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Robust synchronization of arrays of uncertain nonlinear second-order dynamical systems

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Abstract A technique to synchronize arrays of dynamical systems is presented. The arrays are formed by uncertain nonlinear second-order systems, called nodes, where only the generalized position is available. The synchronization technique can be applied to many array topologies where the connections can be unidirectional or bidirectional with different weights; this produces a connection matrix that it is not necessarily symmetric. The design of the coupling signals is based on a robust discontinuous controller and on an exact deriver that estimates the velocity of each node. We present experimental results to illustrate the performance of the synchronization technique.

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1 Introduction

Synchronization is a dynamical behavior that two or more systems exhibit when a correlated motion between them is established. This phenomenon appears very often in nature, but it can also be induced in a forced way by introducing some coupling elements or input signals in a convenient way. This last case is often denoted as controlled synchronization [1].

Synchronization is necessary in many engineering systems where a collaborative operation is essential. In this case, the synchronization becomes a control objective [2], and the product quality depends heavily on the synchronization performance.

Controlled synchronization has important applications; for example, private communication systems [3], multi-robot systems, multi-finger robot hands, and teleoperation master–slave systems, among others. Some important papers on this topic are [4–6].

In the last years, many synchronization techniques have been proposed, dealing with the master–slave scheme and with arrays or networks of dynamical systems; some important papers are [6–15]. Some of these techniques have been developed to synchronize chaotic systems using classic control techniques, e.g. linear feedback [16–18], sliding mode control [20], and observed based synchronization [19]. These works consider ideal conditions like identical systems and availability of all state variables. Moreover, disturbances and uncertainties are not considered.

In [4], a synchronization technique for arrays of mechanical systems with partial measurement of the state vector is proposed. The technique requires an estimation of the velocity and acceleration, which adds significant complexity to the solution. The technique can be applied to arrays of identical systems, and an exact model is assumed to be known.

Synchronization of systems with parameter uncertainties is considered in [5]; here, an adaptive control is used to synchronize two robots with kinematic constraints. This technique assumes availability of the full state and no external disturbances. In [6], a synchronization technique for particular configurations of exactly known Lagrangian system networks is proposed. This technique, based on the contraction stability analysis, guarantees global, exponential convergence.

Nonlinear second-order systems are important in synchronization research because they can model many important phenomena, and may display diverse behaviors like equilibrium points, periodic orbits and, for non-autonomous systems, chaotic dynamics. In practice, many systems may be modeled by a second-order model; some examples are the artificial neurons and one-degree-of-freedom (1DOF) mechanical systems. nDOF mechanical systems can be seen as well as a set of coupled, second-order systems.

Two recent papers about synchronization of second-order systems are [21] and [22]. In [21], a consensus analysis for special topologies in networks of second-order systems is presented. This work considers identical nodes, without uncertainties or disturbances, and complete availability of the state vector is assumed. There is no reference system in the network, and therefore, when the consensus is presented, the solutions of each node must be a possible trajectory of an isolated node. The coupling signals are a linear combination of errors between positions and velocities of the connected nodes. The results are interesting but they are difficult to apply in practice.

Some important works on synchronization of arrays with higher order nodes are [13–15]: they present important results on the relation between synchronization and graph topology and on the stability of the synchronization in the network. However, they consider well established topologies that produce symmetric or asymmetric connection matrices; also, they consider identical nodes without uncertainties.

In [22] the synchronization problem of uncertain second-order systems in normal form is studied. In this work the arrays have a reference system, and the connections are unidirectional. The nodes may be affected by disturbances, and some model uncertainties are accepted. The coupling signals are synthesized based on neuronal networks that estimate the unknown terms in all nodes.

In this paper we present a synchronization technique for arrays of uncertain, second-order dynamical systems which may have unidirectional or bidirectional connections with different weights. The array may have or not a reference system; hence, the arrays considered in [21] and [22] are special cases. The nodes can be different, can be affected with external disturbances, and some parametric uncertainty is tolerated. Also, we assume that the generalized position is the only measured variable.

