Nonlinear Dyn (2012) 67:2759–2777
DOI 10.1007/s11071-011-0187-z

ORIGINAL PAPER

Optimal design of multi-parametric nonlinear systems using a parametric continuation based Genetic Algorithm approach

Bipin Balaram · M.D. Narayanan · P.K. Rajendrakumar

Received: 9 July 2010 / Accepted: 28 July 2011 / Published online: 6 September 2011

Abstract In this paper, a procedure for the optimal design of multi-parametric nonlinear systems is presented which makes use of a parametric continuation strategy based on simple shooting method. Shooting method is used to determine the periodic solutions of the nonlinear system and multi-parametric continuation is then employed to trace the change in the system dynamics as the design parameters are varied. The information on the variation of system dynamics with the value of the parameter vector is then used to find out the exact parameter values for which the system attains the required response. This involves a multi-parametric optimisation procedure which is accomplished by the coupling of parameter continuation with different search algorithms. Genetic Algorithm as well as Gradient Search methods are coupled with parametric continuation to develop an optimisation scheme. Furthermore, in the coupling of continuation and Genetic Algorithm, a "norm-minimising" strategy is developed and made use of minimising the use of continuation. The optimisation procedure developed is applied to the Duffing oscillator for the minimisation of the system acceleration with nonlinear stiffness and damping coefficient as the parameters and the results are reported. It is also briefly indicated how the proposed method can be successfully used to tune nonlinear vibration absorbers.

Keywords Shooting method · Parametric continuation · Genetic Algorithms · Nonlinear systems

B. Balaram (✉) · M.D. Narayanan · P.K. Rajendrakumar
Department of Mechanical Engineering, NIT, Calicut, India
e-mail: bipinbalaram@nitc.ac.in

1 Introduction

Mechanical systems can exhibit nonlinear characteristics due to geometry, material properties or boundary conditions. The traditional technique for the dynamic analysis of such systems has been a direct numerical integration. This involves considerable computational effort due to the need to go past transient solutions in order to find the steady-state response. Furthermore, in the design of mechanical systems which exhibit nonlinear characteristics, there is a need to understand the change in the dynamic behaviour of the system as a particular design parameter is changed. This is done with the specific aim of obtaining the parameter value for which some system performance index is optimised. To do this using numerical integration is a cumbersome exercise as explicit solutions have to be generated corresponding to each parameter value which requires a huge initial map to be swept to obtain accurate results. As the size of the problem increases, the computational cost increases exponentially. Parametric continuation strategy offers an alternate and more robust approach to the aforementioned problem as it

aims at obtaining the solutions at a certain parameter value with the help of the solutions generated at its neighbouring parameter value. The broad strategy here is to quantify the change in solution characteristics with the change in the value of the parameter and to use this information to generate the solution for a range of parameter values.

The continuation technique is generally carried out in two distinct phases: the first phase, known as the shooting technique, tries to find the missing initial conditions of the problem corresponding to the periodic solution and the second phase investigates the change in this periodic solution and the fixed points when a system parameter is varied infinitesimally. Shooting technique will involve the application of numerical integration technique to convert the original nonlinear differential equations to nonlinear algebraic equations. Newton–Raphson method is then used to solve these equations to obtain the complete set of initial conditions corresponding to the periodic solution. Throughout the shooting method the system parameter is kept constant so that the initial conditions obtained are valid for a specific value of the system parameter. The second phase studies the change in the initial conditions and system dynamics as this value is changed. Here the system parameter acquires the status of a variable. After obtaining the system dynamics for different parameter regimes, identification of optimal parameter value is carried out by which the required optimal system response is obtained.

In this paper, an optimisation strategy based on the use of specific search algorithms coupled with parametric continuation has been proposed for the design optimisation of multi-parametric nonlinear systems. To deal with case of multi-parametric systems, continuation will have to be carried out, generally, on an $(m+1)$-dimensional parameter space (if m numbers of system parameters are considered). To this end, a continuation method which can deal with m parameters has been developed in this paper; using this continuation can be carried out along the line connecting any two points on an $(m+1)$-dimensional parameter space. The optimisation procedure involves the search for that point on this parameter space which furnishes the optimal system response and is carried out here with the use of two different search methodologies: Genetic Algorithms (GA) and Gradient Search methods. During the search operation, the search algorithm typically demands the system response at a certain point on the m-dimensional parameter space which is provided by the use of continuation. A shortest distance approach is developed and employed here to keep the steps needed in continuation to the possible minimum. The strategy developed is used to optimise the values of nonlinear stiffness and damping in the Duffing oscillator and the correctness of the results are verified. To ascertain the computational efficiency of the proposed method, the optimisation is carried out without the help of continuation strategy, using GA coupled with shooting method alone and the results and computational time are compared. To check the effect of using a shortest distance approach in continuation the results of the optimisation process using GA and continuation alone are analysed and comparisons are carried out.

2 Parametric continuation strategy

The idea of solution continuation has long been known and exploited in various branches of mathematics and mechanics. It is worth mentioning that this very idea actually underlies the well-known perturbation method whose first applications go back to the pioneering works of Henri Poincaré. In the mathematical theory of nonlinear differential equations, solution continuation has repeatedly been used to prove the existence of solutions of nonlinear equations [1]. The first use of continuation idea for computational purposes appears to be due to Lahaye [2] who proposed a method for solving transcendental equations by introducing a parameter p into the equation and constructing a solution for each p_i by the Newton–Raphson method using the solution of the previous value p_{i-1} as a starting approximation. Another formulation of the continuation method was given by Davidenko [3]. He was the first to realise the process of solution continuation as a process of moving and applied adequate mathematical apparatus of differential equation to it. Morozov successfully used the continuation strategy in the theory of finite deflection of plates [4].

In the later half of the 1990s investigations related to the application of path following to nonlinear dynamical systems began to appear. Kubicek and Marek [5] developed a comprehensive numerical technique based on the shooting method to compute bifurcation points. Solution continuation was done using the augmented Jacobian matrix and the Jacobian matrix was

then solved using Gauss elimination. They also presented the application of the developed technique to partial differential equations. Seydel [6] presented algorithms for the calculation of nonlinear dynamic system bifurcation behaviour. It also discussed numerous examples to illustrate the use of parametric continuation in dynamical systems. Continuation method was used for the solution of numerous problems in nonlinear mechanics, especially in dealing with nonlinear deformation problems, by Grigolyuk and Shalashilin [7]. Closely related to continuation technique is the asymptotic numerical method which aims at the computation of the solution path $U(\lambda)$ for a continuous nonlinear problem where U is the unknown and λ is a scalar parameter [8]. U is normally approximated by vector fields U_n which are solutions of a recurrent sequence of linear problems. These linear problems are solved by a classical discretisation technique, generally by finite element method. Cochelin [9] proposed an efficient continuation method within this framework without any iterative correction step. Noor and Peters [10] used this technique to determine equilibrium paths of nonlinear elastic structures.

Application of this technique to mechanical systems has been reported in recent years. Padmanabhan and Singh [11] have used the technique developed by Kubicek and Marek for the solving the dynamics of Impact Oscillator. The continuation parameters used were both frequency and the input force amplitude. The continuation strategy was based on swapping of the last column in the Jacobian with the right-hand side when a singularity is encountered due to the presence of bifurcations. In a subsequent paper Padmanabhan and Singh [12] show the capability of the algorithm to study systems with multi-harmonic excitations. Differences in the system response due to single harmonic and multi-harmonic excitations were discussed in detail. In both these papers, the nonlinearity was due to clearances in the system leading to impacts. Noah and Sundararajan [13] have used the shooting-based arc-length continuation methods to study rotor bearing systems. Instability issues and bifurcations in the rotor orbits are reported in detail in this paper. More recently the work by Shalashilin and Kuznetsov [14] discussed issues pertaining to the best continuation parameter and introduce the concept of optimal parameterisation.

2.1 Shooting method for periodic solution

The general governing equation for a periodically forced nonlinear system with N degrees of freedom and with a fundamental excitation frequency Ω can be written in the state space form as

$$\dot{\mathbf{X}} = \mathbf{F}(\mathbf{X}; \Omega\tau) \tag{1}$$

where the state vector is defined as

$$\mathbf{X} = [x_1, x_2, \ldots, x_{2N}]^T,$$

$$x_{i+1} = \dot{x}_i, \quad i = 1, 3, 5, \ldots, 2N-1.$$

Here x_i's and x_{i+1}'s ($i = 1, 2, \ldots, 2N-1$) represent the displacements and velocities, respectively, of the nonlinear system. $\mathbf{F}$ represents the force vector which includes the linear, nonlinear and periodic excitation components. In order to solve (1) it is proposed to use the shooting method which transforms the initial value problem into a two-point boundary value problem. The periodicity conditions can be written as

$$x_i(0) = x_i(\tau_0), \quad \tau_0 = 2\pi/\Omega, \; i = 1, 2, \ldots, 2N. \tag{2}$$

