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Perturbation to symmetry and adiabatic invariants of discrete nonholonomic nonconservative mechanical system

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Abstract Perturbation to Noether symmetries and adiabatic invariants of discrete nonholonomic nonconservative mechanical systems on an uniform lattice are investigated. Firstly, we review Noether symmetry and conservation laws of a nonholonomic nonconservative system. Secondly, we study continuous Noether symmetry of a discrete nonholonomic system, give the Noether symmetry criterion and theorem of discrete corresponding holonomic system and nonholonomic system. Thirdly, we study perturbation to Noether symmetry of the discrete nonholonomic nonconservative system, give the criterion of perturbation to Noether symmetry for this system, and based on the definition of adiabatic invariants, we construct the theorem under which can lead to Noether adiabatic invariants for this system, and the forms of discrete Noether adiabatic invariants are given. Finally, we give an example to illustrate our results.

Keywords Symmetrical perturbation · Adiabatic invariant · Discrete nonholonomic dynamical system

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1 Introduction

Symmetries of difference equations and discrete mechanical systems have been extensively investigated in recent years. There are mainly two opinions about this research. One possibility is to consider the difference equations on a fixed lattice and consider only transformations that do not act on the lattice [1–5]. The second possibility is to consider group transformations acting both on the difference equations and on the lattice [6, 7], and the symmetries of the original differential equation are preserved in such a way. The research on symmetries of the discrete constraint system is also studied. Leyendecker et al. [8–11] studied the integrators and Fu and Zhang [12, 13] studied the symmetry of discrete constrained system. However, there are few researches on perturbation of a discrete constraint system.

As we know, even tiny changes in symmetry, which is called perturbation to symmetry, are of great important for physical systems. Based on the definition of adiabatic invariants, the relationship of perturbation to symmetry and adiabatic invariants are constructed, which offer an opportunity for the quasiintegrability for dynamical systems. So, perturbation to symmetry and adiabatic invariants has become a hot subject [14, 15] recently. In Baikov et al. [16], the notion of approximate conservation laws is introduced with specific regard to approximate Noether symmetries; Kara et al. [17–19] extend Baikov's ideals. Perturbation to

Lie symmetry and adiabatic invariants [20–26] are also investigated.

However, studies about perturbation to symmetry, are most considered in continuous systems. Now, the discrete mechanics have a vigorous development and symmetry theory that is applied to it. So, in this paper, we will study perturbation to symmetry and adiabatic invariants of a discrete nonholonomic nonconservative mechanical system with the second method.

The structure of this paper will be as follows. Firstly, we will outline the Noether symmetry of the nonholonomic system. Secondly, we will give continuous Noether symmetry of the discrete nonholonomic system. Thirdly, we will study perturbation to symmetry and adiabatic invariants for discrete nonholonomic systems, and finally, we will give an example to illustrate the application of our results.

2 Noether symmetry and exact invariants of nonholonomic nonconservative mechanical systems

Since Noether unveiled the profound relations between symmetries and conservation laws, a lot of researches about them were done [27–29]. It has also been extended to the nonholonomic system [30–32]. In this section, we will review some results about Noether symmetry of the continuous nonholonomic system.

2.1 Equation of motion of the nonholonomic system of Chetaev type

Suppose that the configuration of a mechanical system is determined by n generalized coordination q_s ($s = 1,\dots,n$), the Lagrangian of the system is $L(t,\boldsymbol{q},\dot{\boldsymbol{q}})$, where $\dot{\boldsymbol{q}} = D(\boldsymbol{q}), \ddot{\boldsymbol{q}} = D(\dot{\boldsymbol{q}})$, D is the first-order linear differential operator

$$D = \frac{\partial}{\partial t} + \dot{q}_s\frac{\partial}{\partial q_s} + \ddot{q}_s\frac{\partial}{\partial \dot{q}_s} + \cdots \quad (s = 1,\dots,n) \qquad (1)$$

The system is subject to g ideal bilateral nonholonomic constraints of Chetaev's type

$$f_\beta(t,\boldsymbol{q},\dot{\boldsymbol{q}}) = 0 \quad (\beta = 1,\dots,g) \qquad (2)$$

and the constraints are independent to each other. The limitation of constraints (2) act on the virtual displacement which satisfies the Appell–Chetaev condition

$$\sum_{s=1}^{n}\frac{\partial f_\beta}{\partial \dot{q}_s}\delta q_s = 0 \qquad (3)$$

Based on the D'Alembert–Lagrange principle and the Appell–Chetaev condition (3), and making use of the Lagrange multiplier method, the motion of the equation of the system can be obtained

$$D\frac{\partial L}{\partial \dot{q}_s} - \frac{\partial L}{\partial q_s} = Q_s + \lambda_\beta\frac{\partial f_\beta}{\partial \dot{q}_s} \qquad (4)$$

where L is Lagrangian of the system, Q_s are nonconservative forces, and λ_β are constrained multipliers. Suppose the system is nonsingular, that is,

$$\det\left(\frac{\partial^2 L}{\partial \dot{q}_s\partial \dot{q}_k}\right) \neq 0 \qquad (5)$$

We can solve λ_β as function of $t, \boldsymbol{q}, \dot{\boldsymbol{q}}$ from (2) and (4). Then (4) can be written as

$$D\frac{\partial L}{\partial \dot{q}_s} - \frac{\partial L}{\partial q_s} = Q_s + \Lambda_s \qquad (6)$$

where

$$\Lambda_s = \Lambda_s(t,\boldsymbol{q},\dot{\boldsymbol{q}}) = \lambda_\beta\frac{\partial f_\beta}{\partial \dot{q}_s} \qquad (7)$$

we call (6) the corresponding holonomic system to nonholonomic system (2), (4).