Based on the array topology, we establish a sufficient condition for synchronizability. If the array is synchronizable, the coupling signals are designed based on exact derivers with finite time convergence to obtain the generalized velocities, and a second-order sliding mode control technique is used to obtain the synchronization in the array. This controller provides a good robustness to the closed-loop system.

The organization of the paper is as follows. Section 2 includes some preliminary definitions and the statement of the synchronization objective. In Sect. 3, the synchronization technique is described. Here, a sufficient condition on synchronizability is established. Also, in this section, a methodology to design the coupling signals is presented. In Sect. 4 we present a strategy to implement the coupling signals, based on exact derivers with finite-time convergence, and using a second-order sliding mode controller. In Sect. 5 we include some experimental results to illustrate the performance of the synchronization technique. Finally, in Sect. 6, some conclusions are presented.

2 Synchronization objective

Consider k nonlinear systems, called nodes, described by the differential equation

$$\Sigma_i: \quad \frac{d^2 y_i}{dt^2} - f_i(y_i, \dot{y}_i) = u_i + v_i + \gamma_i(t, y_i, \dot{y}_i), \quad (1)$$

for $i = 1, \ldots, k$, where y_i is the output, $f_i(x_i)$ is a known, Lipschitz function, v_i is a coupling input signal, and u_i is a control input. The term $\gamma_i(t, y_i, \dot{y}_i)$ includes external disturbances and terms due to parameter uncertainties. It is considered smooth in t, y_i and $\dot{y}_i$, and satisfies

$$\|\gamma_i(t, y_i, \dot{y}_i)\| \le \rho_{0_i} + \rho_{1_i}\|(y_i, \dot{y}_i)\| \tag{2}$$

for some positive numbers ρ_{0_i}, ρ_{1_i}.

Condition 1 *We consider that u_i is a smooth, control signal that, when $v_i = 0$, produces a bounded behavior of system i, for any disturbance γ_i satisfying (2).*

Remark 1 As a consequence of Condition 1, when $v_i = 0$, the disturbance term and the function f_i satisfy

$$\|\gamma_i(t, y_i, \dot{y}_i)\| \le \rho_i, \tag{3}$$

and

$$\|f_i(y_i, \dot{y}_i)\| \le \delta_i. \tag{4}$$

A state representation of system (1) is

$$\Sigma_i\colon \quad \begin{cases} \dot{x}_{1i} = x_{2i}, \\ \dot{x}_{2i} = f_i(x_i) + \gamma_i(t, x_i) + u_i + v_i, \\ y_i = x_{1i}, \end{cases} \tag{5}$$

for $i = 1, \ldots, k$, where $x_i = (x_{1i}, x_{2i})$ is the state vector of node i.

These nodes form an array defined by a connection graph; an example of these graphs is shown in Fig. 1. The spheres represent the nodes Σ_i and the lines represent a coupling. These lines have a particular direction, represented by an arrow which defines the information flow. The meaning of a coupling line is the availability of information, i.e., an arrow from Σ_i to Σ_j indicates that the output y_i of the ith node is available for the jth node. It is important to note that, in this work, we consider arrays where there are no isolate nodes.

Based on the preceding definitions, the problem to be solved is to design the coupling signals v_i such that the objectives

$$\lim_{t\to\infty}\|y_i(t) - y_j(t)\| = 0, \quad \forall i, j \in \{1, \ldots, k\},\ i \ne j, \tag{6}$$

are satisfied for all $x_i(0) \in \Omega_i \subset \mathbb{R}^2$.