But the set of initial conditions for which the equations yield periodic solutions is unknown, so the first step would be to guess the initial conditions randomly so that $x_i(0) = \eta_i$. With the help of this guessed set of initial conditions, the set of equations described by (1) can be solved by using any numerical integration scheme. The values of $x_i(\tau_0)$ depend upon the accuracy of the initial conditions and implicitly on the value of a parameter α; thus,

$$x_i(\tau_0) = \phi_i(\eta; \alpha). \tag{3}$$

Here, α could typically represent any design variable like spring stiffness or damping coefficient, or external variable like amplitude or frequency of the periodic forcing function. The periodicity boundary condition requires that the following equation be satisfied:

$$G_i(\eta; \alpha) = \phi_i(\eta; \alpha) - \eta_i = 0. \tag{4}$$

For a given fixed value of the parameter α the above equation can be solved by the Newton–Raphson method with the Jacobian matrix

$$J_{ij} = \frac{\partial G_i}{\partial \eta_j} = \left\{ \frac{\partial \phi_i}{\partial \eta_j} - \delta_{ij} \right\} \tag{5}$$

where δ_{ij} denotes the Kronecker delta. For finding this Jacobian matrix, the unknown $\frac{\partial \phi_i}{\partial \eta_j}$ which represents the rate of change of the final conditions with respect to the initial conditions is required. Differentiating $x_i(\tau)$ with respect to η yields

$$\frac{\partial x_i(\tau)}{\partial \eta_j} = \mu_{ij}(\tau). \tag{6}$$

Differentiating with respect to τ gives

$$\dot{\mu}_{ij}(\tau) = \frac{\partial}{\partial \tau}\left\{\frac{\partial x_i(\tau)}{\partial \eta_j}\right\} = \frac{\partial}{\partial \eta_j}\left\{\frac{\partial x_i(\tau)}{\partial \tau}\right\} = \sum_{k=1}^{2N} \frac{\partial F_i}{\partial x_k}\mu_{kj}. \tag{7}$$

Here, μ_{ij} can be obtained by solving the above set of equations along with the initial conditions $\mu_{ij}(0) = \delta_{ij}$ using any numerical integration scheme. Now the required Jacobian can be evaluated as

$$J_{ij} = \frac{\partial G_i}{\partial \eta_j} = \left\{\frac{\partial \phi_i}{\partial \eta_j} - \delta_{ij}\right\} = \mu_{ij}(\tau_0) - \delta_{ij}. \tag{8}$$

This Jacobian is used to solve (4) by Newton–Raphson method which yields a new set of better initial conditions, η_i's. Now we start again and solve the parent equation (1) with the help of the new set of initial conditions and this iterative process continues till convergence or till a periodic solution surfaces. This results in the following equation being satisfied:

$$\eta^0 = \phi\left(\eta^0; \alpha^*\right). \tag{9}$$

The solution η^0 is then the fixed point of the iteration process described above for a fixed value of the parameter $\alpha = \alpha^*$. Now the primary interest is in quantifying the change in this initial condition vector when the parameter value is changed infinitesimally which is realised by the method of solution continuation.

2.2 Parametric continuation

Now, one would like to know how the fixed points or the periodic solutions change as the parameter of interest is varied from α^*, for which the solution has already been determined. For this purpose, the trajectory of the solution branch in the neighbourhood of α^* can be obtained from (4) by a Taylor series expansion in α which is truncated to the first term:

$$\sum_{k=1}^{2N}\left\{\frac{\partial \phi_i}{\partial \eta_k} - \delta_{ik}\right\}\frac{\partial \eta_k}{\partial \alpha} + \frac{\partial G_i}{\partial \alpha} = 0. \tag{10}$$

Note that the term in the parentheses represents the Jacobian matrix elements given by (7). So as long as this Jacobian is non-singular at α^*, we may compute the tangents $\frac{\partial \eta_k}{\partial \alpha}$ which quantify the rate of change of initial conditions with respect to the change in the variable α. In order to get the terms $\frac{\delta G_i}{\delta \alpha}$, we differentiate $x_i(\tau)$ with respect to the parameter:

$$\frac{\partial x_i(\tau)}{\partial \alpha} = \psi_i(\tau). \tag{11}$$

Now, differentiating with respect to τ, gives:

$$\dot{\psi}_i(\tau) = \frac{\partial}{\partial \tau}\left\{\frac{\partial x_i(\tau)}{\partial \alpha}\right\} = \frac{\partial}{\partial \alpha}\left\{\frac{\partial x_i(\tau)}{\partial \tau}\right\} = \sum_{k=1}^{2N}\frac{\partial F_i}{\partial x_k}\psi_k + \frac{\partial F_i}{\partial \alpha}. \tag{12}$$

With an appropriate numerical integration scheme, the set of equations given by (12) along with the initial conditions given by $\psi_i(0) = 0$ can be solved. This will yield the individual ψ_i's. Now the missing term in (10) is given by $\frac{\delta G_i}{\delta \alpha} = \psi_i(\tau_0)$. From the calculated fixed points at $\alpha = \alpha^*$ one may estimate the new set of initial conditions at $\alpha^1 = \alpha^* + \Delta\alpha$ as

$$\eta_i^{1*} = \eta_i^0 + \left.\frac{\partial \eta_i}{\partial \alpha}\right|_{\alpha^*}\Delta\alpha. \tag{13}$$

2.3 Continuation for multi-parametric systems

In almost all practical design problems, the system response will depend on more than one design parameter. The case of multi-parametric system can be dealt with using the above-mentioned continuation technique by following a "looping" approach; for example, if there are two design parameters, continuation is carried out for the first parameter alone keeping the second parameter fixed and this is done for a range of the second parameter values. This method, apart from being computationally cumbersome, is not suited for the objective of this work which is to combine continuation with different search strategies for obtaining the parameter values furnishing optimal response (for

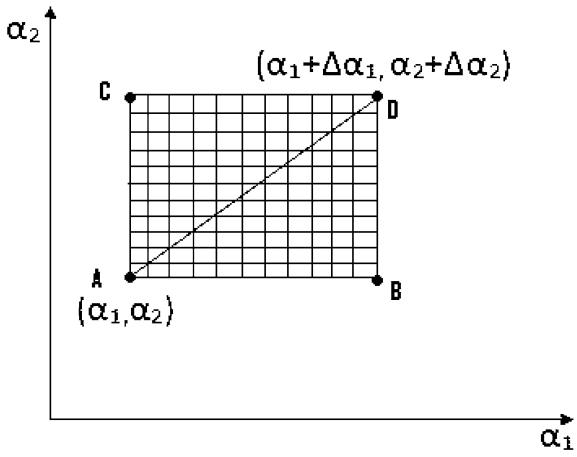


Fig. 1 Continuation for a system with two parameters

further discussions on these lines, see next section). So, a continuation strategy which deals with m number of parameters is developed here. This means that the parameter α given in Sect. 2.2 will become a vector $\alpha = (\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_m)$ and the change of system dynamics when this vector is changed to $\alpha + \Delta\alpha = (\alpha_1 + \Delta\alpha_1, \alpha_2 + \Delta\alpha_2, \ldots, \alpha_m + \Delta\alpha_m)$, where the $\Delta\alpha_i$'s that will in general be unequal, need to be quantified. For this, the continuation needs to be carried out on an $(m+1)$-dimensional space (m parameters and one system response measure).

The geometrical difference in the approaches mentioned above is clearly conveyed in Fig. 1. The case of a system whose dynamics depends upon two parameters α_1 and α_2 is shown here. Suppose that the use of shooting method at the point $A(\alpha_1, \alpha_2)$ has already furnished the periodic solution for this particular set of parameter values. Now one is interested in arriving at the periodic solutions at the point $D(\alpha_1 + \Delta\alpha_1, \alpha_2 + \Delta\alpha_2)$ using the already known solution at the point A and without explicitly solving the system at D. Using the "looping" approach mentioned above, this is attained by continuation along the parameter α_1 alone for different values of α_2. This amounts to keeping α_2 constant to start with and continuing along the parameter α_1 (i.e. along the line AB) in small increments. Then a small increment in α_2 is given and the continuation is carried out for α_1 again at this new fixed value of α_2 to generate the next line parallel to AB shown in the figure. So, it is clear from the figure that the continuation process using this strategy involves the generation of the whole grid $ABCD$. Evidently, the generation of the whole grid is a wasteful process as we are interested in the solution at the point D alone; the best option would be to move along the line AD. To do this, the rate of change of the initial conditions with respect to both the parameters will have to be quantified. If the system being studied is of 2 d.o.f., two initial conditions η_1 and η_2 are needed to solve the system and thus the following terms will have to be quantified: $\frac{\partial\eta_1}{\partial\alpha_1}$, $\frac{\partial\eta_1}{\partial\alpha_2}$, $\frac{\partial\eta_2}{\partial\alpha_1}$, $\frac{\partial\eta_2}{\partial\alpha_1}$. Moreover, it is clear from the above discussion that the computational complexity of the "looping" approach will be $O(n^2)$, whereas the complexity of the algorithm which carries out continuation along the line AD will only be $O(n)$.

Now, consider the case of continuation for a system with m parameters; as already mentioned, in this case the vector α will be $\alpha = (\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_m)$. This will not bring about any change in the shooting method procedures as here α is considered a constant and all the calculations are done for some fixed $\alpha = \alpha^*$. But in the continuation stage, α attains the role of a variable and therefore changes will be needed. The algebraic equation (4) becomes

$$G_i(\eta; \alpha) = \phi_i(\eta; \alpha) - \eta_i = 0$$

where $\alpha = (\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_m)$. The differential form now becomes

$$J\frac{\partial\eta}{\partial\alpha} + \frac{\partial G}{\partial\alpha} = 0 \tag{14}$$

where the Jacobian $J = \frac{\partial G}{\partial\eta}$ remains the same as in the single-parameter case. But as α is an m-dimensional vector now, the other terms in the differential form change. So,

$$J = \frac{\partial G}{\partial\eta} = \begin{pmatrix} \frac{\partial G_1}{\delta\eta_1} & \cdots & \frac{\partial G_1}{\delta\eta_{2N}} \\ \vdots & \ddots & \vdots \\ \frac{\partial G_{2N}}{\delta\eta_1} & \cdots & \frac{\partial G_{2N}}{\delta\eta_{2N}} \end{pmatrix},$$
$$\frac{\partial\eta}{\partial\alpha} = \begin{pmatrix} \frac{\partial\eta_1}{\partial\alpha_1} & \cdots & \frac{\partial\eta_1}{\partial\alpha_m} \\ \vdots & \ddots & \vdots \\ \frac{\partial\eta_{2N}}{\partial\alpha_1} & \cdots & \frac{\partial\eta_{2N}}{\partial\alpha_m} \end{pmatrix}. \tag{15}$$

Note that $\frac{\partial\eta}{\partial\alpha}$ now involves $(2N \times m)$ terms including the cross terms. This gives us the rate of change

of each of the initial conditions with respect to all m system parameters. Now

$$\frac{\partial G}{\partial \alpha} = \begin{pmatrix} \frac{\partial G_1}{\partial \alpha_1} & \cdots & \frac{\partial G_1}{\partial \alpha_m} \\ \vdots & \ddots & \vdots \\ \frac{\partial G_{2N}}{\partial \alpha_1} & \cdots & \frac{\partial G_{2N}}{\partial \alpha_m} \end{pmatrix}. \tag{16}$$

If we denote $\frac{\delta G}{\delta \alpha} = J_\alpha$ because of its resemblance with the actual Jacobian, we have the differential form as

$$J\frac{\partial \eta}{\partial \alpha} + J_\alpha = 0. \tag{17}$$

We need to solve the above equation for $\frac{\partial \eta}{\partial \alpha}$; thus, it constitutes a system of $(2N \times m)$ equations with $(2N \times m)$ variables $\frac{\partial \eta_i}{\partial \alpha_j}$ with $i = 1, 2, \ldots, 2N$ and $j = 1, 2, \ldots, m$.

