Nonlinear Dyn (2012) 67:2847–2856
DOI 10.1007/s11071-011-0193-1

ORIGINAL PAPER

On the global stabilization of Takagi–Sugeno fuzzy cascaded systems

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Received: 7 January 2011 / Accepted: 3 August 2011 / Published online: 15 September 2011

Abstract This paper deals with the problem of the global stabilization for a class of cascade nonlinear control systems. It is well known that, in general, the global asymptotic stability of the cascaded subsystems does not imply the global asymptotic stability of the composite closed-loop system. In this paper, we give additional sufficient conditions for the global stabilization of a cascade nonlinear system. In particular, we consider a class of Takagi–Sugeno (TS) fuzzy cascaded systems. Using the so-called parallel distributed compensation (PDC) controller, we prove that this class of systems can be globally asymptotically stable. An illustrative example is given to show the applicability of the main result.

Keywords Fuzzy systems · PDC controller · Cascaded systems · Lyapunov stability

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1 Introduction

This work studies the problem of the global stabilization of nonlinear cascaded systems of the form

$$\begin{cases} \dot{x}_1 = f(x_1, x_2) \\ \dot{x}_2 = g(x_1, x_2, u) \end{cases} \tag{1.1}$$

where $x_1 \in \mathbb{R}^n$, $x_2 \in \mathbb{R}^q$ and $u \in \mathbb{R}^m$. The functions f and g are supposed $\mathcal{C}^\infty$ and to satisfy $f(0,0) = 0$, $g(0,0,0) = 0$.

Let

$$\dot{x}_1 = f(x_1, x_2). \tag{1.2}$$

It is well known that if the differential equation

$$\dot{x}_1 = f(x_1, 0) \tag{1.3}$$

has $x_1 = 0$ as an equilibrium point globally asymptotically stable, if the system

$$\dot{x}_2 = g(x_1, x_2, u) \tag{1.4}$$

is globally asymptotically stabilized at the origin, uniformly on x_1 by a feedback law $u(x_1, x_2)$, and if all the orbits of the closed-loop system

$$\begin{cases} \dot{x}_1 = f(x_1, x_2) \\ \dot{x}_2 = g(x_1, x_2, u(x_1, x_2)) \end{cases}$$

are bounded, then $(x_1, x_2) = (0, 0)$ is an equilibrium point globally asymptotically stable for (1.1) [13, 15].

As a first step, we need to note that, due to the regularity of f, it is always possible to decompose $f(x_1, x_2)$ into the form

$$f(x_1, x_2) = f(x_1, 0) + \psi(x_1, x_2)x_2$$

where $\varphi(x_1, x_2) = \psi(x_1, x_2)x_2$ is called the interconnection term.

However, it is not easy to determine a priori that every solution to (1.2) is bounded or not, and in addition the key problem remains to be solved: what conditions can ensure the boundedness of solutions of (1.2). Therefore, studying global boundedness of a cascaded system (1.2) is a challenging problem, and this problem is of its own interest as well as of significance in the research on global asymptotic stability and global stabilization of cascaded systems in the literature [1, 4, 5, 14, 17, 18]. One way to solve this problem is to use the now well known property of input-to-state-stability (ISS) introduced by [19]: the system (1.2) is input-to-state-stable (ISS) if there exist functions $\beta \in \mathcal{KL}$ and $\alpha \in \mathcal{K}$ such that for each bounded input $x_2(.)$ and each initial condition $x_1(0)$, the solution $x_1(t)$ exists for all $t \geq 0$ and is bounded by

$$\|x_1(t)\| \leq \beta(\|x_1(0)\|, t) + \alpha\Big(\sup_{0\leq\tau\leq t}\|x_2(\tau)\|\Big).$$

In a recent result given in [2], the ISS property is characterized by the existence of an ISS-Lyapunov function introduced in [3], in the sense that the system (1.2) is ISS if and only if there exists a C^1 positive definite radially unbounded function V such that

$$\|x_1\| \geq \alpha_1(\|x_2\|) \Longrightarrow \frac{\partial V}{\partial x_1} f(x_1, x_2) \leq -\alpha_2(\|x_1\|), \tag{1.5}$$

where α_1 and α_2 are two class $\mathcal{K}$ functions. Such a V is called an ISS Lyapunov function.

Therefore, under the stated assumptions, if the system (1.2), with x_2 as input, is input-to-state-stable and the origin of (1.4) is globally asymptotically stabilized, uniformly on x_1, by a feedback law $u(x_1, x_2)$, then the origin of the cascade system (1.1) is globally asymptotically stable.

However, if the input-state-stability property is not assumed for (1.2), the situation is complicated. For example, let us consider the following nonlinear planar cascade system:

$$\begin{cases} \dot{x}_1 = -x_1 + x_1 \log(x_1^2)x_2 \\ \dot{x}_2 = \sin(x_1)x_2 + u. \end{cases} \tag{1.6}$$

We can see that the interconnection term $\psi(x_1, x_2)x_2 = x_1 \log(x_1^2)x_2$ has a nonlinear growth faster than the linear one in x_1. Otherwise, the destabilizing perturbation $x_1 \log(x_1^2)x_2$ dominates the stabilizing term $-x_1$. Then the system $\dot{x}_1 = -x_1 + x_1 \log(x_1^2)x_2$ is not ISS.