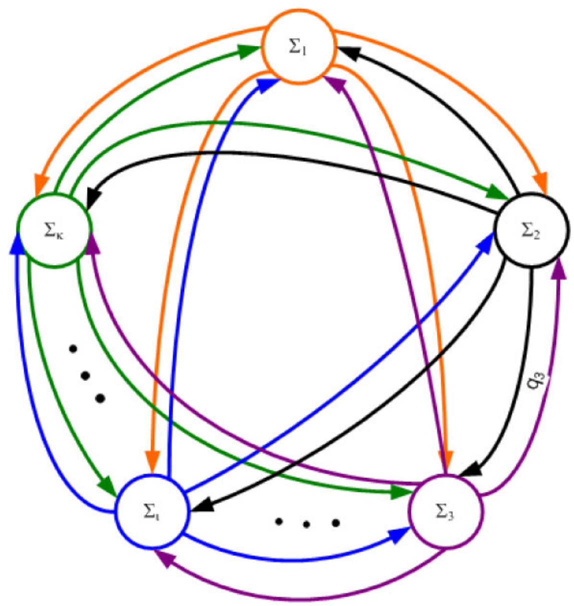


Fig. 1 An example of a connection graph

3 Design of the coupling signals

We establish the following definitions of synchronization, based on [2], to obtain a systematic design of the coupling signals and satisfy the objective given in (6).

Consider a set of k functionals $\epsilon_i : Y_1 \times Y_2 \times \cdots \times Y_k \to \Re$, $i = 1, \ldots, k$, where Y_i are the sets of all output functions.

Definition 1 We call the outputs $y_1(t), \ldots, y_k(t)$ of systems $\Sigma_1, \ldots, \Sigma_k$, as synchronized with respect to the functionals $\epsilon_1, \ldots, \epsilon_k$ if

$$\epsilon_i\big(y_i(t), \ldots, y_k(t)\big) = 0, \quad i = 1, \ldots, k. \tag{7}$$

Definition 2 We call the outputs $y_1(t), \ldots, y_k(t)$ of systems $\Sigma_1, \ldots, \Sigma_k$, with initial conditions $x_1(0), \ldots, x_k(0)$, as asymptotically synchronized with respect to the functionals $\epsilon_1, \ldots, \epsilon_k$ if

$$\lim_{t\to\infty}\epsilon_i\big(y_i(t), \ldots, y_k(t)\big) = 0, \quad i = 1, \ldots, k. \tag{8}$$

Now we propose the functionals

$$\epsilon_i(\cdot) = \sum_{j=1, j\ne i}^{k} \beta_{ij}(x_{1i} - x_{1j}), \quad i = 1, \ldots, k, \tag{9}$$

where β_{ij} is a constant that represents the coupling force from node j to node i. If in the connection graph,

Fig. 1, the node i does not receive information from node j, then $\beta_{ij} = 0$. Each functional ϵ_i is a linear combination of the errors among the output of node i and the output of each node connected to it.

Using (9) we can define two problems: the first one is to find the array topologies where the objective (6) can be satisfied when all functionals ϵ_i vanish. The second problem is to find the coupling signals that asymptotically stabilize the origin of the dynamics of these functionals. Hence we establish the following definition.

Definition 3 Let us consider the array built with systems (5) and with a connection configuration defined by the functionals (9). We call this array synchronizable if there exist coupling signals v_i such that the objective (6) can be satisfied.

Rewriting (9) in a matrix form, we have

$$\epsilon = \Theta x_1, \tag{10}$$

where

$$x_1 = [x_{11}, x_{12}, \ldots, x_{1k}]^T,$$
$$\epsilon = [\epsilon_1\ \epsilon_2 \cdots \epsilon_k]^T,$$

$$\Theta = \begin{bmatrix} \sum_{j=1,j\neq 1}^{k} \beta_{1j} & -\beta_{12} & \cdots & -\beta_{1k} \\ -\beta_{21} & \sum_{j=1,j\neq 2}^{k} \beta_{2j} & \cdots & -\beta_{2k} \\ \vdots & & \ddots & \vdots \\ -\beta_{k1} & -\beta_{k2} & \cdots & \sum_{j=1,j\neq k}^{k} \beta_{kj} \end{bmatrix}. \tag{11}$$

The matrix Θ is denoted as the connection matrix. Note that Θ is square and the sum of the elements in a row is zero. Also, note that this matrix is not necessarily symmetric.