Now the task is to get hold of the individual terms of J and J_α. The elements of the Jacobian matrix $J_{ij} = \frac{\partial G_i}{\partial \eta_j}$ undergo no change as they remain unaffected by the change in the dimension of α. So all the J_{ij}'s are computed using (8) itself. The individual entries of J_α are

$$J_{\alpha(ij)} = \frac{\partial G_i}{\partial \alpha_j} \tag{18}$$

where $i = 1, 2, \ldots, 2N$ and $j = 1, 2, \ldots, m$. Define new variables

$$\psi_{ij}(\tau) = \frac{\partial x_i(\tau)}{\partial \alpha_j}. \tag{19}$$

Differentiating once

$$\dot{\psi}_{ij}(\tau) = \frac{\partial}{\partial \alpha_j}\left(\frac{\partial x_i}{\partial \tau}\right) = \sum_{k=1}^{m} \frac{\partial \mathbf{F}_i}{\partial x_k}\psi_{kj} + \frac{\partial \mathbf{F}_i}{\partial \alpha_k} \tag{20}$$

and solving (20) with appropriate initial conditions gives us all the ψ_{ij}'s recalling that

$$J_{\alpha(ij)} = \frac{\partial G_i}{\partial \alpha_j} = \left(\frac{\partial \phi_i}{\partial \alpha_j} - \frac{\partial \eta_i}{\partial \alpha_j}\right) = \psi_{ij}(\tau_0) - \frac{\partial \eta_i}{\partial \alpha_j}.$$

Writing in the matrix form (replace $2N = n$ for notational convenience)

$$J = \begin{pmatrix} \mu_{11}(\tau_0) - 1 & \ldots & \mu_{1n}(\tau_0) \\ \vdots & \ddots & \vdots \\ \mu_{n1}(\tau_0) & \ldots & \mu_{nn}(\tau_0) - 1 \end{pmatrix} \tag{21}$$

and

$$\begin{aligned} J_\alpha &= \begin{pmatrix} \psi_{11}(\tau_0) - \frac{\partial \eta_1}{\delta \alpha_1} & \ldots & \psi_{1m}(\tau_0) - \frac{\partial \eta_1}{\delta \alpha_m} \\ \vdots & \ddots & \vdots \\ \psi_{n1}(\tau_0) - \frac{\partial \eta_n}{\delta \alpha_1} & \ldots & \psi_{nm}(\tau_0) - \frac{\partial \eta_n}{\delta \alpha_m} \end{pmatrix} \\ &= \begin{pmatrix} \psi_{11}(\tau_0) & \ldots & \psi_{1m}(\tau_0) \\ \vdots & \ddots & \vdots \\ \psi_{n1}(\tau_0) & \ldots & \psi_{nm}(\tau_0) \end{pmatrix} \\ &\quad - \begin{pmatrix} \frac{\partial \eta_1}{\delta \alpha_1} & \cdots & \frac{\partial \eta_1}{\delta \alpha_m} \\ \vdots & \ddots & \vdots \\ \frac{\partial \eta_n}{\delta \alpha_1} & \cdots & \frac{\partial \eta_n}{\delta \alpha_m} \end{pmatrix}. \end{aligned} \tag{22}$$

Equation (14) now becomes

$$\begin{aligned} &\begin{pmatrix} \mu_{11} - 1 & \ldots & \mu_{1n} \\ \vdots & \ddots & \vdots \\ \mu_{n1} & \ldots & \mu_{nn} - 1 \end{pmatrix}_{\tau_0} \left[\frac{\partial \eta}{\partial \alpha}\right] \\ &\quad + \begin{pmatrix} \psi_{11} & \ldots & \psi_{1m} \\ \vdots & \ddots & \vdots \\ \psi_{n1} & \ldots & \psi_{nm} \end{pmatrix}_{\tau_0} - \left[\frac{\partial \eta}{\partial \alpha}\right] = 0 \end{aligned} \tag{23}$$

where $[\frac{\partial \eta}{\partial \alpha}]$ is given by (15). Solving the above equation gives us all the $\frac{\delta \eta_i}{\delta \alpha_j}$'s which means that we have the rate of change of each initial condition with respect to each parameter. Then, the new initial conditions will be

$$\eta_i^{\text{new}} = \eta_i^{\text{old}} + \sum_{j=1}^{m} \frac{\partial \eta_i}{\partial \alpha_j}\Delta\alpha_j. \tag{24}$$

This will provide an estimate of the new initial conditions at the point $(\alpha + \Delta\alpha)$ in the parameter space.

3 Formulation of the optimisation problem

The main aim of this work is to make use of the parametric continuation method in the optimal design of multi-parametric nonlinear systems. This involves the determination of the optimal value of the m-dimensional parameter vector so that the nonlinear system gives the desired performance. In the single-parameter case, the optimal value of the parameter can easily be inferred from the continuation curve itself; but as the number of parameters increases, the

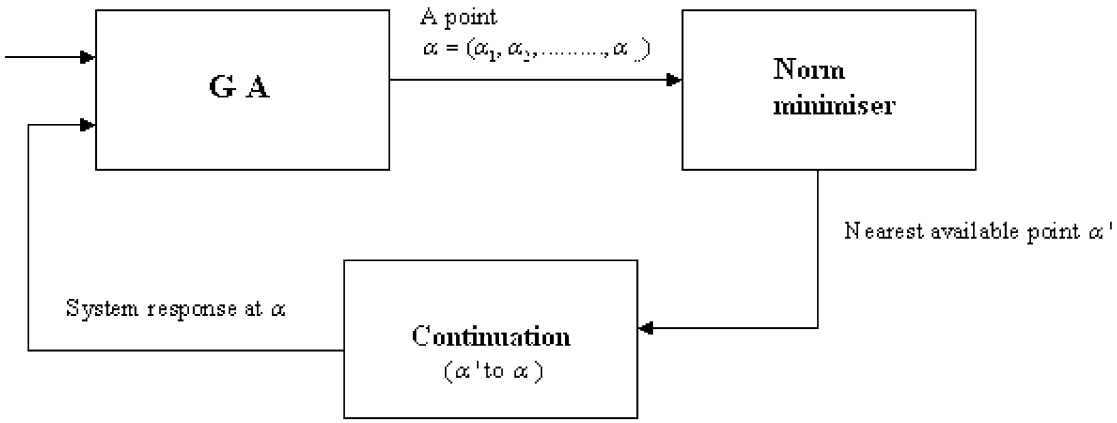


Fig. 2 The scheme for optimisation of multi-parametric systems using GA along with continuation

determination of optimal parameter value becomes a difficult task as the search is to be carried out in higher dimensional spaces. In this case, the use of a specialised search algorithm which can deal with the nonlinear nature of the problem cannot be avoided. This work proposes the use of two different search strategies coupled with continuation, namely, the Gradient Search method and GA. These two strategies were chosen because of their diametrically opposite character; Gradient search is normally a local search method whereas the GA is a global search strategy [16]. Thus, we may arrive at local optimum in the neighbourhood of a specific point in parameter space by using Gradient methods in a short time. But if we are interested in global optimal values, then we will have to resort to the use of GA and obviously it will be more time-consuming than local search. The optimisation problem can be formulated as follows:

Minimise the function $R = f(\alpha_1, \alpha_2, \ldots, \alpha_m)$

subject to the constraints $a_i \leq \alpha_i \leq b_i$.

3.1 Coupling of continuation and GA

The role of GA in the whole scheme is made clear in Fig. 2. For explaining how GA is combined with parameter continuation, let us consider GA as a toolbox. A typical iteration of the whole scheme sketched in Fig. 2 will be as follows. The GA, after carrying out a search based on its internal logic of natural selection, furnishes a point $\alpha = (\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_m)$ in the m-dimensional parameter space. The multi-parametric

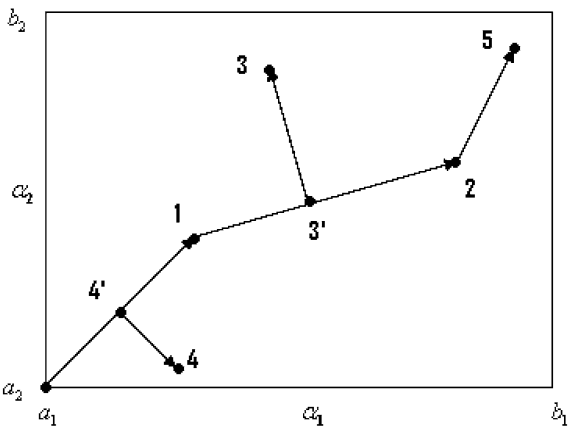


Fig. 3 Continuation in a 2-dimensional parameter space with the help of norm minimiser

continuation has to be used to furnish the system response measure at this particular parameter values to the GA toolbox. To speed up this process of continuation, a module termed "norm minimiser" is used in the scheme. In the first iteration of the whole scheme, continuation begins from the point (a_1, a_2) as it is the lower limit given by the constraints. With reference to Fig. 3, which shows the case of a system with two parameters α_1 and α_2, the point 1 is the first point furnished by the GA and the system response measure at 1 is found out by continuing from the point (a_1, a_2). This continuation is done in small steps and at each step the parameter vector (α_1, α_2) along with the response measure at that point is stored in the database. Based on the value at 1, the GA toolbox furnishes the next point 2. Once the point 2 is fed into the

"norm minimiser", it searches the database to locate the nearest point available to 2. This amounts to finding the Euclidean norm between the point 2 and all the points available in the database and selecting the minimum, hence the name "norm minimiser". In Fig. 3, this closest point comes out to be the point 1; so multi-parametric continuation is carried out from point 1 to 2 as shown. Let us say that the next point generated by the GA is 3; the norm minimiser locates the nearest available point in the database (3′ in the figure) and continuation is carried out from that point to 3. This process continues till convergence criterion of the GA is met; then we would have converged to the optimal parameter vector.