Therefore, in order to prove that all solutions remain bounded under the cascade interconnection, one remarkable result was given in [14], where the interconnection term satisfies the linear growth condition in x_1, i.e., there exist two class $\mathcal{K}$ functions γ_1 and γ_2, differentiable at $x_2 = 0$, such that

$$\|\psi(x_1, x_2)x_2\| \leq \gamma_1(\|x_2\|)\|x_1\| + \gamma_2(\|x_2\|),$$

which is restrictive in general. This can be viewed in example (1.6) where the interconnection term does not satisfy the linear growth condition in x_1. For these reasons, we consider in particular a class of nonlinear dynamical TS fuzzy cascaded systems. Indeed, TS fuzzy models [10, 16, 22] are nonlinear systems described by a set of if-then rules which give local linear representations of an underlying system. Such models can approximate a wide class of nonlinear systems. They can even describe exactly certain nonlinear systems [11, 20]. A recent work on the output control problem of a class of uncertain SISO nonlinear systems is investigated based on an indirect adaptive fuzzy approach is [24].

In this paper, by using the idea of the sector nonlinearity given in [21], we suppose that we can represent exactly (1.1) into a TS fuzzy cascaded system which has the following ith rule local model

$$\begin{cases} \dot{x}_1 = A_i x_1 + \psi_i(x_1, x_2)x_2 \\ \dot{x}_2 = C_i x_2 + B_i u \end{cases}$$

where $A_i \in \mathbb{R}^{n\times n}$, $C_i \in \mathbb{R}^{q\times q}$ and $B_i \in \mathbb{R}^{q\times m}$, we call

$$\varphi_i(x_1, x_2) = \psi_i(x_1, x_2)x_2$$

the ith interconnection term. By means of the concept of the parallel distributed compensation (PDC) approaches, which was proposed in [8], we prove that the TS fuzzy cascaded system (1.1) can be globally asymptotically stable. In addition, this method of stabilization is conceptually simple and straightforward because the linear feedback control techniques can be utilized. Various non-parallel distribution compensation (PDC) fuzzy controllers, which can better utilize the characteristic of the Parameter Dependent Lyapunov Function (PDLF), are proposed by [7] to close the feedback loop.

Note that the design problem of stabilization can be transformed into a convex problem [9], which is efficiently solved by linear matrix inequalities optimization. If the solution is feasible, meaning that the stabilization constraints are met, then local state feedback gains are obtained.

2 Global stabilization and boundedness of solutions

Consider the fuzzy dynamic cascade model of TS described by the following fuzzy if-then rules:

If z_1 is F_{i1} and $\dots$ and z_p is F_{ip}

$$\text{then } \begin{cases} \dot{x}_1 = A_i x_1 + \psi_i(x_1, x_2)x_2 \\ \dot{x}_2 = C_i x_2 + B_i u \end{cases}$$

where F_{ij} $(j = 1, \dots, p)$ is the fuzzy set and r is the number of model rules (if-then), $r \geq 2$. $x_1 \in \mathbb{R}^n$ and $x_2 \in \mathbb{R}^q$ are the state vectors, $u \in \mathbb{R}^m$ is the input vector. The matrices A_i, B_i and C_i are of appropriate dimension and $z_1 \sim z_p$ are the premise variables which are measurable and depend on x_1. We will use z to denote the vector containing all the individual elements $z_1, \dots, z_p$. By using the fuzzy inference method with a singleton fuzzifier, product inference, and center average defuzzifier, the global fuzzy cascaded model can be expressed as

$$\begin{cases} \dot{x}_1 = \sum_{i=1}^r h_i(z) A_i x_1 + \sum_{i=1}^r h_i(z)\psi_i(x_1, x_2)x_2 \\ \dot{x}_2 = \sum_{i=1}^r h_i(z)(C_i x_2 + B_i u) \end{cases} \tag{2.1}$$

where

$$h_i(z) = \frac{W_i(z)}{\sum_{i=1}^r W_i(z)},$$

and

$$W_i(z) = \prod_{j=1}^p F_{ij}(z_j), \quad \text{for all } i = 1, \dots, r.$$

It is assumed that $h_i(z) > 0$, for all $i = 1, \dots, r$ and $\sum_{i=1}^r h_i(z) = 1$, for all $t \geq 0$.

Let

$$\dot{x}_1 = \sum_{i=1}^r h_i(z) A_i x_1, \tag{2.2}$$

$$\dot{x}_1 = \sum_{i=1}^r h_i(z) A_i x_1 + \sum_{i=1}^r h_i(z)\psi_i(x_1, x_2)x_2, \tag{2.3}$$

and

$$\dot{x}_2 = \sum_{i=1}^r h_i(z)(C_i x_2 + B_i u). \tag{2.4}$$

Let us consider the following assumptions.

($\mathcal{A}_1$) The pairs (C_i, B_i) are controllable, for all $i = 1, \dots, r$.
Many published results, concerning the control of the fuzzy system, are based on the parallel distributed compensation (PDC) principle [6, 9, 23]. In our case, the fuzzy cascaded system is assumed to be locally controllable. The design of the fuzzy controller shares the same antecedent as the fuzzy cascaded system and employs a linear state feedback control in the consequent part. For each local dynamics the controller is defined as

($\mathcal{A}_2$) If z_1 is F_{i1} and $\dots$ and z_p is F_{ip} then $u = -K_i x_2$, where $K_i \in \mathbb{R}^{q\times m}$ is the gain matrix.

Based on theses assumptions, each cascaded subsystem is locally controllable and state feedback gains are determined for every local cascaded system.