2.2 Variation of Hamilton action

Hamilton action is defined as a integral of Lagrangian $L = L(t,\boldsymbol{q},\dot{\boldsymbol{q}})$ between the interval of time $[t_1,t_2]$, which can be expressed as

$$S(\gamma) = \int_{t_1}^{t_2} L(t,\boldsymbol{q},\dot{\boldsymbol{q}})\,dt \qquad (8)$$

where γ is a curve. Let us introduce in infinitesimal transformation

$$t^* = t + \Delta t, \qquad q_s^* = q_s + \Delta q_s \quad (s = 1,2,\dots,n) \qquad (9)$$

and its expanded style is

$$\begin{aligned} t^* &= t + \epsilon\xi_{00}(t,q_s,\dot{q}_s), \\ q_s^* &= q_s + \epsilon\xi_{s0}(t,q_s,\dot{q}_s) \quad (s = 1,\dots,n) \end{aligned} \qquad (10)$$

which constitute of the single parameter Lie group of infinitesimal transformation, where $\xi_{00}(t, q_s, \dot{q}_s)$, $\xi_{s0}(t, q_s, \dot{q}_s)$ are infinitesimal transformation generators, and ϵ is infinitesimal parameter. Through the transformation (9), the curve γ goes to a neighborhood curve γ^*. The corresponding Hamilton action becomes

$$S(\gamma^*) = \int_{t_1^*}^{t_2^*} L\big(t^*, \boldsymbol{q}^*, \dot{\boldsymbol{q}}^*\big)\, dt^* \tag{11}$$

The variation of action S is the linear part to ϵ for the difference of $\Delta S = S(\gamma^*) - S(\gamma)$, Δ is the total variation operator, so we can get

$$\Delta S = \int_{t_1}^{t_2} \big[\Delta L + L(\Delta t)^{\cdot}\big]\, dt \tag{12}$$

Based on the relation formula between total variation (Δ) and isochronous variation(δ)

$$\begin{aligned} \Delta q_s &= \delta q_s + \dot{q}_s \Delta t, \\ \Delta \dot{q}_s &= \delta \dot{q}_s + \ddot{q}_s \Delta t \end{aligned} \tag{13}$$

and considering the commutative relation of isochronous variation that under Hölder definition

$$\delta \dot{q}_s = D(\delta q_s) \tag{14}$$

we can deduce

$$\begin{aligned} \Delta S = \int_{t_1}^{t_2} \bigg[& D\bigg(L(\Delta t) + \frac{\partial L}{\partial \dot{q}_s}\delta q_s\bigg) \\ & + \bigg(\frac{\partial L}{\partial q_s} - D\frac{\partial L}{\partial \dot{q}_s}\bigg)\delta q_s\bigg]\, dt \end{aligned} \tag{15}$$

Note that

$$\delta q_s = \Delta q_s - \dot{q}_s \Delta t = \varepsilon(\xi_{s0} - \dot{q}_s \xi_{00}) \tag{16}$$

the (15) can be written as

$$\begin{aligned} \Delta S = \int_{t_1}^{t_2} \varepsilon \bigg[& D\bigg(L\xi_{00} + \frac{\partial L}{\partial \dot{q}_s}\bar{\xi}_{s0}\bigg) \\ & + \bigg(\frac{\partial L}{\partial q_s} - D\frac{\partial L}{\partial \dot{q}_s}\bigg)\bar{\xi}_{s0}\bigg]\, dt \end{aligned} \tag{17}$$

where $\bar{\xi}_{s0} = \xi_{s0} - \dot{q}_s \xi_{00}$. Equations (15) and (17) are the basic style of Hamilton action variation.

2.3 Noether symmetry of corresponding holonomic system

Definition 1 If Hamilton action is invariant under the infinitesimal group transformation, that is, the following condition is satisfied for every infinitesimal transformation

$$\Delta S = -\int_{t_1}^{t_2} \big[D(\Delta G) + (Q_s + \Lambda_s)\delta q_s\big]\, dt \tag{18}$$

where $G = G(t, \mathbf{q}, \dot{\mathbf{q}})$, $Q_s \delta q_s$ is the summation of virtual work of the nonconservative force, $\Lambda_s \delta q_s$ is the summation of virtual work of the nonholonomic constraint reaction; the infinitesimal transformation (9) is called the generalized quasitransformation.

We can get the following criteria from Definitions (18) and (9) and (10).

Criterion 1 *If infinitesimal transformation (9) satisfies the following condition,*

$$\begin{aligned} & \frac{\partial L}{\partial t}\Delta t + \frac{\partial L}{\partial q_s}\Delta q_s + \frac{\partial L}{\partial \dot{q}_s}\Delta \dot{q}_s + LD\Delta t \\ & \quad + (Q_s + \Lambda_s)(\Delta q_s - \dot{q}_s \Delta t) = -D(\Delta G) \end{aligned} \tag{19}$$

it is called the Noether quasisymmetry transformation of the corresponding holonomic system (6).

Criterion 2 *If infinitesimal transformation (10) satisfies the following condition,*

$$\begin{aligned} & D\bigg(L\xi_{00} + \frac{\partial L}{\partial \dot{q}_s}\bar{\xi}_{s0}\bigg) \\ & \quad + \bigg(\frac{\partial L}{\partial q_s} - D\frac{\partial L}{\partial \dot{q}_s} + Q_s + \Lambda_s\bigg)\bar{\xi}_{s0} = -DG \end{aligned} \tag{20}$$

where $\Delta G = \epsilon G$, it is called the Noether quasisymmetry transformation of the corresponding holonomic system (6).