3.2 Coupling of continuation and Gradient search

The Gradient-based search strategies usually rely on the gradient of the function f to be optimised with respect to each of its parameters. The gradient of the function f which depends on m variables is defined as an m-component vector given by

$$\nabla f = \begin{Bmatrix} \frac{\partial f}{\partial \alpha_1} \\ \vdots \\ \frac{\partial f}{\partial \alpha_m} \end{Bmatrix}.$$

The gradient has the very important property that the negative of the gradient vector denotes the direction of steepest descent [17]. Taking the case of two variables, let us illustrate how we may find the gradient of a function f using continuation. For finding ∇f at a point P (refer to Fig. 4) in the parameter space, we need $\frac{\partial f}{\partial \alpha_1}$ and $\frac{\partial f}{\partial \alpha_2}$ at the point P which can be found out as follows:

$$\left.\frac{\partial f}{\partial \alpha_1}\right|_P = \frac{f(2) - f(1)}{\text{dist}(2,1)}$$
$$\left.\frac{\partial f}{\partial \alpha_2}\right|_P = \frac{f(3) - f(4)}{\text{dist}(3,4)}$$

where $f(1)$, $f(2)$, etc. are the values of the function at points 1 and 2 respectively and dist$(2, 1)$ denotes the Euclidean distance between points 2 and 1. In our case, the function f will be some dynamical property of the system which we need to minimise, say acceleration. Thus, if we need the gradient of f at the point P, we first find $f(1), \ldots, f(4)$ by carrying out continuation in the α_1–α_2 space from P. Once we have the function values at these points, the gradient can be found out very easily from the above relation. Then the search proceeds in the direction of the negative gradient and this continues till local optimum is obtained.

The main contributions of this work are the coupled use of search strategies and parametric continuation towards the optimisation of multi-parametric nonlinear systems and the introduction of the above-mentioned norm minimising procedure to minimise the number of iterations in continuation and thus to expedite the optimisation process. Two different analyses can be carried out to see the effect of these on the optimisation. First, optimal parameter values of the same system can be arrived at by the use of GA and shooting method alone. Here, the system response demanded by the GA at a point in the parameter space is provided by the use of shooting method at that point. The comparison of the results obtained by this method and the newly proposed algorithm will give a qualitative idea about the efficiency of the new method. Second, to see the advantages, if any, of using the proposed "norm minimiser" along with continuation, we may perform the optimisation employing GA and continuation alone. In this case, the system response at a point will be found out in each iteration using continuation from a constant point (this point can be taken as the zero vector). A comparison with this result will bring out the effect of using the norm minimiser.

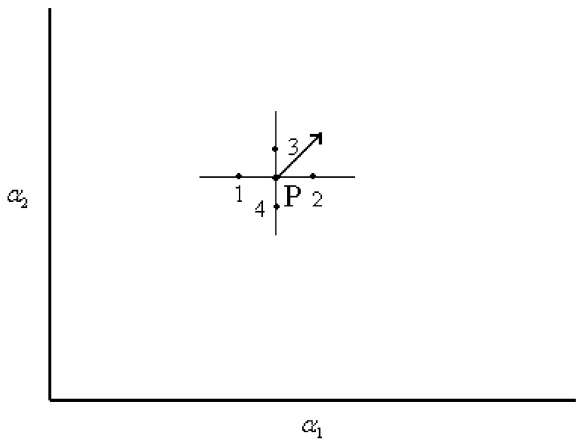


Fig. 4 Steepest descent search in a 2-parameter space

4 Application to Duffing oscillator

4.1 Single-parameter continuation

In the present work, a MATLAB based computational scheme has been developed to apply the above technique to a commonly used single-degree-of-freedom (SDOF) Duffing oscillator whose governing equation is given by

$$\ddot{x} + \xi\dot{x} + kx + \beta x^3 = f\cos(\Omega\tau). \tag{25}$$

Here, k is the linear stiffness, ξ is the viscous damping coefficient, β is the nonlinear stiffness, f is the external excitation amplitude and Ω is the excitation frequency.

4.1.1 Continuation in frequency

In order to ascertain the correctness of the algorithm as well as the computational apparatus, first, taking frequency as the parameter, the amplitude values obtained from the present continuation procedure are compared with the standard theoretical amplitude values available for the Duffing oscillator [18]. The governing equation can be written in the state space form as

$$\begin{Bmatrix}\dot{x}_1\\ \dot{x}_2\end{Bmatrix} = \begin{pmatrix}0 & 1\\ -k & -\xi\end{pmatrix}\begin{Bmatrix}x_1\\ x_2\end{Bmatrix} + \begin{Bmatrix}0\\ -\beta x_1^3\end{Bmatrix} + \begin{Bmatrix}0\\ f\cos(\Omega\tau)\end{Bmatrix}. \tag{26}$$

The variational equations defining μ_{ij} can be obtained as

$$\begin{pmatrix}\dot{\mu}_{11} & \dot{\mu}_{12}\\ \dot{\mu}_{21} & \dot{\mu}_{22}\end{pmatrix} = \begin{pmatrix}0 & 1\\ -k-3\beta x_1^2 & -\xi\end{pmatrix}\begin{pmatrix}\mu_{11} & \mu_{12}\\ \mu_{21} & \mu_{22}\end{pmatrix}. \tag{27}$$

Similarly, the governing equations for ψ_i's may be obtained, treating the frequency Ω as the parameter, as

$$\begin{Bmatrix}\dot{\psi}_1\\ \dot{\psi}_2\end{Bmatrix} = \begin{pmatrix}0 & 1\\ -k-3\beta x_1^2 & -\xi\end{pmatrix}\begin{Bmatrix}\psi_1\\ \psi_2\end{Bmatrix} + \begin{Bmatrix}0\\ -f\tau\sin(\Omega\tau)\end{Bmatrix}. \tag{28}$$

Now consider the case of the hardening spring for which the nonlinear stiffness is positive. Figure 5(a) shows the frequency continuation curve for the linear Duffing oscillator with parameter values $\xi = 0.05$, $f = 0.2$, $k = 1$, $\beta = 0.5$.

Figure 5(b) shows the comparison results for softening spring with negative nonlinear stiffness, the parameter values being $\xi = 0.05$, $f = 0.2$, $k = 1$, $\beta = -0.5$. Both the comparison plots show that there is a very close agreement between the theoretical values and those obtained by the continuation procedure. It is to be noted that in both the cases the continuation curves were obtained by forward sweep; i.e. the frequency value was positively incremented. In both the cases the other branch of the solution can easily be obtained by sweepback.

4.1.2 Continuation in nonlinear stiffness

Now, the case of single-parameter continuation in nonlinear Duffing oscillator is considered to ascertain the shape of the single-parameter curve and its trend with respect to different parameters. If the nonlinear stiffness is considered as the parameter, the governing equation for ψ_i's will be

$$\begin{Bmatrix}\dot{\psi}_1\\ \dot{\psi}_2\end{Bmatrix} = \begin{pmatrix}0 & 1\\ -k-3\beta x_1^2 & -\xi\end{pmatrix}\begin{Bmatrix}\psi_1\\ \psi_2\end{Bmatrix} + \begin{Bmatrix}0\\ -x_1^3\end{Bmatrix}. \tag{29}$$

Figure 6(a) shows the continuation diagram with nonlinear stiffness as the parameter for Duffing oscillator with parameter values $f = 0.3$, $k = 1$, $\xi = 0.15$ for three different values of forcing frequency which are greater than the natural frequency of the unforced linear system which is obviously $\sqrt{k/m} = 1$. The y axis shows the acceleration values (the root mean square (RMS) values are taken here). As is evident from the figure, the acceleration of the system registers a small increase with the introduction of nonlinearity into the system ($\beta = 0$ corresponds to the linear system) when the forcing frequency under consideration is well removed from the resonant region. As the frequency gets closer to the resonant region, it is seen that the increase in acceleration becomes more pronounced and is characterised by sudden jumps to higher values as the stiffness is varied. It is a well-known fact that the nonlinear effect becomes more relevant in the resonant regime and this explains the steep rise for the curve for $\Omega = 1.3$; yet it should be noted that the global trend of increase in acceleration is not altered.