The goal of this paper is to find some conditions on the ith interconnection term such that the closed-loop fuzzy cascaded system (2.1) is globally asymptotically stable. Notice that our stabilization analysis and design assume that the ith interconnection term has a nonlinear growth in x_1 such that neither the ISS property nor the linear growth condition in x_1 are satisfied for (2.3).

2.1 Main results

Let us consider the following assumptions.

($\mathcal{H}_1$) There exists a common positive symmetric definite matrix P that satisfies the following inequality:

$$A_i^T P + P A_i < 0, \quad \text{for all } i = 1, \dots, r.$$

Remark 2.1 Note that assumption $(\mathcal{H}_1)$ may be very severe because all local models seem to be stable. But readers should not forget that in this paper not all nonlinearities are transformed into the matrices A_i. Indeed, the term ψ_i may include unbounded nonlinearities, or bounded ones, such that only the stable part of the model is included in A_i.

$(\mathcal{H}_2)$ There exist positive symmetric and definite matrices P, $\tilde{Q}_{ii}$ and $\tilde{Q}_{ij}$ $(i<j)$, such that the following inequalities [12] hold:

$$\Upsilon_{ii} < -\tilde{Q}_{ii}, \quad \text{for all } i = 1,\dots,r,$$

and

$$\Upsilon_{ij} + \Upsilon_{ji} < -\tilde{Q}_{ij} - \tilde{Q}_{ji}, \quad 1 \le i < j \le r,$$

where $\Upsilon_{ij} = (C_i - B_i K_j)^T \tilde{P} + \tilde{P}(C_i - B_i K_j)$.

$(\mathcal{H}_3)$ $\|\varphi_i(x_1,x_2)\| \le G_i(x_2)H_i(x_1)$, such that: for $i=1,\dots,r$, H_i is a positive bounded function and G_i is a positive continuous function with $G_i(0)=0$ and satisfies the local Lipschitz condition.

Theorem 2.1 *Under assumptions $(\mathcal{H}_1)$, $(\mathcal{H}_2)$, and $(\mathcal{H}_3)$, the system (2.1) in closed loop with the feedback $u(x_1,x_2) = -\sum_{i=1}^r h_i(z)K_i x_2$ is globally asymptotically stable.*

Remark 2.2 Using as Lyapunov function the candidate $\mathcal{V}(x_1) = x_1^T P x_1$ for the system (2.3), it is easy to see that the system (2.2) is globally asymptotically stable.

Proof Consider the Lyapunov function candidate $\tilde{\mathcal{V}}(x_2) = x_2^T \tilde{P} x_2$ for the subsystem (2.4). Its derivative with respect to time is given by

$$\begin{aligned}\dot{\tilde{\mathcal{V}}}(x_2) &= \dot{x}_2^T \tilde{P} x_2 + x_2^T \tilde{P} \dot{x}_2 \\ &= x_2^T \sum_{i=1}^r h_i^2(z)\Upsilon_{ii} x_2 \\ &\quad + x_2^T \sum_{i<j} h_i(z)h_j(z)(\Upsilon_{ij} + \Upsilon_{ji})x_2 \\ &\le -x_2^T \sum_{i=1}^r h_i^2(z)\tilde{Q}_{ii} x_2 \\ &\quad - x_2^T \sum_{i<j} h_i(z)h_j(z)(\tilde{Q}_{ij} + \tilde{Q}_{ji})x_2.\end{aligned}$$

Let $\delta = \inf\{\lambda_{\min}(Q_{ij}), i,j = 1,\dots,r\}$; $\lambda_{\min}$ denotes the smallest eigenvalue of the matrices Q_{ij}, $i,j = 1,\dots,r$.

Then

$$\begin{aligned}\dot{\tilde{\mathcal{V}}}(x_2) &\le -\delta \sum_{i=1}^r \sum_{j=1}^r h_i(z)h_j(z)\|x_2\|^2 \\ &\le -\delta\|x_2\|^2,\end{aligned}$$

which implies that the closed-loop system (2.4) is globally exponentially stable, uniformly on x_1, with $u = -\sum_{i=1}^r h_i(z)K_i x_2$, and x_2 verifies the following estimation:

$$\|x_2(t)\| \le \frac{\lambda_{\max}^{\frac{1}{2}}(P)}{\lambda_{\min}^{\frac{1}{2}}(P)} \|x_2(0)\| \exp\left(-\frac{\delta}{2\lambda_{\max}(P)} t\right),$$

for all $t \ge 0$.

Then, taking into account the above estimation and the fact that the system (2.2) is globally asymptotically stable, it suffices to prove the boundedness of the component $x_1(t)$ of any trajectory $(x_1(t), x_2(t))$, $t \ge 0$, of the system (2.1).