We can get easily Noether identity of the corresponding holonomic system (6) from (19) or (20) as

$$\begin{aligned} & \frac{\partial L}{\partial t}\xi_{00} + \frac{\partial L}{\partial q_s}\xi_{s0} + \frac{\partial L}{\partial \dot{q}_s}\dot{\xi}_{s0} + \bigg(L - \frac{\partial L}{\partial \dot{q}_s}\dot{q}_s\bigg)\dot{\xi}_{00} \\ & \quad + (Q_s + \Lambda_s)(\xi_{s0} - \dot{q}_s \xi_{00}) = -DG \end{aligned} \tag{21}$$

which can be obtained by straight deduction.

Theorem 1 *If the infinitesimal group transformation (10) is the Noether quasisymmetry transformation, that is, it satisfies criterion* 1 *or* 2, *or Noether identity* (21), *the corresponding holonomic system* (6) *has the following Noether invariants*:

$$I_{00} = L\xi_{00} + \frac{\partial L}{\partial \dot{q}_s}\bar{\xi}_{s0} + G = \text{const} \tag{22}$$

2.4 Noether symmetry of nonholonomic system

The limitation of the nonholonomic constraint (2) acts on infinitesimal transformation and can be expressed as

$$\frac{\partial f_\beta}{\partial \dot{q}_s}\bar{\xi}_{s0} = 0 \tag{23}$$

which can be also called the Appell–Chetaev condition.

Definition 2 If infinitesimal transformation (10) is a Noether quasisymmetry transformation, and it satisfies the Appell–Chetaev condition (23), the transformation is called the Noether quasisymmetry transformation of the nonholonomic system.

Theorem 2 *For the nonholonomic system* (2), (4), *if the infinitesimal transformation* (10) *is a Noether quasi-symmetry transformation of the nonholonomic system, the system has the same style invariants as* (22).

Proof Making use of the definition of the Noether quasisymmetry of the nonholonomic system and (4), it is easy to verify the theorem. □

3 Noether symmetry and exact invariants of discrete nonholonomic nonconservative mechanical systems

3.1 Discrete version of Hamilton action variation

Let us introduce total shift and linear difference operators which work as

$$\underset{\pm h}{S} f(t) = f(t_\pm), \qquad \underset{\pm h}{D} = \frac{\underset{\pm h}{S} - 1}{\pm h_\pm} \tag{24}$$

the operators $\underset{-h}{S}$, $\underset{-h}{S}$, $\underset{+h}{D}$, and $\underset{-h}{D}$ commute in any combination, while $\underset{+h}{D} = \underset{-h}{D}\,\underset{+h}{S}$, $\underset{-h}{D} = \underset{+h}{D}\,\underset{-h}{S}$, and possess corresponding finite-difference Leibniz rule

$$\begin{aligned} \underset{+h}{D}(FG) &= \underset{+h}{D}(F)G + F\underset{+h}{D}(G) \\ &\quad + h\underset{+h}{D}(F)\underset{+h}{D}(G) \\ \underset{-h}{D}(FG) &= \underset{-h}{D}(F)G + F\underset{-h}{D}(G) \\ &\quad - h\underset{-h}{D}(F)\underset{-h}{D}(G) \end{aligned} \tag{25}$$

We point out that we will use invariant lattice h in t-direction for convenience in the following. The discrete Hamilton action of the system can be defined by

$$M_d = \sum L_d\big(t, q^s, q_k^s\big)h \tag{26}$$

where $L_d(t, q^s, q_k^s)$ is the corresponding discrete Lagrangian and

$$\begin{aligned} q_k^s &= \underset{+h}{D}\big(q^s\big) = \frac{q^{s+} - q^s}{t^+ - t} = \frac{q^{s+} - q^s}{h} \\ q_{\bar{k}}^{s} &= \underset{-h}{D}\big(q^s\big) = \frac{q^s - q^{s-}}{t - t^-} = \frac{q^s - q^{s-}}{h} \end{aligned} \tag{27}$$

$$\underset{+h}{S} q^s = q^{s+}, \qquad \underset{-h}{S} q^s = q^{s-} \tag{28}$$

Under the infinitesimal transformation (10), the discrete Hamilton action becomes

$$M_d^* = \sum L_d\big(t^*, q^{s*}, q_k^{s*}\big)h^* \tag{29}$$

where $h^* = (1 + \underset{+h}{D}(\Delta t))h = (1 + \epsilon\underset{+h}{D}(\xi_{00}))h$ and Δ express the total variation. The discrete analogue of Appell–Chetaev's condition with respect to discrete nonholonomic constraints which restrict generators of infinitesimal transformations can be written as

$$\sum_{s=1}^{n} \frac{\partial f_\beta^d}{\partial q_k^s}\bar{\xi}_{s0} = 0 \tag{30}$$

where $\bar{\xi}_{s0} = \xi_{s0} - q_k^s\xi_{00}$, which come from discrete virtual displacements variation $\delta_d q^s = \Delta q^s - q_k^s\Delta t = \epsilon[\xi_{s0} - q_k^s\xi_{00}] = \epsilon\bar{\xi}_{s0}$.