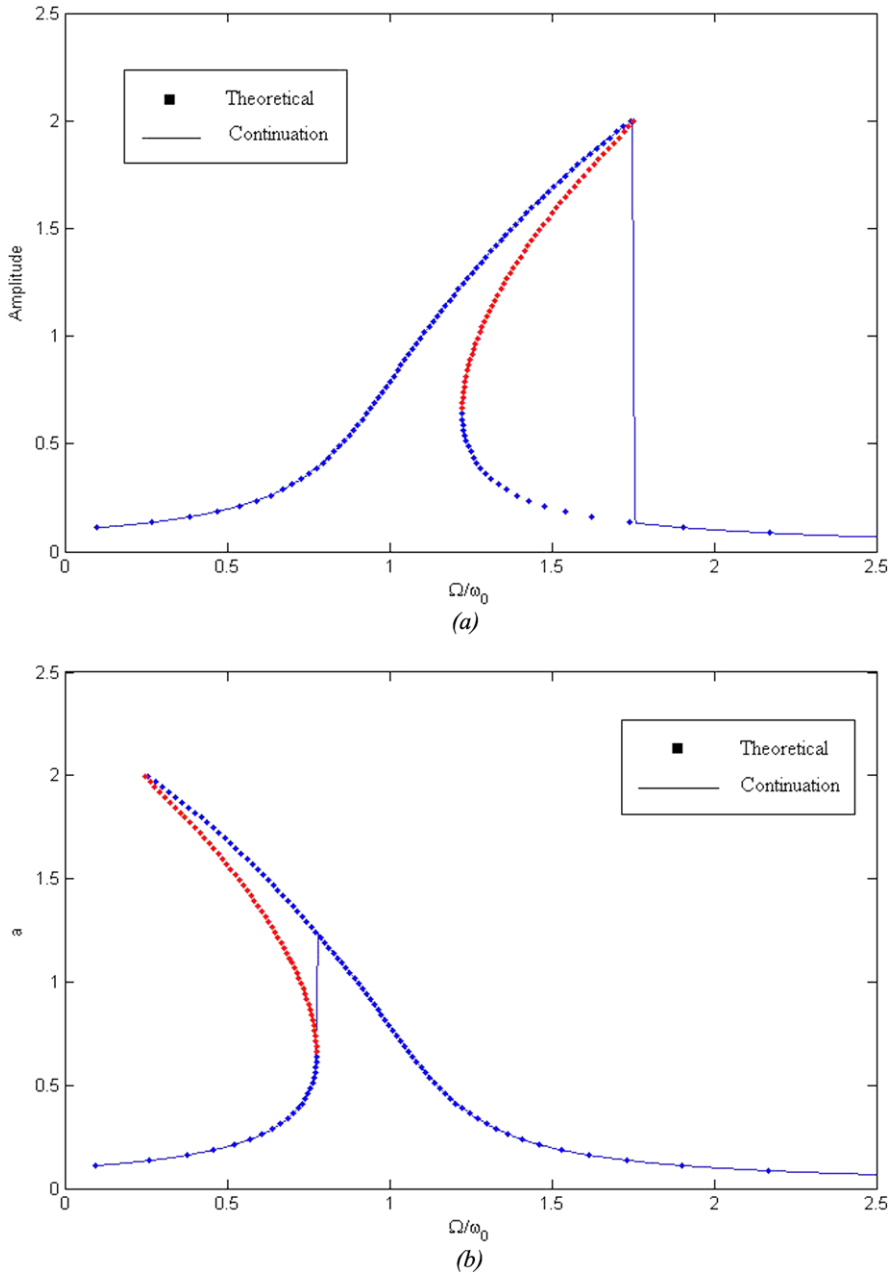
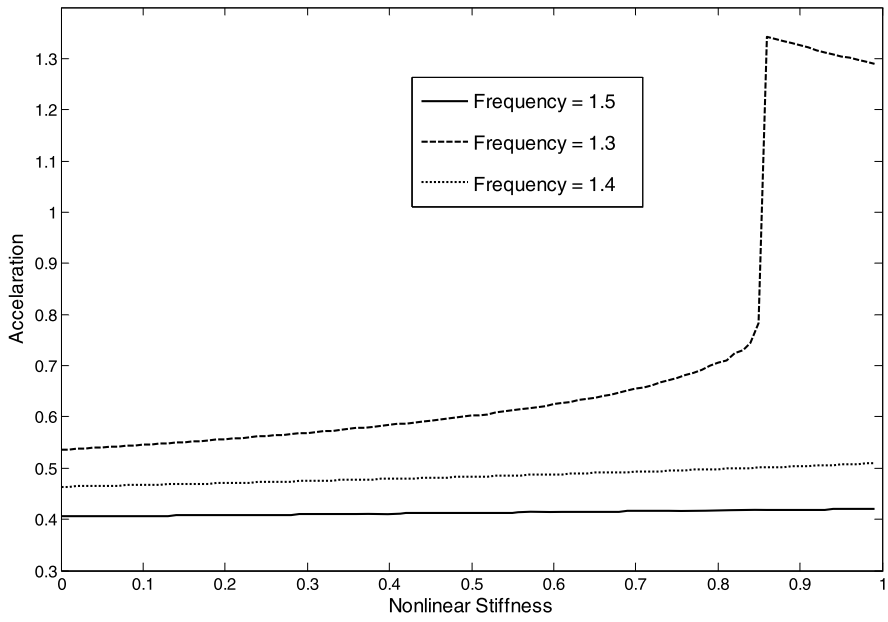
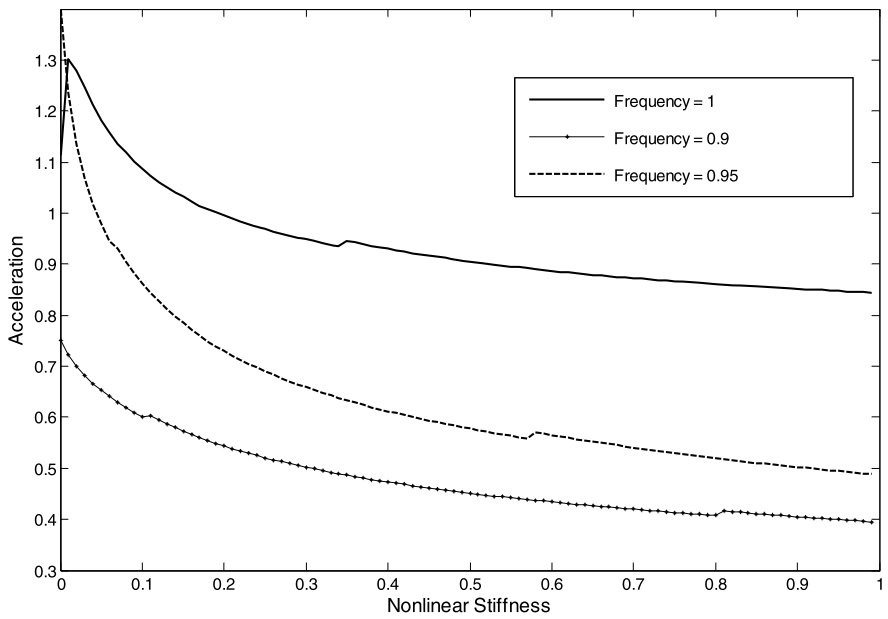


Fig. 5 Comparison of frequency response curves for the Duffing oscillator for parameter values $\xi = 0.05$, $f = 0.2$, $k = 1$, $\beta = 0.5$ and $\xi = 0.05$, $f = 0.2$, $k = 1$, $\beta = -0.5$

Figure 6(b) shows the case when the forcing frequency is equal to or less than the natural resonant frequency. For the case where $\Omega = 1$ we see that the acceleration begins to increase as small nonlinearity is introduced but then reverses the trend and decreases after that; thus it is seen that there is a reversal in the trend when one passes the resonant frequency. The figure shows that for frequencies less than the resonant one, the decreasing trend continues. From a system design perspective, the above analysis reveals that the presence of nonlinearity in the system can be exploited; we see that if the minimisation of acceleration is one of our design objectives, it is beneficial to introduce nonlinear stiffness into the system provided we operate at frequencies lower than the resonant frequency. If the operating range is higher than the resonant one, it is better to stick to the linear case as it corresponds to minimum acceleration. The curves



(a)



(b)

Fig. 6 (**a**) Continuation curve for nonlinear stiffness in Duffing oscillator for parameter values $f = 0.3$, $k = 1$, $\xi = 0.15$ and for three frequency values which are greater than the natural frequency of the unforced linear system ($\sqrt{k/m} = 1$). (**b**) Continuation curve for nonlinear stiffness in Duffing oscillator for parameter values $f = 0.3$, $k = 1$, $\xi = 0.15$ and for three frequency values which are less than the natural frequency of the unforced linear system

given above closely reflect the "jump" behaviour of the Duffing oscillator as given in works like [15]. Due to the jump phenomenon, the resonance gets shifted and thus the increase in amplitude occurs after passing the natural frequency; in the above plot we see that the sudden increase occurs at frequency 1.3 rather than the natural frequency 1.