Suppose that $\|x_1\| > c$, where $c > 0$, i.e., x_1 is unbounded. The derivative of $\mathcal{V}$ along the trajectories of system (2.3) is given by

$$\begin{aligned}\dot{\mathcal{V}}(x_1) &= \nabla\mathcal{V}(x_1)\left(\sum_{i=1}^r h_i(z)A_i x_1\right) \\ &\quad + \nabla\mathcal{V}(x_1)\left(\sum_{i=1}^r h_i(z)\varphi_i(x_1,x_2)\right) \\ &\le \nabla\mathcal{V}(x_1)\left(\sum_{i=1}^r h_i(z)\varphi_i(x_1,x_2)\right) \\ &\le \|\nabla\mathcal{V}(x_1)\| \sum_{i=1}^r \|\varphi_i(x_1,x_2)\|.\end{aligned}$$

Since

$$\lambda_{\min}(P)\|x_1\|^2 \le V(x_1) = x_1^T P x_1 \le \lambda_{\max}(P)\|x_1\|^2,$$

and

$$\|\nabla\mathcal{V}(x_1)\| \le 2\lambda_{\max}(P)\|x\|,$$

we have

$$\|\nabla\mathcal{V}(x_1)\| \le 2\frac{\lambda_{\max}(P)}{\lambda_{\min}^{\frac{1}{2}}(P)}\mathcal{V}^{\frac{1}{2}}(x_1).$$

Also, we have

$$\|\varphi_i(x_1,x_2)\| \le G_i(x_2)H_i(x_1),$$

and then

$$\dot{\mathcal{V}}(x_1) \le 2\frac{\lambda_{\max}(P)}{\lambda_{\min}^{\frac{1}{2}}(P)}\mathcal{V}^{\frac{1}{2}}(x_1)\sum_{i=1}^{r} G_i(x_2)H_i(x_1).$$

Since, for all $i=1,\dots,r$, H_i is bounded, there exists $M>0$, such that $\|H_i(x_1)\| \le M$, for all x_1 and for all $i=1,\dots,r$.

It follows that

$$\dot{\mathcal{V}}(x_1) \le 2M\frac{\lambda_{\max}(P)}{\lambda_{\min}^{\frac{1}{2}}(P)}\mathcal{V}^{\frac{1}{2}}(x_1)\sum_{i=1}^{r} G_i(x_2).$$

Thus,

$$\frac{d\mathcal{V}(x_1)}{\mathcal{V}^{\frac{1}{2}}(x_1)} \le 2M\frac{\lambda_{\max}(P)}{\lambda_{\min}^{\frac{1}{2}}(P)}\sum_{i=1}^{r} G_i(x_2)\,dt.$$

Integrating between 0 and t, one obtains for all $t \ge 0$,

$$\int_0^t \frac{d\mathcal{V}(x_1)}{2\mathcal{V}^{\frac{1}{2}}(x_1)} \le M\frac{\lambda_{\max}(P)}{\lambda_{\min}^{\frac{1}{2}}(P)}\sum_{i=1}^{r}\int_0^t G_i(x_2)\,ds.$$

This implies that

$$\mathcal{V}^{\frac{1}{2}}(x_1) \le 2M\frac{\lambda_{\max}(P)}{\lambda_{\min}^{\frac{1}{2}}(P)}\sum_{i=1}^{r}\int_0^t G_i(x_2)\,ds.$$

We prove that $\int_0^t \|G_i(x_2(t))\|\,ds < \infty$, for all $t \ge 0$ and for all $i=1,\dots,r$.

Using the fact that G_i satisfies the local Lipschitz condition, there exists for all $i=1,\dots,r$, a neighborhood of 0, $V_i(0)$, where G_i satisfies the Lipschitz condition.

Let

$$W = \bigcap_{i=1}^{r} V_i(0).$$

Then, there exists $T \ge 0$ such that $x_2(t) \in W$ for all $t \ge T$.

Let ξ_i, $i=1,\dots,r$, is the Lipschitz constant which is associated to G_i.

Therefore, for all $t \ge T$ we have

$$\begin{aligned}
&\int_T^t \|G_i(x_2(t))\|\,ds \\
&\quad\le \xi_i \int_T^t \|x_2(t)\|\,ds \\
&\quad\le \xi_i \frac{\lambda_{\max}^{\frac{1}{2}}(P)}{\lambda_{\min}^{\frac{1}{2}}(P)}\|x_2(0)\|\int_T^t \exp\left(\frac{-\delta}{2\lambda_{\max}(P)}t\right)ds < \infty.
\end{aligned}$$

It follows that for all $t \ge 0$,

$$\begin{aligned}
\int_0^t \|G_i(x_2(t))\|\,ds &= \int_0^T \|G_i(x_2(t))\|\,dt \\
&\quad + \int_T^t \|G_i(x_2(t))\|\,ds < \infty.
\end{aligned}$$

We deduce that $\mathcal{V}(x_1)$ is bounded. It follows that x_1 must be bounded. Hence, the system (2.1) in closed loop with the feedback $u(x_1,x_2) = -\sum_{i=1}^{r} h_i(z)K_i x_2$ is globally asymptotically stable. □

Now, dropping the locally Lipschitz condition of the function G_i, other additional conditions have to be taken into account.

Then let us assume the following assumptions.

($\mathcal{H}'_1$) There exist positive symmetric definite matrices P and Q_i such that the following inequality holds for all $i=1,\dots,r$:

$$A_i^T P + PA_i < -Q_i.$$

($\mathcal{H}'_2$) There exists a common positive symmetric definite matrix $\tilde{P}$ that satisfies the following inequalities:

$$\Upsilon_{ii} < 0, \quad \text{for all } i=1,\dots,r,$$

and

$$\Upsilon_{ij} + \Upsilon_{ji} < 0, \quad 1 \le i < j \le r.$$

($\mathcal{H}'_3$) $\|\varphi_i(x_1,x_2)\| \le G_i(x_2)H_i(x_1)$, such as: for $i=1,\dots,r$, H_i is a positive continuous bounded function and G_i is a positive continuous function with $G_i(0)=0$.