So, the discrete version of Hamilton action variation is

$$\Delta M_d = M_d^* - M_d = \sum \big[\Delta L_d + L_d \underset{+h}{D}(\Delta t)\big] h \tag{31}$$

or

$$\Delta M_d = \sum \epsilon \bigg[\frac{\partial L_d}{\partial t}\xi_{00} + \frac{\partial L_d}{\partial q^s}\xi_{s0} + \frac{\partial L_d}{\partial q_k^s}\big(\underset{+h}{D}(\xi_{s0}) - q_k^s \underset{+h}{D}(\xi_{00})\big) + L_d \underset{+h}{D}(\xi_{00}) \bigg] h \tag{32}$$

3.2 Noether symmetry of discrete corresponding holonomic system

Definition 3 If Hamilton action is generalized quasi-invariant under the infinitesimal transformation (10), i.e.,

$$\Delta M_d = -\sum_{s=1}^{n} \big\{ \underset{+h}{D}(\Delta G_d) + \big(\Lambda_d^s + Q_d^s\big)\delta_d q^s \big\} h \tag{33}$$

where $G_d = G_d(t, q^s, q_k^s)$, $Q_d^s = Q_d^s(t, q^s, q_k^s)$ are the discrete gauge function and discrete nonconservative forces. $\Lambda_d^s = \lambda_\beta^d \frac{\partial f_\beta^d}{\partial q_k^s}$ are forces corresponding to discrete nonholonomic constraints (and λ_β^d is discrete constraint multiplication), $\Lambda_d^s \delta_d q^s$ are the discrete analogue of the virtual work for the nonholonomic constraint force. Then we call (10) the discrete analogue of generalized Noether quasisymmetry transformation.

Expanding (33) and limit it to the first order of ϵ, we have

$$\frac{\partial L_d}{\partial t}\Delta t + \frac{\partial L_d}{\partial q^s}\Delta q^s + \frac{\partial L_d}{\partial q_k^s}\big(\underset{+h}{D}(\Delta q^s) - q_k^s \underset{+h}{D}(\Delta t)\big) + L_d \underset{+h}{D}(\Delta t) + \big(\Lambda_d^s + Q_d^s\big)\delta_d q^s + \underset{+h}{D}(\Delta G_d) = 0 \tag{34}$$

in the course of upward calculation, we have used the relation $\Delta q^s = \delta_d q^s + q_k^s \Delta t$, $\Delta q_k^s = \delta_d q_k^s + (\overset{s}{\bar{q}}_{kk})\Delta t$ (where $\overset{s}{\bar{q}}_{kk} = \underset{-h}{D}\underset{+h}{D} q^s$) and the commute relation $\underset{+h}{D}(\delta_d q^s) = (\delta_d q_k^s)$, and the Leibniz rule of (forward) difference as the discrete derivative.

For the infinitesimal transformation (10), (34) can be written as

$$\frac{\partial L_d}{\partial t}\xi_{00} + \frac{\partial L_d}{\partial q^s}\xi_{s0} + \frac{\partial L_d}{\partial q_k^s}\big(\underset{+h}{D}(\xi_{s0}) - q_k^s \underset{+h}{D}(\xi_{00})\big) + L_d \underset{+h}{D}(\xi_{00}) + \big(\Lambda_d^s + Q_d^s\big)\big(\xi_{s0} - q_k^s \xi_{00}\big) + \underset{+h}{D}(G_d) = 0 \tag{35}$$

we have made use of $\Delta G_d = \epsilon G_d$.

Equation (35) can be expressed as

$$\begin{aligned} &\frac{\partial L_d}{\partial t}\xi_{00} + \frac{\partial L_d}{\partial q^s}\xi_{s0} + \frac{\partial L_d}{\partial q_k^s}\big(\underset{+h}{D}(\xi_{s0}) - q_k^s \underset{+h}{D}(\xi_{00})\big) + L_d \underset{+h}{D}(\xi_{00}) \\ &\quad + \big(\Lambda_d^s + Q_d^s\big)\big(\xi_{s0} - q_k^s \xi_{00}\big) + \underset{+h}{D}(G_d) \\ &= \xi_{00}\bigg\{ \frac{\partial L_d}{\partial t} + \underset{-h}{D}\bigg(q_k^s \frac{\partial L_d}{\partial q_k^s} - L_d \bigg) - q_k^s\big(Q_d^s + \Lambda_d^s\big) \bigg\} \\ &\quad + \xi_{s0}\bigg\{ \frac{\partial L_d}{\partial q^s} - \underset{-h}{D}\bigg(\frac{\partial L_d}{\partial q_k^s}\bigg) + \Lambda_d^s + Q_d^s \bigg\} \\ &\quad + \underset{+h}{D}\bigg\{ \xi_{00} \underset{-h}{S}(L_d) + \big(\xi_{s0} - \underset{-h}{S}(q_k^s)\xi_{00}\big) \underset{-h}{S}\bigg(\frac{\partial L_d}{\partial q_k^s}\bigg) + G_d \bigg\} = 0 \end{aligned} \tag{36}$$

If there exists some equations,

$$\frac{\partial L_d}{\partial t} + \underset{-h}{D}\bigg(q_k^s \frac{\partial L_d}{\partial q_k^s} - L_d \bigg) - q_k^s\big(Q_d^s + \Lambda_d^s\big) = 0 \tag{37}$$

$$\frac{\partial L_d}{\partial q^s} - \underset{-h}{D}\bigg(\frac{\partial L_d}{\partial q_k^s}\bigg) + \Lambda_d^s + Q_d^s = 0 \tag{38}$$

which are called the generalized quasiextremal equation for this discrete system, and in fact (38) is the difference analogues of the corresponding holonomic system (6) and (37) is the discrete energy equation; then this system possesses the discrete analogue of conservation law

$$\underset{+h}{D}\bigg\{ \xi_{00}^\alpha \underset{-h}{S}(L_d) + \big(\xi_{s0}^\alpha - \underset{-h}{S}(q_k^s)\xi_{00}^\alpha\big) \underset{-h}{S}\bigg(\frac{\partial L_d}{\partial q_k^s}\bigg) + G_d^\alpha \bigg\} = 0 \tag{39}$$

namely exact invariants

$$I_d^0 = \xi_{00}^\alpha \underset{-h}{S}(L_d) + \big(\xi_{s0}^\alpha - (\overset{s}{q_k^-})\xi_{00}^\alpha\big) \underset{-h}{S}\left(\frac{\partial L_d}{\partial q_k^s}\right) + G_d^\alpha = \text{const} \tag{40}$$

The difference equations (39) and (40) are called the difference analogue of Noether conservation laws associated with such a discrete corresponding holonomic system. The difference equations (39) and (40) form the invariant schemes on regular lattice h.