We establish now a sufficient condition for array (5) to be synchronizable.

Theorem 1 *If the rank of the connection matrix Θ of (10) is $k-1$, then the array (5) is synchronizable.*

Proof First we show that the null space of Θ, (10), is the diagonal of $\mathbb{R}^k$, and then we calculate the coupling signals that force array (5) to be synchronized. Take the equation

$$0 = \Theta x_1. \tag{12}$$

If $\text{rank}\{\Theta\} = k-1$, a Gaussian elimination yields the matrix Ξ,

$$\Xi = \begin{bmatrix} 1 & -a_{12} & \cdots & -a_{1i} & \cdots & -a_{1\,k-1} & -a_{1k} \\ 0 & 1 & \cdots & -a_{2i} & \cdots & -a_{2\,k-1} & -a_{2k} \\ \vdots & & \ddots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & \cdots & -a_{i\,k-1} & -a_{ik} \\ \vdots & & \cdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & \cdots & 1 & -1 \\ 0 & 0 & \cdots & 0 & \cdots & 0 & 0 \end{bmatrix}, \tag{13}$$

where the sum of the elements in each row is zero. From (13) it is easy to see that the null space of Ξ, which defines the solution for (12), is given by $x_{11} = x_{12} = \cdots = x_{1k}$.

Remark 2 If $\text{rank}\{\Theta\} = k$, the unique solution of system (12) is $x_{11} = x_{12} = \cdots = x_{1k} = 0$, which reduces the possible solutions for the nodes (5) satisfying the objective (6).

With the matrix Ξ we define a new system of linear equations, equivalent to system (10):

$$\tilde{\epsilon} = \Xi x_1.$$

Now let us define $r = [\tilde{\epsilon}_1 \ \cdots \ \tilde{\epsilon}_{k-1}] \in \mathbb{R}^{k-1}$, so

$$r = \Phi x_1, \tag{14}$$

where

$$\Phi = \begin{bmatrix} 1 & -a_{12} & \cdots & -a_{1i} & \cdots & -a_{1\,k} \\ 0 & 1 & \cdots & -a_{2i} & \cdots & -a_{2\,k} \\ \vdots & & \ddots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 & \cdots & -a_{i\,k} \\ \vdots & & \cdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 1 & -1 \end{bmatrix}.$$

The dynamics of the variables r are given by

$$\frac{d}{dt}\begin{bmatrix} r \\ \dot{r} \end{bmatrix} = \begin{bmatrix} \dot{r} \\ \Phi \dot{x}_2 \end{bmatrix}, \tag{15}$$

where $\dot{x}_2$ is defined as

$$\begin{bmatrix} \dot{x}_{21} \\ \vdots \\ \dot{x}_{2k} \end{bmatrix} = \begin{bmatrix} f_1(x_1) + \gamma_1(t, x_1) + u_1 \\ \vdots \\ f_k(x_k) + \gamma_k(t, x_k) + u_k \end{bmatrix} + \begin{bmatrix} v_1 \\ \vdots \\ v_k \end{bmatrix},$$

or in compact form,

$$\dot{x}_2 = F(\cdot) + \Gamma(\cdot) + v,$$

where

$$F(\cdot) = \left[f_1(x_1) + u_1, \ldots, f_k(x_k) + u_k\right]^T,$$
$$\Gamma(\cdot) = \left[\gamma_1(t, x_1), \ldots, \gamma_k(t, x_k)\right]^T,$$
$$v = (v_1, \ldots, v_k)^T.$$