4.1.3 Continuation in damping

If the damping coefficient is considered as the parameter to be varied, the governing equation for ψ_i's becomes

$$\begin{Bmatrix} \dot{\psi}_1 \\ \dot{\psi}_2 \end{Bmatrix} = \begin{pmatrix} 0 & 1 \\ -k - 3\beta x_1^2 & -\xi \end{pmatrix} \begin{Bmatrix} \psi_1 \\ \psi_2 \end{Bmatrix} + \begin{Bmatrix} 0 \\ -x_2 \end{Bmatrix}. \quad (30)$$

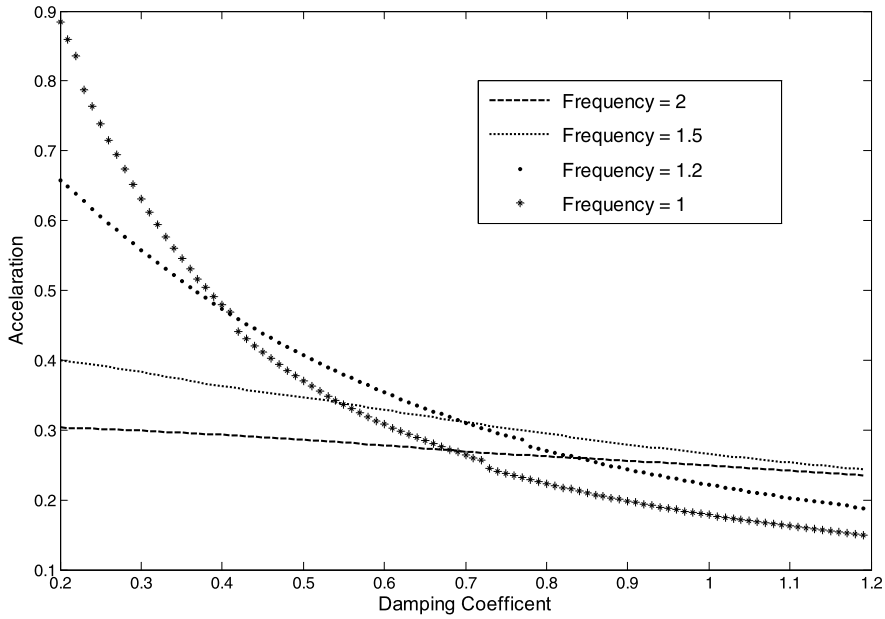


Fig. 7 Continuation curve for damping coefficient in Duffing oscillator for parameter values $\beta = 0.1$, $k = 1$, $f = 0.3$ and for four different frequency values

Figure 7 shows the variation of system acceleration with respect to the value of damping coefficient for the values $\beta = 0.1$, $k = 1$, $f = 0.3$. The plots are given for four different values of external frequency as is shown in the figure. It is easily inferred from the figure that the general trend is for the system acceleration to decrease with increasing value of the damping coefficient in the case of Duffing oscillator. This result is along expected lines keeping in mind the fact that the energy dissipation increases with increasing damping factor and this results in the decrease in the amplitude as well as acceleration. For higher values of excitation frequency, this decrease is minimal and almost linear but it can be seen that for lower values of forcing frequency the nonlinear aspects become more pronounced and the variation is no longer linear. In fact, the rate of decrease of acceleration increases drastically for lower frequencies; furthermore, the general trend remains the same throughout and does not seem to depend upon the natural frequency as in the case of nonlinear stiffness. But another point becomes clear when we examine the curves corresponding to the frequency values 1 and 1.2. When the damping value is below 0.4, the system forced with frequency 1.2 gives lesser acceleration but once this damping value is passed, it is desirable to have frequency as 1 as it gives lower acceleration. Thus it is clear that the minimisation of acceleration, in this case, depends upon the range of damping and the value of the excitation frequency.

4.2 Multi-parametric continuation

Now the case of multi-parametric continuation is considered with the parameter vector $\alpha = [\beta, \xi]$. The variational equations in the shooting method remain unchanged and the μ_{ij}'s are given by (16) itself. But the continuation equation has to be changed and (20) gives

$$\begin{pmatrix} \psi'_{11} & \psi'_{12} \\ \psi'_{21} & \psi'_{22} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -k - 3\beta x_1^2 & -\xi \end{pmatrix} \begin{pmatrix} \psi_{11} & \psi_{12} \\ \psi_{21} & \psi_{22} \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ -x_1^3 & -x_2 \end{pmatrix}. \tag{31}$$

Solving (31) gives us all the ψ_{ij}'s. Thus (23) can now be solved for all $\frac{\partial \eta_i}{\partial \alpha_j}$'s which is the quantification of the rate of change of the system response with respect to each parameter value. The new initial conditions for shooting are given by (24).

4.2.1 Application of GA

This multi-parametric continuation is now coupled with the use of GA as explained in Sect. 3. The minimisation of RMS acceleration is selected as the optimisation criterion and the value of the parameter vector which gives minimum acceleration is found out.

Table 1 The minimum RMS acceleration values and their corresponding parameter vector for Duffing oscillator at a forcing frequency of 1.2

	Range for nonlinear stiffness	Range for damping coefficient	Optimal nonlinear stiffness	Optimal damping	RMS acceleration
1	0.2–2	0.2–2	0.246	1.999	0.167
2	0.1–0.5	0.1–0.5	0.118	0.479	0.287
3	0–0.5	0.1–1	0.024	0.995	0.240

Table 2 Comparison between the proposed method and the others which do not use norm minimiser

Method used	Optimal parameter vector	RMS acceleration	Computational time (s)
GA + Shooting	(0.027, 0.958)	0.244	212
GA + Continuation from zero vector	(0.020, 0.990)	0.245	240
Proposed method	(0.024, 0.995)	0.241	113

The results quoted here are for a frequency value of 1.2 and are given in Table 1. Case 1 in the table refers to the following optimisation problem:

$$\text{Minimise } R = \text{Acceleration (nonlinear stiffness } (\alpha_1), \text{ damping } (\alpha_2))$$

$$\text{subject to the conditions } 0.2 \le \alpha_1 \le 2 \quad \text{and} \quad 0.2 \le \alpha_2 \le 2.$$

From Fig. 6(a) it is clear that at $f = 1.2$, the acceleration increases with increasing nonlinear stiffness, and from Fig. 7 it can be seen that system acceleration decreases with increase in damping. Therefore, the results obtained by the use of multi-parametric continuation coupled with GA and given in Table 1 closely match these trends.

As explained in Sect. 3, the optimisation of the same system is now carried out using GA coupled with shooting method and GA along with continuation from a fixed point for comparison purposes. The results are given in Table 2.

It is seen that when the optimisation process involves only GA and shooting procedure, the accuracy is severely compromised. When we use GA and continuation from zero vector, we get results very close to those of the proposed method, but at the expense of almost double the computational time. Computational time when using shooting method is lower than when using continuation from zero as it does not carry the continuation procedure with it which, as is evident from the table, severely restricts its potential of predicting the exact dynamics of the problem.

In Fig. 8 the best 10 individual function values in each GA iteration are plotted; it is clearly seen from the plot that with iteration the number of individual values giving higher acceleration values steadily decreases indicating convergence to the minimum acceleration. The ranges for stiffness and damping were set as 0.1–1 and 0.1–0.7.

Figure 9 shows the convergence of GA-based method in the parameter space. The parameter vector for the minimum acceleration value in each iteration is plotted here. The rounded point is the optimal parameter vector (0.1075, 0.6788).

4.2.2 Application of Gradient search

The same optimisation problem which was solved using GA (above) is now tackled with the help of Gradient search. But here the constraint ranges used are 0.95–0.1 for stiffness and 0.5–2.65 for damping. It is clearly seen from Fig. 10 that we get the same behaviour as in the case of GA; the main difference being that this strategy consumes much less time, obviously because it is a local search.

The 2D version of the Gradient search shown in Fig. 11 confirms to our analysis in the single-parameter case and with GA.

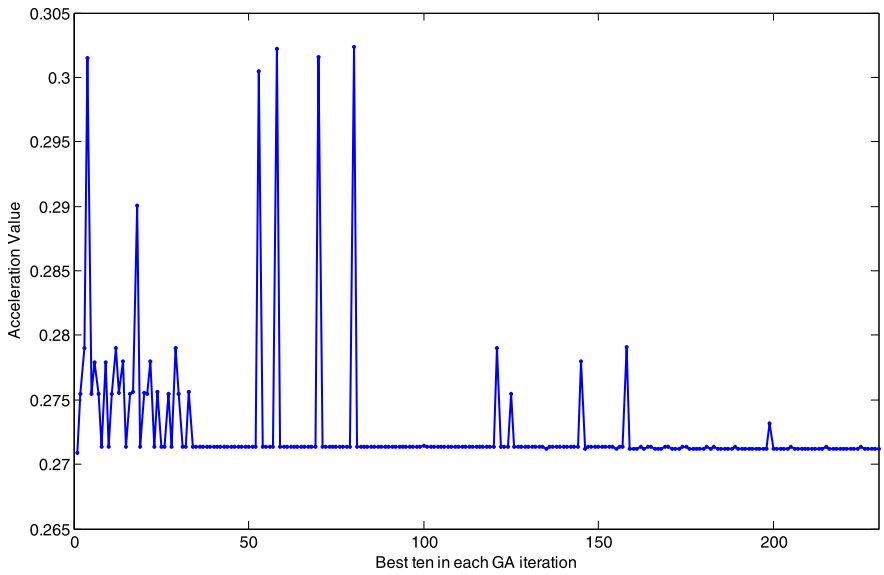


Fig. 8 Best 10 function values in each GA iteration. The minimum acceleration value obtained is 0.2712 and the optimal vector is (0.1075, 0.6788)

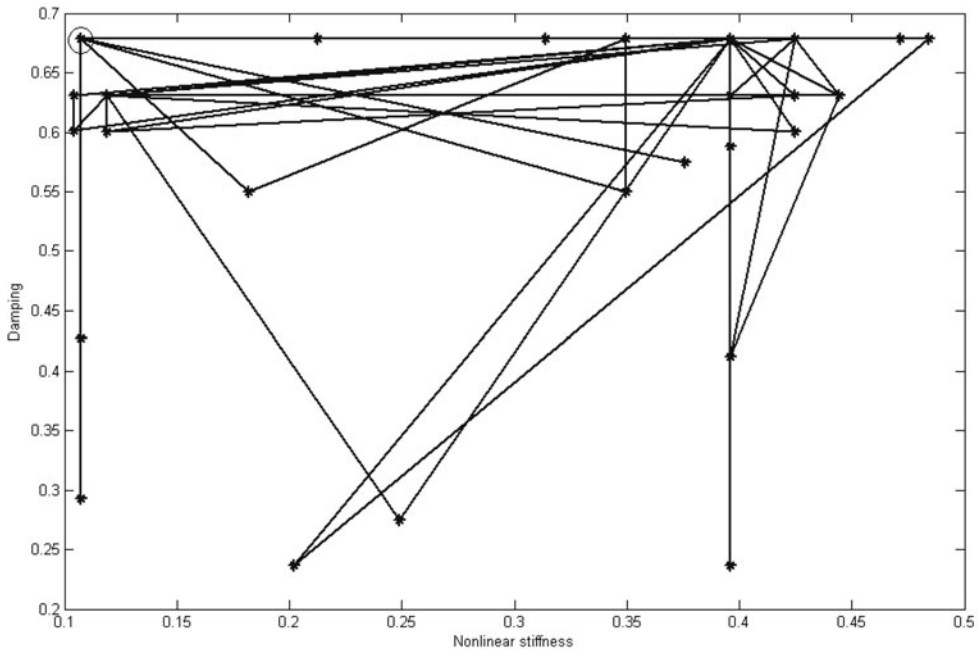


Fig. 9 Convergence in the parameter space

5 Application to vibration absorbers

The Vibration absorber or the tuned mass damper (TMD) is one popular device for passive vibration mitigation of mechanical structures. Theoretical tuning methodologies for linear TMDs have been developed which give the optimal values of the absorber mass and the absorber damping [19]. Realising that the linear TMD is effective only when it is precisely tuned to the frequency of a vibration mode, the development of nonlinear vibration absorbers effective in larger frequency range has been undertaken [20]. One of the main advantages of introducing nonlinearity into absorbers is that the absorption bandwidth can