Theorem 2.2 *Under the assumptions* $(\mathcal{H}'_1)$, $(\mathcal{H}'_2)$, *and* $(\mathcal{H}'_3)$, *the system* (2.1) *in closed loop with the feedback* $u(x_1, x_2) = -\sum_{i=1}^{r} h_i(z)K_i x_2$ *is globally asymptotically stable.*

Remark 2.3 Also we can see as Lyapunov function the candidate $\mathcal{V}(x_1) = x_1^T P x_1$ for the system (2.3). The derivative of $\mathcal{V}$ along the trajectories of system (2.3) satisfies

$$\dot{\mathcal{V}}(x_1) \leq -\sum_{i=1}^{r} \lambda_{\min}(Q_i)\|x_1\|^2.$$

It means that the system (2.2) is globally asymptotically stable.

Proof By $(\mathcal{H}'_2)$, the subsystem (2.4) is globally asymptotically stable, uniformly on x_1.

Since, for all $i = 1, \dots, r$, H_i is a positive continuous bounded function, there exists a positive constant $M > 0$ such that

$$\|H_i(x_1)\| \leq M, \quad \text{for all } i = 1, \dots, r \text{ and for all } x_1.$$

Since, the equilibrium of (2.4) is globally asymptotically stable, uniformly on x_1, there exists $T > 0$ such that

$$G_i(x_2) \leq \frac{\sum_{i=1}^{r} \lambda_{\min}(Q_i)}{4rM\lambda_{\max}(P)},$$

$$\text{for all } t > T \text{ and for all } i = 1, \dots, r,$$

where $\lambda_{\min}(Q_i)$ denotes the smallest eigenvalue of the matrices Q_i for all $i = 1, \dots, r$, and $\lambda_{\max}(P)$ denotes the largest eigenvalue of the matrix P.

Consider the Lyapunov function candidate $\mathcal{V}(x_1) = x_1^T P x_1$ for the system (2.3). Its derivative with respect to time is given by

$$\dot{\mathcal{V}}(x_1) = \nabla\mathcal{V}(x_1)\left(\sum_{i=1}^{r} h_i(z)A_i x_1\right) + \nabla\mathcal{V}(x_1)\left(\sum_{i=1}^{r} h_i(z)\varphi_i(x_1, x_2)\right).$$

Suppose that

$$\|x_1\| > c, \quad \text{where } c > 0.$$

On the one hand, we have

$$\|\nabla\mathcal{V}(x_1)\| \leq 2\lambda_{\max}(P)\|x_1\|.$$

Thus,

$$\dot{\mathcal{V}}(x_1) \leq -\sum_{i=1}^{r} \lambda_{\min}(Q_i)\|x_1\|^2 + \frac{1}{2}\sum_{i=1}^{r} \lambda_{\min}(Q_i)\|x_1\|^2,$$

$$\text{for all } t > T \text{ and } \|x_1\| > \max(1, c).$$

It follows that $\mathcal{V}(x_1)$ is bounded for $\|x_1(t)\| > \max(1, c)$ and $t > T$. If $\|x_1(t)\| \leq \max(1, c)$ for $t > T$, $\mathcal{V}(x_1)$ is bounded by definition. Then $\mathcal{V}(x_1)$ is bounded for all $t > T$.

On the other hand, we can see that

$$\begin{aligned}\dot{\mathcal{V}}(x_1) &= \nabla\mathcal{V}(x_1)\left(\sum_{i=1}^{r} h_i(z)A_i x_1\right) \\ &\quad + \nabla\mathcal{V}(x_1)\left(\sum_{i=1}^{r} h_i(z)\varphi_i(x_1, x_2)\right) \\ &\leq \|\nabla\mathcal{V}(x_1)\| \sum_{i=1}^{r} G_i(x_2)H_i(x_1) \\ &\leq 2M\lambda_{\max}(P)\|x_1\| \sum_{i=1}^{r} G_i(x_2).\end{aligned}$$

Thus,

$$\dot{\mathcal{V}}(x_1) \leq \frac{2M\lambda_{\max}(P)}{\lambda_{\min}^{\frac{1}{2}}(P)}\mathcal{V}^{\frac{1}{2}}(x_1)\sum_{i=1}^{r} G_i(x_2).$$

It follows that

$$\begin{aligned}\int_0^{\tau} \frac{d\mathcal{V}(x_1(t))}{2\mathcal{V}^{\frac{1}{2}}(x_1(t))} &\leq \frac{M\lambda_{\max}(P)}{\lambda_{\min}^{\frac{1}{2}}(P)} \int_0^{\tau} \sum_{i=1}^{r} G_i\big(x_2(t)\big)\,dt \\ &\leq \frac{2M\lambda_{\max}(P)}{\lambda_{\min}^{\frac{1}{2}}(P)} bT, \quad \text{for } 0 \leq \tau \leq T,\end{aligned}$$

with $b = \max_{\tau\in[0,T]} \sum_{i=1}^{r} G_i(x_2(\tau))$.

This implies that $\mathcal{V}(x_1)$ is bounded for all $t \in [0, T]$.

This contradicts the fact that $\mathcal{V}$ is radially unbounded, so the component x_1 of any trajectory $(x_1(t), x_2(t))$, $t \geq 0$ is bounded for all $t \geq 0$. Then the system (2.1) in closed loop with the feedback $u(x_1, x_2) = -\sum_{i=1}^{r} h_i(z)K_i x_2$ is globally asymptotically stable. □

Remark 2.4 The conclusion that we can obtain from Theorem 2.1 and Theorem 2.2 is that the ith interconnection term has a nonlinear bounded function on x_1.