Then system (15) can be rewritten as

$$\frac{d}{dt}\begin{bmatrix} r \\ \dot{r} \end{bmatrix} = \begin{bmatrix} \dot{r} \\ \Phi[F(\cdot) + \Gamma(\cdot)] + \mathcal{U} \end{bmatrix}, \tag{16}$$

where $\mathcal{U} = \Phi v$ is the vector of control inputs for the system. The problem now is to propose a control $\mathcal{U}$ that stabilizes the origin of (16). For example, a possible control input can be

$$\mathcal{U} = -\Phi\left[F(\cdot) + \Gamma(\cdot)\right] - (K_p r + K_v \dot{r}), \tag{17}$$

where K_p and K_v are $(k-1) \times (k-1)$-positive definite matrices. □

Note that Φ is a $(k-1) \times k$-matrix; hence, normally we will have an infinite number of choices for v.

4 Implementation of the coupling signals

The control input (17) depends on the terms F, Γ, and $\dot{r}$, which are not exactly known in practice. In this section we present a technique to implement the control input (17) taking into account this situation. This technique is based on a discontinuous deriver with finite-time convergence, based on the deriver proposed in [23], and a discontinuous control based on the controller proposed in [24].

The implementation of the coupling signals has two stages. In the first stage all the nodes operate disconnected, that is, we set $v_i = 0$ for all i a time long enough such that the derivers attain their steady state. After this time, the coupling signals are activated.

4.1 A deriver with convergence in finite time

Let us consider the first equation of each node,

$$\dot{x}_{1i} = x_{2i}.$$

The deriver is given by

$$\begin{aligned} \dot{\hat{x}}_i &= z_i + c_{1i}|x_{1i} - \hat{x}_i|^{\frac{1}{2}} \operatorname{sign}(x_{1i} - \hat{x}_i), \\ \dot{z}_i &= u_i + v_i + c_{2i} \operatorname{sign}(x_{1i} - \hat{x}_i), \end{aligned} \tag{18}$$

where c_{1i} and c_{2i} are positive constants. The solutions of system (18) are defined in the Filippov's sense [25]. To analyze the deriver's performance we define the error $e_i = x_{1i} - \hat{x}_i$, whose dynamics are given by

$$\dot{e}_i = x_{2i} - z_i - c_{1i}|e_i|^{\frac{1}{2}} \operatorname{sign}(e_i),$$
$$\dot{z}_i = u_i + v_i + c_{2i} \operatorname{sign}(e_i).$$

A change of variables $\upsilon_{1i} = e_i$, $\upsilon_{2i} = x_{2i} - z_i$ leads to

$$\begin{aligned} \dot{\upsilon}_{1i} &= \upsilon_{2i} - c_{1i}|\upsilon_{1i}|^{\frac{1}{2}} \operatorname{sign}(\upsilon_{1i}), \\ \dot{\upsilon}_{2i} &= f_i(x_{1i}, x_{2i}) + \gamma_i(t, x_i) - c_{2i} \operatorname{sign}(\upsilon_1). \end{aligned} \tag{19}$$

From (3) and (4) we have

$$\left|f_i(x_{1i}, x_{2i}) + \gamma_i(t, x_i)\right| \le \rho_i + \delta_i.$$

Therefore, we can use Theorem 1, given in [26], to calculate the constants c_{1i} and c_{2i} satisfying

$$\begin{aligned} c_{1i} &> \sqrt{\frac{2}{c_{2i} + \delta_i}} \frac{(c_{2i} + (\rho_i + \delta_i))(1 + p)}{1 - p}, \\ c_{2i} &> \rho_i + \delta_i, \end{aligned} \tag{20}$$

for some constant $p \in (0, 1)$, to guarantee that the trajectories of system (19) converge to the origin in finite time. This means that $\hat{x}_i = \hat{x}_{1i}$ and $z_i = x_{2i}$ in finite time.

4.2 Design of the control signal

Once the deriver described in the previous section converges to the system state, the control input v is activated. Hence, in this stage we consider that the velocity in all nodes is available.