We should point out that the difference equations (38), which are obtained in this progress, preserve the Noether symmetry, and the (37) "disappears" in continuous limit since the operator in brackets tends to zero as $h \to 0$.

From Definition (33), we have the following criterion.

Criterion 3 *If the infinitesimal transformation* (9) *satisfy* (34), *it is called the generalized Noether quasisymmetry transformation.*

Criterion 4 *If the infinitesimal transformation* (10) *satisfy* (35), *it is called the generalized Noether quasisymmetry transformation.*

We call (35) the discrete analogue of generalized Noether-type identity for this discrete corresponding holonomic system.

Theorem 3 *If the infinitesimal transformations Lie group* (9) *for the discrete system* (38) *on an uniform mesh h, are Noether quasisymmetry transformation, then the discrete corresponding holonomic system possesses the discrete analogue of Noether conserved quantities* (40).

Theorem 4 *If the infinitesimal transformations Lie group* (10) *for the discrete system* (38)*on an uniform mesh h, are Noether quasisymmetry transformation, then the discrete corresponding holonomic system possesses the discrete analogue of Noether conserved quantities* (40).

We call Theorems 3 and 4 the discrete analogue of generalized Noether theorems associated with discrete corresponding holonomic systems (38).

3.3 Discrete Noether symmetry of nonholonomic system

The discrete nonholonomic constraints of Chetaev's type can be written as

$$f_\beta^d\big(t, q_s, q_k^s\big) = 0 \quad (\beta = 1, \dots, g) \tag{41}$$

The corresponding discrete Appell–Chetaev condition is (30).

Definition 4 If infinitesimal transformation (10) is Noether quasisymmetry transformation, and it satisfies the Appell–Chetaev condition (30), the transformation is called Noether quasisymmetry transformation of discrete nonholonomic system (38), (41).

Theorem 5 *For the discrete nonholonomic system* (41), (38), *if the infinitesimal transformation* (10) *is Noether quasisymmetry transformation of discrete nonholonomic system, the system has the same style of discrete Noether invariants as* (40).

Proof Making use of Definition 4 of Noether quasisymmetry of the nonholonomic system and (38), it is easy to verify the theorem. □

4 Perturbation to symmetry and adiabatic invariants of discrete nonholonomic nonconservative mechanical systems

Suppose the systems (38) and (37) are perturbed by small quantity $\epsilon W_d^s = \epsilon W_d^s(t, q^s, q_d^s)$, then the equations of the discrete nonholonomic mechanical systems become

$$\underset{-h}{D}\left(\frac{\partial L_d}{\partial q_k^s}\right) - \frac{\partial L_d}{\partial q^s} = \Lambda_d^s + Q_d^s + \epsilon W_d^s \tag{42}$$

and

$$\frac{\partial L_d}{\partial t} + \underset{-h}{D}\left(q_k^s \frac{\partial L_d}{\partial q_k^s} - L_d\right) - q_k^s\big(Q_d^s + \Lambda_d^s + \epsilon W_d^s\big) = 0 \tag{43}$$

Due to the action of ϵW_d^s, the primary symmetries and invariants of systems (38) and (37) may vary. Assume the variation is a small perturbation based on the symmetrical transformation of the initial system, then

$\xi_0^\alpha, \xi_s^\alpha$ which denote the new generators after being perturbed can be expressed as

$$\begin{aligned}\xi_0 &= \xi_{00} + \epsilon\xi_{01} + \epsilon^2\xi_{02} + \cdots \\ \xi_s &= \xi_{s0} + \epsilon\xi_{s1} + \epsilon^2\xi_{s2} + \cdots\end{aligned} \tag{44}$$

and the new generators satisfy

$$\begin{aligned}&\frac{\partial L_d}{\partial t}\xi_0 + \frac{\partial L_d}{\partial q^s}\xi_s + \frac{\partial L_d}{\partial q_k^s}\big(\underset{+h}{D}(\xi_s) - q_k^s \underset{+h}{D}(\xi_0)\big) \\ &\quad + L_d \underset{+h}{D}(\xi_0) + \big(\Lambda_d^s + Q_d^s\big)\big(\xi_s - q_k^s\xi_0\big) \\ &\quad + \epsilon W_d^s\big(\xi_s - q_k^s\xi_0\big) + \underset{+h}{D}(G_d) = 0\end{aligned} \tag{45}$$

if we make

$$G_d = G_{d0}^\alpha + \epsilon G_d + \epsilon^2 G_{d2} + \cdots \tag{46}$$

and substitute (44) and (46) into (45), we have

$$\begin{aligned}&\frac{\partial L_d}{\partial t}\xi_{0m} + \frac{\partial L_d}{\partial q^s}\xi_{sm} + \frac{\partial L_d}{\partial q_k^s}\big(\underset{+h}{D}(\xi_{sm}) - q_k^s \underset{+h}{D}(\xi_{0m})\big) \\ &\quad + L_d \underset{+h}{D}(\xi_{0m}) + \big(\Lambda_d^s + Q_d^s\big)\big(\xi_{sm} - q_k^s\xi_{0m}\big) \\ &\quad + W_s^s\big(\xi_{sm-1} - q_k^s\xi_{0m-1}\big) + \underset{+h}{D}(G_{dm}) = 0 \\ &\quad (m = 0, 1, 2, \ldots)\end{aligned} \tag{47}$$

when $m = 0$, the condition $W_d^s = 0$.