Then the question which can be asked is: are these results valuable when the ith interconnection term has unbounded functions on x_1?

Example 2.1 We consider the following planar system:

$$\begin{cases} \dot{x}_{1_1} = -2x_{1_1} - \frac{1}{\pi}x_{1_1}\log(x_{1_1}^2 + x_{1_2}^2)\arctan(x_{1_1})x_{2_1} \\ \dot{x}_{1_2} = x_{1_1}\arctan(x_{1_1}) - x_{1_2} \\ \qquad + \frac{1}{2}x_{1_2}\log(x_{1_1}^2 + x_{1_2}^2)x_{2_2} \\ \dot{x}_{2_1} = -x_{2_2} - u \\ \dot{x}_{2_2} = -2x_{2_2} + x_{2_1}\arctan(x_{1_1}) - u. \end{cases} \tag{2.5}$$

Let

$$\begin{cases} \dot{x}_{1_1} = -2x_{1_1} \\ \dot{x}_{1_2} = x_{1_1}\arctan(x_{1_1}) - x_{1_2}. \end{cases} \tag{2.6}$$

$$\begin{cases} \dot{x}_{1_1} = -2x_{1_1} - \frac{1}{\pi}x_{1_1}\log(x_{1_1}^2 + x_{1_2}^2)\arctan(x_{1_1})x_{2_1} \\ \dot{x}_{1_2} = x_{1_1}\arctan(x_{1_1}) - x_{1_2} \\ \qquad + \frac{1}{2}x_{1_2}\log(x_{1_1}^2 + x_{1_2}^2)x_{2_2}. \end{cases} \tag{2.7}$$

$$\begin{cases} \dot{x}_{2_1} = -x_{2_2} - u \\ \dot{x}_{2_2} = -2x_{2_2} + x_{2_1}\arctan(x_{1_1}) - u. \end{cases} \tag{2.8}$$

Let $x_1 = [x_{1_1}\ x_{1_2}]^T$ and $x_2 = [x_{2_1}\ x_{2_2}]^T$.

It is clear to see that neither the ISS property nor the linear growth condition in x_1 are verified for (2.7).

Using the idea of the sector nonlinearity, one can represent exactly the system by the following two-rule model:

Rule 1: If z is F_{11} then $\begin{cases} \dot{x}_1 = A_1x_1 + \psi_1(x_1, x_2)x_2 \\ \dot{x}_2 = C_1x_2 + B_1u \end{cases}$

Rule 2: If z is F_{21} then $\begin{cases} \dot{x}_1 = A_2x_1 + \psi_2(x_1, x_2)x_2 \\ \dot{x}_2 = C_2x_2 + B_2u \end{cases}$

where

$$A_1 = \begin{bmatrix} -2 & 0 \\ -\frac{\pi}{2} & -1 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} -2 & 0 \\ \frac{\pi}{2} & -1 \end{bmatrix},$$

$$C_1 = \begin{bmatrix} 0 & -1 \\ -\frac{\pi}{2} & -2 \end{bmatrix}, \qquad C_2 = \begin{bmatrix} 0 & -1 \\ \frac{\pi}{2} & -2 \end{bmatrix},$$

$$B_1 = B_2 = \begin{bmatrix} -1 \\ -1 \end{bmatrix},$$

$$F_{11} = \frac{\frac{\pi}{2} - \arctan(x_{1_1})}{\pi} \quad \text{and}$$

$$F_{21} = \frac{\arctan(x_{1_1}) + \frac{\pi}{2}}{\pi}.$$

We define the membership functions as

$$h_1(x_1) = \frac{\frac{\pi}{2} - \arctan(x_{1_1})}{\pi} \quad \text{and}$$

$$h_2(x_1) = \frac{\arctan(x_{1_1}) + \frac{\pi}{2}}{\pi}.$$

Using an LMI optimization algorithm, we obtain

$$P = \begin{bmatrix} 13.2640 & 0.0000 \\ 0.0000 & 12.6093 \end{bmatrix}$$

and the following matrices:

$$Q_1 = 10^3 \times \begin{bmatrix} 4.6671 & 1.6563 \\ 1.6563 & 2.0048 \end{bmatrix},$$

$$Q_2 = 10^3 \times \begin{bmatrix} 4.6671 & -1.6563 \\ -1.6563 & 2.0048 \end{bmatrix}.$$

By taking $\mathcal{V}(x_1) = x_1^T P x_1$ as a Lyapunov function candidate for (2.6), it is easy to show that the system (2.6) is globally asymptotically stable.