Let us consider the system (16). Because $\Gamma(\cdot)$ is not exactly known and $\dot{r}$ is estimated with the deriver, we propose that the control input $\mathcal{U}$, given by (17), have the form

$$\mathcal{U} = -\Phi F(\cdot) - \left[K_p r + K_v \dot{\hat{r}} + K_s \operatorname{sign}(r)\right],$$

where K_p, K_v, and K_s are diagonal definite positive matrices, $\dot{\hat{r}}$ is given by

$$\dot{\hat{r}} = \Phi z,$$
$$z = (z_1 \ \cdots \ z_k)^T,$$

and sign(r) is defined as

$$\text{sign}(r) = \big[\text{sign}(r_1)\ \dots\ \text{sign}(r_k)\big]^T.$$

The closed-loop system is given by

$$\frac{d}{dt}\begin{bmatrix} r \\ \dot{r} \end{bmatrix} = \begin{bmatrix} \dot{r} \\ \Phi\Gamma(\cdot) - K_p r - K_v \dot{\hat{r}} - K_s\,\text{sign}(r) \end{bmatrix}. \tag{21}$$

This system can be seen as a set of $k-1$ second-order subsystems with the form

$$\frac{d}{dt}\begin{bmatrix} r_i \\ \dot{r}_i \end{bmatrix} = \begin{bmatrix} \dot{r}_i \\ \mu_i(\cdot) - k_{pi} r_i - k_{vi} \dot{\hat{r}}_i - k_{si}\,\text{sign}(r_i) \end{bmatrix}, \tag{22}$$

where

$$\mu_i(\cdot) = \phi_{1i}\gamma_1(\cdot) + \cdots + \phi_{ki}\gamma_k(\cdot),$$

and

$$\big|\mu_i(\cdot)\big| = \big|\phi_{1i}\gamma_1(\cdot) + \cdots + \phi_{ki}\gamma_k(\cdot)\big| \le \sum_{j=1}^{k} \phi_{ij}\rho_j.$$

We can then use the results presented in [24], where it is shown that there exists a set of constants k_{pi}, k_{vi}, and k_{si} such that the origin will be an asymptotically stable equilibrium point with a basin of attraction large enough to be useful in practice. These conditions are

$$k_{pi} > 0,$$
$$k_{vi} > 0,$$
$$k_{si} > \alpha_i,$$

where

$$\alpha_j > \lambda_{\max}(P_i)\sqrt{\frac{\lambda_{\max}(P_i)}{\lambda_{\min}(P_i)}}\left(\frac{k_{pi}\sum_{j=1}^{k}\phi_{ij}\rho_j}{\theta}\right),$$

$0 < \theta < 1$, and P_i is a matrix defined by

$$P_i = \begin{bmatrix} 0 & 1 \\ -k_{pi} & -k_{vi} \end{bmatrix}.$$

Therefore, we can guarantee the convergence of r to zero and, in this way, the objective (6) will be satisfied.

5 Synchronization of four mechanical systems

In this section the experimental results of the synchronization of an array composed by four 1-DOF mechanical systems is presented. Systems Σ_1 and Σ_2 are mass–spring–damper systems like the one shown in Fig. 2; system Σ_3 is an axis of the x–y mechanical system shown in Fig. 3; system Σ_4 is a horizontal pendulum, which is the first link of the SCARA robot shown in Fig. 4. All the positions are measured, but the system parameters are all unknown. Input signals u_1 and u_2, which drive systems Σ_1 and Σ_2, respectively, are set to $1.5\sin(t)$ volts. Signals u_3 and u_4 are control inputs of systems Σ_3 and Σ_4. They are synthesized by feedback controllers to make the systems