Correspondingly, the discrete Appell-Chetaev condition becomes

$$\sum_{s=1}^{n} \frac{\partial f_\beta^d}{\partial q_k^s}\bar{\xi}_{sm} = 0 \tag{48}$$

where $\bar{\xi}_{sm} = \xi_{sm} - q_k^s\xi_{0m}$, and the generalized Noether-type operators for discrete perturbed system can be written as

$$\begin{aligned}X &= \xi_0\frac{\partial}{\partial t} + \xi_s\frac{\partial}{\partial q^s} + \big[\underset{+h}{D}(\xi_s) - q_k^s \underset{+h}{D}(\xi_0)\big]\frac{\partial}{\partial q_k^s} \\ &\quad + \cdots + h\underset{+h}{D}(\xi_0)\frac{\partial}{\partial h}\end{aligned} \tag{49}$$

Substituting (44) into (49), we have

$$X = \epsilon^m X_m \tag{50}$$

where

$$\begin{aligned}X_m &= \xi_{0m}\frac{\partial}{\partial t} + \xi_{sm}\frac{\partial}{\partial q^s} + \big[\underset{+h}{D}(\xi_{sm}) - q_k^s \underset{+h}{D}(\xi_{0m})\big]\frac{\partial}{\partial q_k^s} \\ &\quad + \cdots + h\underset{+h}{D}(\xi_{0m})\frac{\partial}{\partial h}\end{aligned} \tag{51}$$

So, we can give the criterion of perturbation to Noether symmetry of this system.

Criterion 5 *For the perturbed discrete system* (42), (41), *if the infinitesimal transformation generators* $\xi_{0m}^\alpha, \xi_{sm}^\alpha$ *satisfy* (47) *and* (48), *and there exist gauge function* $G_{dm}^\alpha = G_{dm}^\alpha(t, q^s, q_k^s)$, *the corresponding variety of Noether symmetry of the nonholonomic nonconservative mechanical system is called perturbation to Noether symmetry.*

According the definition of adiabatic invariants in [15], we can give the definition of discrete adiabatic invariants as the following definition.

Definition 5 For systems (42) and (41), if a physical quantity $I_d^z(t, q^s, q_d^s, \epsilon)$ satisfies

$$\underset{+h}{D}(I_z) = O\big(\epsilon^{z+1}\big) \tag{52}$$

where

$$I_d^z = I_{d0}^0 + \epsilon I_{d1}^1 + \cdots + \epsilon^z I_{dz}^z \tag{53}$$

I_d^z is called a zth-order adiabatic invariants of the system.

Basing on Definition 5 and criterion 5, we have the following theorem.

Theorem 6 *For the systems* (42) *with the nonholonomic constraint* (41), *which is perturbed by a small physical quantity* ϵW_d^s, *if the generators* $\xi_{0m}^\alpha, \xi_{sm}^\alpha$ *of the infinitesimal transformations are perturbation to Noether symmetry* (*i.e., satisfy* (47), (48), *and there exists a gauge function* $G_{dm}^\alpha = G_{dm}^\alpha(t, q^s, q_k^s)$), *then the system has* z*th-order adiabatic invariants of discrete analogue as the following forms*:

$$\begin{aligned}I_d^z = \sum_{m=0}^{z} \epsilon^m \bigg\{&\xi_{0m}^\alpha \underset{-h}{S}(L_d) \\ &+ \big(\xi_{sm}^\alpha - \big(q_{\overset{-}{k}}^{s}\big)\xi_{0m}^\alpha\big)\underset{-h}{S}\bigg(\frac{\partial L_d}{\partial q_k^s}\bigg) + G_{dm}^\alpha\bigg\}\end{aligned} \tag{54}$$

when $z = 0$, $W_d^s = 0$ holds.

Proof Based on Definition 5 of zth-order adiabatic invariants, making the discrete derivative of I_d^z, and making use of (47) and the Leibniz rule, we have

$$\begin{aligned}
\underset{+h}{D}\left(I_d^z\right) &= \xi_{0m}^{\alpha} \underset{-h}{D}(L_d) + \left(\xi_{sm}^{\alpha} - q_k^s \xi_{0m}^{\alpha}\right) \underset{-h}{D}\left(\frac{\partial L_d}{\partial q_k^s}\right) \\
&\quad + \underset{-h}{D}\left(q_k^s\right) \xi_{0m}^{\alpha} \frac{\partial L_d}{\partial q_k^s} - \xi_{0m}^{\alpha} \frac{\partial L_d}{\partial t} - \xi_{sm}^{\alpha} \frac{\partial L_d}{\partial q^s} \\
&\quad - \left(\Lambda_d^s + Q_d^s\right)\left(\xi_{sm}^{\alpha} - q_k^s \xi_{0m}^{\alpha}\right) \\
&\quad - W_d^s\left(\xi_{sm-1}^{\alpha} - q_k^s \xi_{0m-1}^{\alpha}\right) \\
&= \xi_{0m}^{\alpha} \left\{ \underset{-h}{D}\left(L_d - q_k^s \frac{\partial L_d}{\partial q_k^s}\right) - \frac{\partial L_d}{\partial t} \right. \\
&\quad \left. + q_k^s\left(Q_d^s + \Lambda_d^s\right) \right\} \\
&\quad + \xi_{sm}^{\alpha} \left\{ \underset{-h}{D}\left(\frac{\partial L_d}{\partial q_k^s}\right) - \frac{\partial L_d}{\partial q^s} - \Lambda_d^s + Q_d^s \right\} \\
&\quad - W_d^s\left(\xi_{sm-1}^{\alpha} - q_k^s \xi_{0m-1}^{\alpha}\right)
\end{aligned} \tag{55}$$