Now, for each subsystem in (2.8), we assume that the following rules are given:

Rule 1: If z is F_{11} then $u(t) = -K_1x_2$

and

Rule 2: If z is F_{21} then $u(t) = -K_2x_2$,

where

$$K_1 = \begin{bmatrix} -2.3463\ 1.6400 \end{bmatrix} \quad \text{and}$$

$$K_2 = \begin{bmatrix} -2.9843\ 1.4334 \end{bmatrix},$$

and w have the following optimization results:

$$\tilde{P} = \begin{bmatrix} 0.0873 & -0.0026 \\ -0.0026 & 0.0173 \end{bmatrix},$$

$$\tilde{Q}_{11} = \begin{bmatrix} 0.1903 & 0.0027 \\ 0.0027 & 0.0070 \end{bmatrix},$$

$$\tilde{Q}_{22} = \begin{bmatrix} 0.2525 & -0.0111 \\ -0.0111 & 0.0101 \end{bmatrix},$$

$$\tilde{Q}_{12} = \begin{bmatrix} 0.2981 & -0.0057 \\ -0.0057 & 0.0120 \end{bmatrix}, \quad \text{and}$$

$$\tilde{Q}_{21} = \begin{bmatrix} 0.2981 & -0.0057 \\ -0.0057 & 0.0120 \end{bmatrix}.$$

It follows that the closed-loop system (2.8) is globally exponentially stable, uniformly on x_1, with the feedback law $u(x_1, x_2) = -\sum_{i=1}^{2} h_i(z) K_i x_2$.

Let

$$\psi_1(x_1, x_2) = \begin{bmatrix} \frac{1}{2} x_{1_1} \log(x_{1_1}^2 + x_{1_2}^2) & 0 \\ 0 & \frac{1}{2} x_{1_2} \log(x_{1_2}^2 + x_{1_1}^2) \end{bmatrix}$$

and

$$\psi_2(x_1, x_2) = \begin{bmatrix} -\frac{1}{2} x_{1_1} \log(x_{1_1}^2 + x_{1_2}^2) & 0 \\ 0 & \frac{1}{2} x_{1_2} \log(x_{1_1}^2 + x_{1_2}^2) \end{bmatrix}.$$

We can see that $\|\psi_1(x_1, x_2)\| = \|\psi_2(x_1, x_2)\| \leq \frac{1}{2}(|x_{1_1}| + |x_{1_2}|) |\log(x_{1_1}^2 + x_{1_2}^2)|$, having a nonlinear unbounded function on x_1. Thus, $(\mathcal{H}_3)$ and $(\mathcal{H}'_3)$ are not satisfied. Then, we cannot apply the above theorems. Nevertheless, it is easy to prove that this system in closed loop with $u(x_1, x_2) = -\sum_{i=1}^{2} h_i(z) K_i x_2$ can be globally asymptotically stable. In the following we will present a more general result to cover the situation discussed in this example.

First of all, we suppose that $(\mathcal{H}_1)$ holds. Then we consider the Lyapunov function candidate

$$\mathcal{V}(x_1) = x_1^T P x_1$$

for the system (2.3).

One can state the following assumption.

$(\mathcal{H}''_3)$ There exists $c > 0$, such that for all $i = 1, \ldots, r$,

$$\nabla \mathcal{V}(x_1) \big(h_i(z) \varphi_i(x_1, x_2) \big) \leq G_i(x_2) H_i \big(\mathcal{V}(x_1) \big), \quad \text{for all } \|x_1\| \geq c,$$

where $H_i : [0, +\infty[\to \mathbb{R}$, is a continuous function satisfying

$$\begin{cases} \int_a^{\infty} \frac{ds}{\sum_{i=1}^{r} H_i(s)} = \infty, & \text{for some } a > 0, \\ H_i(\mathcal{V}(x_1)) > 0, & \text{for all } \|x_1\| > c, \end{cases}$$

and G_i is a continuous function with $G_i(0) = 0$ and satisfies the local Lipschitz condition.

Then we have the following theorem.

Theorem 2.3 *Under assumptions $(\mathcal{H}_1)$, $(\mathcal{H}_2)$ and $(\mathcal{H}''_3)$, the system (2.1) in closed loop with the feedback $u(x_1, x_2) = -\sum_{i=1}^{r} h_i(z) K_i x_2$ is globally asymptotically stable.*

Proof Using the same idea as Theorem 2.1, we proof that

$$\int_0^{+\infty} \big\| G_i\big(x_2(t)\big) \big\| \, dt < \infty, \quad \text{for all } i = 1, \ldots, r.$$

Now, we suppose that x_1 is unbounded. The derivative of $\mathcal{V}$ along the trajectories of system (2.3) is given by

$$\begin{aligned} \dot{\mathcal{V}}(x_1) &= \nabla \mathcal{V}(x_1) \left(\sum_{i=1}^{r} h_i(z) A_i x_1 \right) \\ &\quad + \nabla \mathcal{V}(x_1) \left(\sum_{i=1}^{r} h_i(z) \varphi_i(x_1, x_2) \right) \\ &\leq \nabla \mathcal{V}(x_1) \left(\sum_{i=1}^{r} h_i(z) \varphi_i(x_1, x_2) \right) \\ &\leq \sum_{i=1}^{r} G_i(x_2) H_i(x_1), \quad \text{for all } \|x_1\| \geq c. \end{aligned}$$

Let $t_1 < \bar{t}_1 < t_2 < \bar{t}_2 < \cdots < t_k < \bar{t}_k < \cdots$ be the time values such that $\|x_1\| \geq c$, for all $t \in [t_k, \bar{t}_k]$ and for all $k = 1, 2, \ldots$.

Then,

$$\begin{aligned} &\sum_{k=1}^{+\infty} \int_{t_k}^{\bar{t}_k} \frac{d\mathcal{V}(x_1(t))}{\sum_{i=1}^{r} H_i(\mathcal{V}(x_1(t)))} \\ &\quad \leq \sum_{k=1}^{+\infty} \int_{t_k}^{\bar{t}_k} \sum_{i=1}^{r} G_i\big(x_2(t)\big) \, dt \\ &\quad \leq \int_0^{+\infty} \sum_{i=1}^{r} \big\| G_i\big(x_2(t)\big) \big\| \, dt. \end{aligned}$$

Since x_1 is unbounded, it is easy to see that for every sufficient large integer $N > 0$, there exists an interval $I_N = [t_{kN}, \bar{t}_{kN}]$ such that $\|x_1\| \geq c$ for all $t \in I_N$.

