Fig. 9 Control input (19) in [1] applied for the Lorenz system



ing fractional-order nonlinear system and for showing the asymptotic stability of a fractional-order nonlinear system the fractional Lyapunov stability theorems should be adopted [2, 3].

As a result, based on the above discussions, the global stability of the proposed fractional sliding mode control approach in [1] cannot be ensured. To confirm this claim, we present some numerical simulations as in Sect. 4 in [1].

2 Simulation results

Here, we give some numerical simulations to illustrate that the proposed sliding mode controllers in [1] cannot globally stabilize the fractional-order chaotic system (1). All the system and controller parameters are selected as those in [1] except the system's fractional orders. Numerical simulations are performed using Matlab software with a step time of 0.001.

Case I: Consider the non-commensurate fractional-order Chen system (14) in [1] with $q_1 = 0.99$, $q_2 = 0.9$ and $q_3 = 0.95$. Figure 1 shows the state trajectories of the Chen system with the controller (15) and (16) in [1]. It is seen that the system trajectories go to infinite as time evolves. The time responses of the sliding surface (15) in [1] and control input (16) in [1] are illustrated in Figs. 2 and 3, respectively. Obviously, the sliding surface and control input are impractical.

Case II: Consider the uncertain non-commensurate fractional-order Chen system (14) in [1] with $q_1 = 0.99$, $q_2 = 0.9$ and $q_3 = 0.92$. The state trajectories of the system with the controller (15) and (16) in [1] are depicted in Fig. 4. Figures 5 and 6 reveal the time histories of the sliding surface (15) in [1] and control input (16) in [1], respectively. One can see that the proposed controller in [1] cannot stabilize the uncertain Chen system.

Case III: Consider the commensurate fractional-order Lorenz system (17) in [1] with $q_1 = q_2 = q_3 = 0.95$. The controller (18) and (19) in [1] is applied to obtain the results. Figures 7, 8, 9 illustrate the state trajectories of the controlled Lorenz system, the time response of the sliding surface (18) in [1] and the time history of the control input (19) in [1], respectively. It can be seen that the proposed controller (18)–(19) in [1] does not work for stabilization of the commensurate fractional-order Lorenz chaotic system.

3 Concluding remarks

In this note, we demonstrate that the proposed fractional sliding mode controllers in the paper of Chen et al. [1] are inappropriate.

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