make use of (42) and (43), after deduction, we have

$$\begin{aligned}
\underset{+h}{D}\left(I_d^z\right) = \sum_{m=0}^{z} &\left\{ \epsilon W_d^s\left(\xi_{sm}^{\alpha} - q_k^s \xi_{0m}^{\alpha}\right) \right. \\
&\left. - W_d^s\left(\xi_{sm-1}^{\alpha} - q_k^s \xi_{0m-1}^{\alpha}\right) \right\}
\end{aligned}$$

expending the above formula and making summation, we have

$$\underset{+h}{D}\left(I_d^z\right) = \epsilon^{z+1} \sum_{s=1}^{n} W_d^s\left(\xi_{sz}^{\alpha} - q_k^s \xi_{0z}^{\alpha}\right) \tag{56}$$

It shows that $\underset{+h}{D}(I_d^z)$ is in direct proportion to ϵ^{z+1}, so I_d^z is discrete analogue of zth-order adiabatic invariants for discrete disturbed nonholonomic nonconservative systems (42), (41). □

5 Illustrate example

The dynamical systems with discrete Lagrangian

$$L = \frac{1}{2}\left\{\left(q_k^1\right)^2 + \left(q_k^2\right)^2 + \left(q_k^3\right)^2\right\} - mg\frac{q^{3+} + q^3}{2} \tag{57}$$

subject to nonholonomic constraints

$$f = \left(q_k^1\right)^2 + \left(q_k^2\right)^2 - \left(q_k^3\right)^2 = 0 \tag{58}$$

Let us study its exact invariants and adiabatic invariants.

The equation of motion of system are

$$\begin{aligned}
\overset{1}{q}_{kk} &= 2\lambda_d q_k^1 \\
\overset{2}{q}_{kk} &= 2\lambda_d q_k^2 \\
\overset{3}{q}_{kk} + mg &= -2\lambda_d q_k^2
\end{aligned} \tag{59}$$

where $\overset{s}{q}_{kk} = \underset{-h+h}{D\,D} q^s$. From (58) and (59), we can work out

$$\lambda_d = -\frac{mg}{4q_k^3} \tag{60}$$

so we have

$$\Lambda_1 = -\frac{mgq_k^1}{2q_k^3}, \qquad \Lambda_2 = -\frac{mgq_k^2}{2q_k^3}, \qquad \Lambda_3 = \frac{mg}{2} \tag{61}$$

From the discrete Noether identity (44), we have

$$\begin{aligned}
&-mg\xi_{30}^{\alpha} + mq_k^1\left(\underset{+h}{D}\left(\xi_{10}^{\alpha}\right) - q_k^1 \underset{+h}{D}\left(\xi_{00}^{\alpha}\right)\right) \\
&\quad + mq_k^2\left(\underset{+h}{D}\left(\xi_{20}^{\alpha}\right) - q_k^2 \underset{+h}{D}\left(\xi_{00}^{\alpha}\right)\right) \\
&\quad + mq_k^3\left(\underset{+h}{D}\left(\xi_{30}^{\alpha}\right) - q_k^3 \underset{+h}{D}\left(\xi_{00}^{\alpha}\right)\right) \\
&\quad + \left\{\frac{1}{2}\left\{\left(q_k^1\right)^2 + \left(q_k^2\right)^2 + \left(q_k^3\right)^2\right\}\right. \\
&\quad \left. - mg\frac{q^{3+} + q^3}{2}\right\} \underset{+h}{D}\left(\xi_{00}^{\alpha}\right) - \frac{mgq_k^1}{2q_k^3}\left(\xi_{10}^{\alpha} - q_k^1 \xi_{00}^{\alpha}\right) \\
&\quad - \frac{mgq_k^2}{2q_k^3}\left(\xi_{20}^{\alpha} - q_k^2 \xi_{00}^{\alpha}\right) \\
&\quad + \frac{mg}{2}\left(\xi_{30}^{\alpha} - q_k^3 \xi_{00}^{\alpha}\right) + \underset{+h}{D}\left(G_d^{\alpha}\right) = 0
\end{aligned} \tag{62}$$

It has a group of solutions such as

$$\begin{aligned}
&\xi_{00}^{\alpha} = 0, \qquad \xi_{10}^{\alpha} = \xi_{20}^{\alpha} = 0, \qquad \xi_{30}^{\alpha} = 1 \\
&G_{d1}^{\alpha} = \frac{1}{2}mg\frac{t^{+} + t}{2}
\end{aligned} \tag{63}$$

$$\begin{aligned}
&\xi_{00}^{\alpha} = 1, \qquad \xi_{10}^{\alpha} = \xi_{20}^{\alpha} = \xi_{30}^{\alpha} = 0, \\
&G_{d2}^{\alpha} = 0
\end{aligned} \tag{64}$$

so, (63) and (64) are Noether symmetry for the discrete corresponding holonomic system (38).