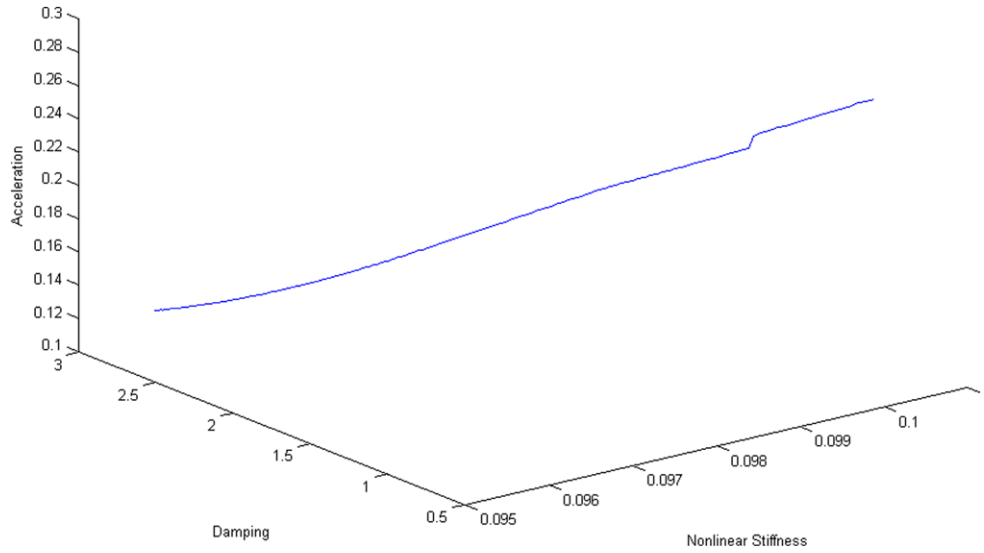


Fig. 10 Local search by Gradient method

be increased, thus facilitating absorption over a larger frequency range [21]. But nonlinearity makes the development of a closed-form tuning strategy impossible and thus the optimal parameter values of the absorber have to be found out using numerical search methods. This brief section aims at utilising the methodology developed here to optimally tune TMDs.

5.1 The linear case

A linear TMD consists of a primary undamped system connected to an absorbing system which is damped. The equations of motion are given by

$$\begin{aligned} &m_1\ddot{x}_1 + k_1x_1 + k_2(x_1 - x_2) + c_2(\dot{x}_1 - \dot{x}_2) \\ &\quad = F_0 \sin \omega t \\ &m_2\ddot{x}_2 + k_2(x_2 - x_1) + c_2(\dot{x}_2 - \dot{x}_1) = 0. \end{aligned} \tag{32}$$

Tuning of an absorber essentially consists of arriving at the optimal values of the absorber mass and absorber damper with the aim to mitigate the vibration at the resonant frequency of the primary system. The theoretical tuning condition requires that [19]

$$f = \frac{1}{1+\mu} \tag{33}$$

where $f = \omega_a/\omega_n$, $\mu = m_2/m_1$. ω_a is the absorber frequency and ω_n is the natural frequency of the primary system. From (33) it is very clear that if the aim is to mitigate the vibration at the resonant frequency of primary system, then the smaller the absorber mass the better. But this tuning condition tells us nothing about the value of absorber damping. For the linear case given by (32), different closed-form expressions for optimal damping have been arrived at for different optimisation criteria [19, 22]. But this approach does not yield results in the nonlinear case wherein it becomes very difficult to arrive at such closed-form expressions; therefore, numerical tuning becomes a necessity.

Let us formulate the tuning problem for the linear case as follows:

$$\text{Minimise } X_1 \text{ at } \frac{\omega}{\omega_n}$$

$$\text{subject to the constraints } \quad a_1 \le m_2 \le b_1 \quad \text{and} \quad a_2 \le c_2 \le b_2.$$

This formulation amounts to the minimisation of the primary structure vibration at its resonant frequency with bounds imposed upon the absorber mass and the absorber damping. As the optimisation parameters are m_2 and c_2, in order to apply the developed optimisation scheme, it is necessary to carry out multi-parametric continuation with the parameter vector being (m_2, c_2). The variational and continuation equations needed for this can be easily obtained from (7) and (12), respectively. We impose the condition $\omega_a/\omega_n = 1$, where ω_a is the absorber natural

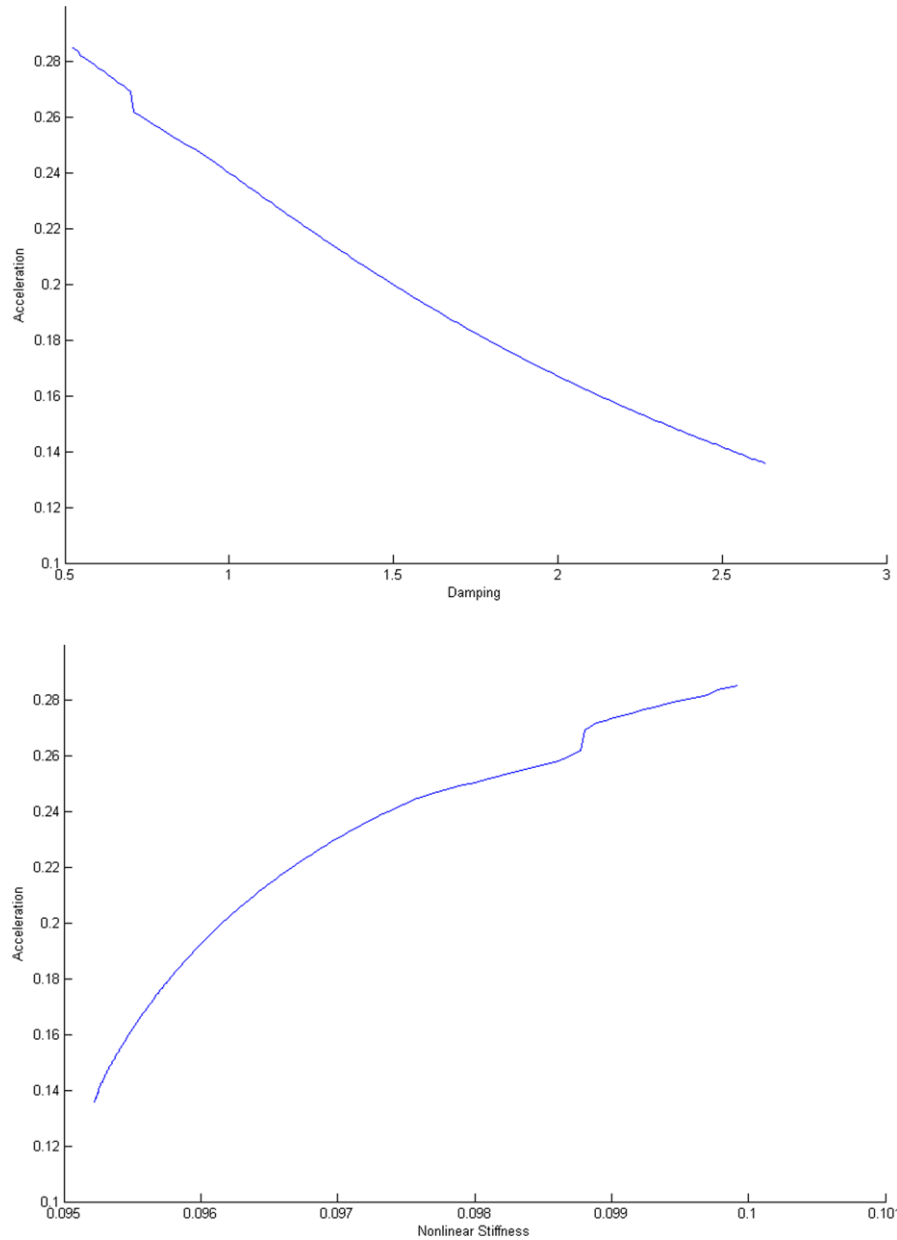


Fig. 11 2D plots for Gradient search

frequency and ω_n is the primary system natural frequency. Also set $m_1 = 1$ and $F_0 = 1$. The optimal parameter values obtained by the use of the above method of continuation in (m_2, c_2) coupled with GA are given in Table 3.