This implies that, for all $N > 0$,

$$\int_{t_{kN}}^{\bar{t}_{kN}} \frac{d\mathcal{V}(x_1(t))}{\sum_{i=1}^{r} H_i(\mathcal{V}(x_1(t)))} \leq \sum_{i=1}^{r} \int_0^{+\infty} \| G_i(x_2(t)) \| \, dt < \infty.$$

This contradiction means that x_1 must be bounded. Then all the orbits of (2.3) are bounded. So, the system (2.1) in closed loop with the feedback $u(x_1, x_2) = -\sum_{i=1}^{r} h_i(z) K_i x_2$ is globally asymptotically stable. □

Let return to the example.

Since

$$\|\nabla\mathcal{V}(x_1)\| \leq \lambda_{\max}(P)\|x_1\|,$$

we have for all $\|x_1\| \geq 1$, and $i = 1, 2$,

$$\nabla\mathcal{V}(x_1)\big(h_i(z)\psi_i(x_1, x_2)x_2\big) \leq \lambda_{\max}(P)\|x_1\|^2 \log\big(\|x_1\|^2\big)\|x_2\|.$$

Moreover,

$$\lambda_{\min}(P) = 12.6093 > 1;$$

then, by using the inequality

$$\lambda_{\min}(P)\|x_1\|^2 \leq V(x_1) = x_1^T P x_1,$$

it follows that

$$\nabla\mathcal{V}(x_1)\big(h_i(z)\psi_i(x_1, x_2)x_2\big) \leq \lambda_{\max}(P)\|x_1\|^2 \log\big(\lambda_{\min}(P)\|x_1\|^2\big)\|x_2\|,$$

where $\lambda_{\min}(P)$ denotes the smallest eigenvalue of the matrix P and $\lambda_{\max}(P)$ denotes the largest eigenvalue of the matrix P.

Hence,

$$\nabla\mathcal{V}(x_1)\big(h_i(z)\psi_i(x_1, x_2)x_2\big) \leq \frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}\mathcal{V}(x_1)\log\big(\mathcal{V}(x_1)\big)\|x_2\|.$$

One can get

$$G_1(x_2) = G_2(x_2) = \frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}\|x_2\|$$

and

$$H_1\big(\mathcal{V}(x_1)\big) = H_2\big(\mathcal{V}(x_1)\big) = \mathcal{V}(x_1)\log\big(\mathcal{V}(x_1)\big).$$

Therefore, it is easy to see that

$$\begin{cases} \int_2^{+\infty} \frac{ds}{\sum_{i=1}^{2} H_i(s)} = \infty \\ H_i(\mathcal{V}(x_1)) > 0, \quad \text{for all } i = 1, 2, \text{ and } \|x_1\| \geq 1. \end{cases}$$

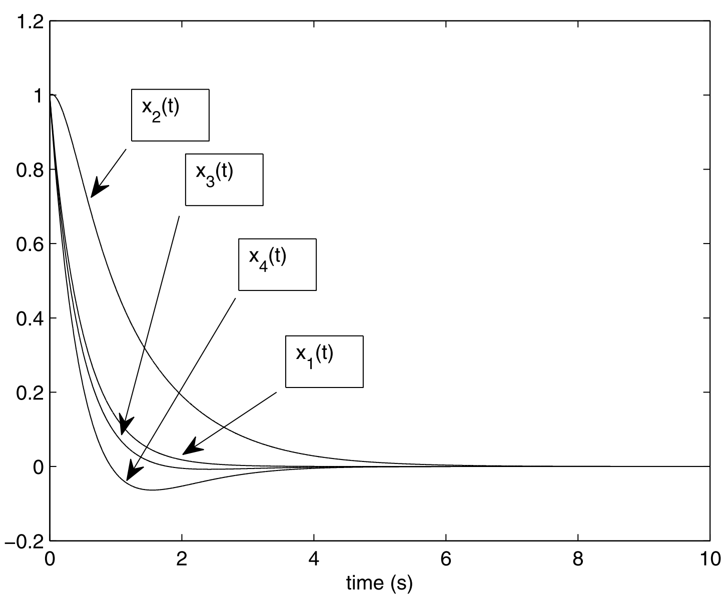

Fig. 1 Trajectories of the closed-loop system

Hence, the conditions of theorem (2.3) are satisfied. It follows that the closed-loop system (2.5) is globally asymptotically stable with $u(x_1, x_2) = -\sum_{i=1}^{2} h_i(z) K_i x_2$.

For simulation we select $x(0) = [1, 1, 1, 1]^T$ as initial condition. The result of the simulation is depicted in Fig. 1.

3 Conclusion

In this paper, we dealt with the analysis problem of nonlinear cascaded systems. In particular, we have interested in a class of TS fuzzy cascaded systems. We have developed sufficient conditions to ensure global asymptotic stability by using a fuzzy feedback law. These results are far from complete, much work is needed to pursue the conditions for the stability of TS fuzzy cascaded systems.

Acknowledgement The authors wish to thank the reviewers for their valuable and careful comments.

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