According to (40), we can get discrete Noether conserved law as

$$I_{d01}^{\alpha}=m\overset{3}{q_k^-}+\frac{1}{2}mg\frac{t^{+}+t}{2}=\text{const} \tag{65}$$

$$\begin{aligned}I_{d02}^{\alpha}&=-\frac{1}{2}m\{(\overset{1}{q_k^-})^2+(\overset{2}{q_k^-})^2-(\overset{3}{q_k^-})^2\}-mg\overset{3}{q^-}\\&=\text{const}\end{aligned} \tag{66}$$

The Appell–Chetaev condition of the discrete nonholonomic constraints are

$$\begin{aligned}&2q_k^1(\xi_{01}^{\alpha}-q_k^1\xi_{00}^{\alpha})+2q_k^2(\xi_{02}^{\alpha}-q_k^2\xi_{00}^{\alpha})\\&\quad-2q_k^3(\xi_{03}^{\alpha}-q_k^3\xi_{00}^{\alpha})=0\end{aligned} \tag{67}$$

it is easy to very (64) satisfy (67), so the (63) is Noether symmetry for nonholonomic mechanical (38)and (41), the corresponding conserved law is (66).

Suppose the system is perturbed by

$$\epsilon W_d^1=-\epsilon q_k^1,\qquad \epsilon W_d^2=-\epsilon q_k^2,\qquad \epsilon W_d^3=-\epsilon q_k^3 \tag{68}$$

Let $m=1$, then (47) give

$$\begin{aligned}&-mg\xi_{31}^{\alpha}+mq_k^1\big(\underset{+h}{D}(\xi_{11}^{\alpha})-q_k^1\underset{+h}{D}(\xi_{01}^{\alpha})\big)\\&\quad+mq_k^2\big(\underset{+h}{D}(\xi_{21}^{\alpha})-q_k^2\underset{+h}{D}(\xi_{01}^{\alpha})\big)\\&\quad+mq_k^3\big(\underset{+h}{D}(\xi_{31}^{\alpha})-q_k^3\underset{+h}{D}(\xi_{01}^{\alpha})\big)\\&\quad+\left\{\frac{1}{2}\{(q_k^1)^2+(q_k^2)^2+(q_k^3)^2\}\right.\\&\quad\left.-mg\frac{q^{3+}+q^3}{2}\right\}\underset{+h}{D}(\xi_{01}^{\alpha})-\frac{mgq_k^1}{2q_k^3}(\xi_{11}^{\alpha}-q_k^1\xi_{01}^{\alpha})\\&\quad-\frac{mgq_k^2}{2q_k^3}(\xi_{21}^{\alpha}-q_k^2\xi_{01}^{\alpha})\\&\quad+\frac{mg}{2}(\xi_{31}^{\alpha}-q_k^3\xi_{01}^{\alpha})-q_k^1(\xi_{10}^{\alpha}-q_k^1\xi_{00}^{\alpha})\\&\quad-q_k^2(\xi_{20}^{\alpha}-q_k^2\xi_{00}^{\alpha})\\&\quad-q_k^3(\xi_{30}^{\alpha}-q_k^3\xi_{00}^{\alpha})+\underset{+h}{D}(G_{d1}^{\alpha})=0\end{aligned} \tag{69}$$

we can work out solutions as

$$\begin{aligned}&\xi_{01}^{\alpha}=0,\qquad \xi_{11}^{\alpha}=\xi_{21}^{\alpha}=0,\qquad \xi_{31}^{\alpha}=1,\\&G_{d11}^{\alpha}=\frac{1}{2}mg\frac{t^{+}+t}{2}-\frac{q^{3+}q^3}{2}\end{aligned} \tag{70}$$

$$\begin{aligned}&\xi_{01}^{\alpha}=1,\qquad \xi_{11}^{\alpha}=\xi_{21}^{\alpha}=0,\qquad \xi_{31}^{\alpha}=0,\\&G_{d12}^{\alpha}=0\end{aligned} \tag{71}$$

The corresponding conserved law is

$$\begin{aligned}I_{d11}^{\alpha}&=m\overset{3}{q_k^-}+\frac{1}{2}mgt+\epsilon\left\{\overset{3}{q_k^-}+(m-1)\frac{q^{3+}+q^3}{2}\right.\\&\quad\left.+\frac{1}{2}mg\frac{t^{+}+t}{2}\right\}=\text{const}\end{aligned} \tag{72}$$

and

$$\begin{aligned}I_{d12}^{\alpha}&=-\frac{1}{2}m\{(\overset{1}{q_k^-})^2+(\overset{2}{q_k^-})^2-(\overset{3}{q_k^-})^2\}-mg\overset{3}{q^-}\\&\quad-\epsilon\left\{\frac{1}{2}m\{(\overset{1}{q_k^-})^2+(\overset{2}{q_k^-})^2-(\overset{3}{q_k^-})^2\}-mg\overset{3}{q^-}\right\}\\&=\text{const}\end{aligned} \tag{73}$$

The Appell–Chetaev condition of discrete perturbed system is

$$\begin{aligned}&2q_k^1(\xi_{11}^{\alpha}-q_k^1\xi_{01}^{\alpha})+2q_k^2(\xi_{12}^{\alpha}-q_k^2\xi_{01}^{\alpha})\\&\quad-2q_k^3(\xi_{13}^{\alpha}-q_k^3\xi_{01}^{\alpha})=0\end{aligned} \tag{74}$$

It is easy to verify (71) and satisfy the above Appell–Chetaev condition (74), so (71) is perturbation to the Noether symmetry generator of the discrete nonholonomic system, and (73) is a corresponding first-order discrete adiabatic invariant. Furthermore, we can obtain higher order adiabatic invariants.

6 Conclusion

In this paper, (1) we obtain Noether exact invariants for the discrete nonholonomic nonconservative systems; (2) we get the theorem under which the perturbation to Noether symmetry can lead to the Noether adiabatic invariants and the forms of the adiabatic invariants for discrete nonholonomic nonconservative systems.

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