The correctness of the above values can easily be ascertained by plotting the amplitude–frequency diagram of the system for the parent mass for the corresponding values of the parameters. This can be done by using Ω as the parameter and in the continuation procedure. The amplitude–frequency plot for the three optimal cases obtained in Table 3 are given in Fig. 12. It is evident from the plots that for all the three cases, the obtained parameter values do indeed mitigate the parent structure vibration at resonance as the amplitude value is very close to zero. It is further seen that the optimal tends lie very close to the lower bound of the range. This is because of the fact that lower absorber mass facilitates higher mitigation. Regarding the damping, this occurs because we are

Table 3 Optimal values of non-dimensional absorber mass and damping for different ranges and the corresponding amplitude

	Range of $\mu = m_2/m_1$	Range of $\varsigma = c_2/c_c$	Optimal μ	Optimal ς	Value of X_1
1	(0.005, 0.05)	(0.002, 0.02)	0.005	0.0021	0.72
2	(0.009, 0.05)	(0.003, 0.02)	0.009	0.0033	0.732
3	(0.013, 0.05)	(0.005, 0.02)	0.014	0.005	0.715

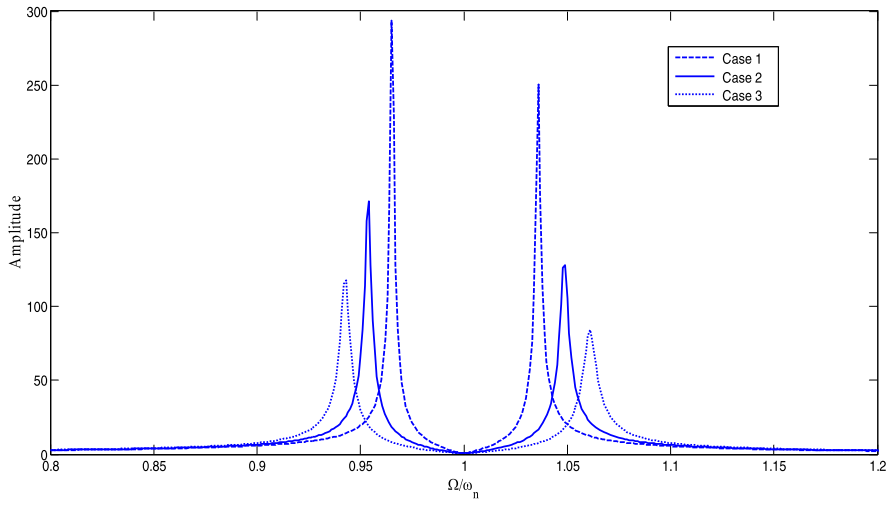


Fig. 12 Amplitude–frequency plots for the three cases mentioned in Table 3

trying to minimise the amplitude at resonance; it is known that for undamped systems this amplitude is zero, hence the trend given by the present analysis is justified.

5.2 Tuning nonlinear absorbers

The main motivation in using nonlinear absorbers is to increase the bandwidth of absorption. We shall consider the case of a nonlinear vibration absorber connected to an essentially nonlinear system. Damping is provided for both the masses to induce energy dissipation. The equations of motion are

$$\begin{aligned} &m_1\ddot{x}_1 + c_1\dot{x}_1 + c_2(\dot{x}_1 - \dot{x}_2) + \beta_1 x_1^3 \\ &\quad + \beta_2(x_1 - x_2)^3 = 0 \\ &m_2\ddot{x}_2 + c_2(\dot{x}_2 - \dot{x}_1) + \beta_2(x_2 - x_1)^3 = 0. \end{aligned} \tag{34}$$

The parameters of the primary systems are $m_1 = 1$ kg and $c_1 = 0.002$ Ns/m. For obvious reasons, a lightweight absorber is selected with $m_2 = 0.05$ kg and $c_2 = 0.002$ Ns/m. We shall provide the primary mass an initial non-zero velocity so that it acts as an impulse for the system; here it is given as $\dot{x}_1(0) = 4$ m/s.

Our aim is to tune the nonlinear absorber stiffness for given values of primary stiffness. As in the nonlinear case vibration mitigation occurs for a range of frequency values, the optimisation problem cannot be phrased in terms of the resonant frequency. So the function to be optimised is normally taken as the total energy absorbed by the absorber (this has to be maximised):

$$E_{\%}(t) = 100\frac{c_2\int_0^t (x_1(\tau) - x_2(\tau))^2\, d\tau}{\frac{1}{2}m_1\dot{x}_1(0)^2}. \tag{35}$$

Using the method developed above, this process can be carried out by taking β_2 as the continuation parameter. The variational and continuation equation can be derived from (7) and (12). The result of GA search coupled with this continuation is given in Table 4. The optimal values obtained by our method are compared with the values obtained by tuning methodology developed elsewhere [23]. Both the values are found to be in very close agreement.

Table 4 Optimal values of nonlinear absorber stiffness and comparison with theoretically tuned values

	Value of β_1	Range of β_2	Optimal β_2 by proposed method	β_2 by theoretical tuning
1	0.3	(0.001, 0.01)	0.0027	0.0025
2	1	(0.001, 0.01)	0.0074	0.0075

6 Conclusion

The present study has focused on the qualitative changes in the system dynamics with variation in the parameter values and investigated the same with the help of a simple shooting method based parametric continuation strategy. An effort has also been taken to use parametric continuation for the determination of parameter values for which the system gives the optimal response. Shooting method was used to generate the periodic solution of the system and the multi-parametric continuation scheme was employed to obtain the change in the system dynamics with the change in parameter values. This information was then used to find the parameter values which generate an optimal system response. For this, well-known search strategies like Genetic Algorithm and Gradient Search method were coupled with parametric continuation. A norm minimisation strategy was used to make good use of the continuation data which we possess and thus to minimise continuation steps needed in each iteration of the search algorithm. The developed procedure was used to minimise acceleration of a Duffing oscillator with nonlinear stiffness and damping coefficient as the parameters. The results indicate that the proposed strategy is robust and also that the norm minimisation introduced does help in improving the search speed. The study also points towards the fact that the introduction of nonlinearity into the system may be used to elicit the required system response; it is seen that under appropriate frequency ranges, nonlinearity has the effect of decreasing the system acceleration. It was further demonstrated how the proposed method can be used as a tool for tuning of nonlinear vibration absorbers.

Future work is oriented towards the application of the developed procedure in real time mechanical systems with an aim to tune typical design parameters to attain optimal system performance. In such a work, it will be beneficial to use the GA and the Gradient search complementarily; GA coupled with continuation can be used to zoom into a global optimal point and Gradient search can then be used to search locally in the neighbourhood to locate the optimal value. This has the potential to speed up the search considerably. Furthermore, in this work, the continuation strategy is based on the simple shooting method which makes the implicit assumption that the response frequency is the same as that of the forcing frequency. In case of systems with strong nonlinearities, this may not be true. To deal with such systems, a continuation scheme based on generalised shooting method which treats the response frequency also as an unknown will have to be developed.

References

1. Ficken, F.: The continuation method for nonlinear functional equations. Comm. Pure Appl. Math. **4**, 435–456 (1951)
2. Lahaye, M.E.: Solution of system of transcendental equations. Bull. Cl. Sci., Acad. R. Belg. **5**, 805–822 (1948)
3. Davidenko, D.F.: On the approximate solution of systems of nonlinear equations. Ukr. Mat. Zh. **5**, 196–206 (1953)
4. Morozov, N.F.: Nonlinear problems in theory of thin plates. Vestn. Leningr. Univ. **19**, 100–124 (1958)
5. Kubicek, M., Marek, M.: Computational Methods in Bifurcation Theory and Dissipative Structures. Springer, New York (1983)
6. Seydel, R.: From Equilibrium to Chaos: Practical Stability and Bifurcation Analysis. Elsevier, New York (1988)
7. Grigolyuk, E.I., Shalashilin, V.I.: Problems of Nonlinear Deformation. Kluwer Academic, Dordrecht (1991)
8. Vannucci, P., Cochelin, B., Damil, N., Potier-Ferry, M.: An asymptotic numerical method to compute bifurcation branches. Int. J. Numer. Methods Eng. **41**, 1365–1389 (1997)
9. Cochelin, B.: A path following technique via an asymptotic numerical method. Comput. Struct. **53**, 1181–1192 (1994)
10. Noor, A.K., Peters, J.M.: Tracing post limit paths with reduced basis technique. Computer Methods in Applied Mechanics **28**, 217–240 (1981)
11. Padmanabhan, C., Singh, R.: Analysis of periodically excited nonlinear systems by a parametric continuation method. J. Sound Vib. **184**, 35–58 (1995)
12. Padmanabhan, C., Singh, R.: Dynamic analysis of a piecewise nonlinear system subjected to dual harmonic excitation using parametric continuation. J. Sound Vib. **184**, 767–799 (1995)

13. Noah, S.T., Sundararajan, P.: Dynamics of forced nonlinear systems using shooting/arc length continuation method—application to rotor systems. ASME J. Vib. Acoust. **119**, 9–20 (1997)
14. Shalashilin, V.I., Kuznetsov, E.B.: Parametric Continuation and Optimal Parameterization in Applied Mathematics and Mechanics. Kluwer Academic, Dordrecht (2003)
15. Burton, T.D.: Introduction to Dynamic System Analysis. McGraw-Hill, New York (1994)
16. Goldberg, D.E.: Genetic Algorithms in Search, Optimisation and Machine Learning. Addison-Wesley, Reading (1989)
17. Rao, S.S.: Engineering Optimization: Theory and Practice. New Age, New Delhi (1996)
18. Nayfeh, A.H., Mook, D.T.: Nonlinear Oscillations. Wiley, New York (1979)
19. Den Hartog, J.P.: Mechanical Vibrations. Dover, New York (1985)
20. Pipes, L.: Analysis of a nonlinear dynamic vibration absorber. J. Appl. Mech. **20**, 515–518 (1953)
21. Hunt, J., Nissen, J.C.: The broadband dynamic vibration absorber. J. Sound Vib. **83**, 573–578 (1982)
22. Asami, T., Nishihara, O.: Closed form exact solution to HINF optimization of dynamic vibration absorbers. J. Vib. Acoust. **125**, 381–411 (2003)
23. Viguie, R., Kerschen, G.: Nonlinear vibration absorber coupled to a nonlinear primary system: a tuning methodology. J. Sound Vib. **326**, 780–793 (2009)