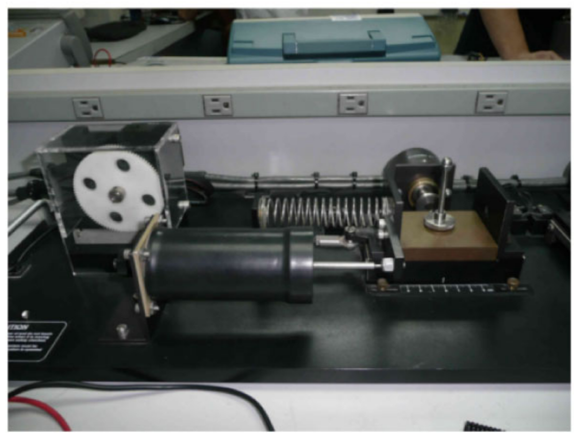


Fig. 2 Mass–spring–damper system that corresponds to systems Σ_1 and Σ_2

Fig. 3 One axis of the x–y mechanical system corresponds to system Σ_3

Fig. 4 SCARA robot. The first link corresponds to system Σ_4

track the reference signal $0.01\sin(t)$, in meters, for Σ_3 and $0.01\sin(t)$, in radians, for Σ_4.

Figure 5 shows the structure of the array. In this experiment, the outputs of systems Σ_1, Σ_2, and Σ_3 are measured in meters, the output of system Σ_4 is given in radians. For simplicity, these units are not indicated in the graphs. The relation between the rectilinear and the angular motions are 1 meter corresponding to 1 radian. Based on the connection graph, the functionals ϵ_i are given by

$$\begin{aligned}
\epsilon_1 &= \beta_{13}(x_{11}-x_{13})+\beta_{14}(x_{11}-x_{14}),\\
\epsilon_2 &= \beta_{21}(x_{12}-x_{11}),\\
\epsilon_3 &= \beta_{31}(x_{13}-x_{11})\\
&\quad+\beta_{32}(x_{13}-x_{12})+\beta_{34}(x_{13}-x_{14}),\\
\epsilon_4 &= \beta_{41}(x_{14}-x_{11})\\
&\quad+\beta_{42}(x_{14}-x_{12})\\
&\quad+\beta_{43}(x_{14}-x_{13}),
\end{aligned} \tag{23}$$

where $\beta_{13}=2$, $\beta_{14}=5$, $\beta_{21}=1$, $\beta_{31}=3$, $\beta_{32}=1.5$, $\beta_{34}=0.5$, $\beta_{41}=3$, $\beta_{42}=1$, and $\beta_{43}=2$. The connection matrix Θ is then

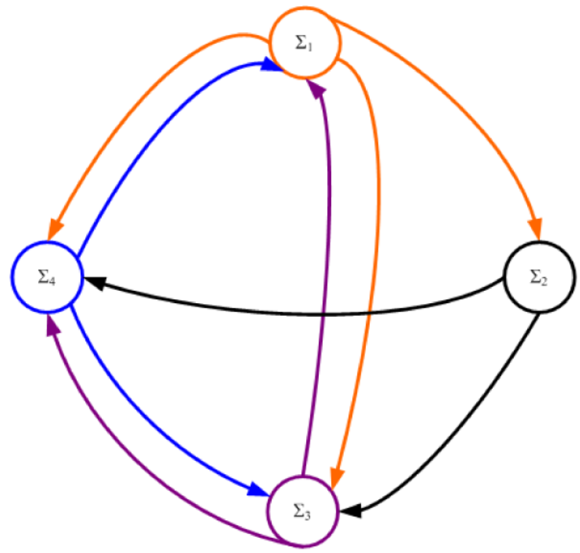


Fig. 5 An array of four mechanical systems

$$\Theta=\begin{bmatrix}7 & 0 & -2 & -5\\ -1 & 1 & 0 & 0\\ -3 & -1.5 & 5 & -0.5\\ -3 & -1 & -2 & 6\end{bmatrix}.$$

After a Gaussian elimination we obtain the matrix Ξ as

$$\Xi=\begin{bmatrix}1 & 0 & -\frac{2}{7} & -\frac{5}{7}\\ 0 & 1 & -\frac{2}{7} & -\frac{5}{7}\\ 0 & 0 & 1 & -1\\ 0 & 0 & 0 & 0\end{bmatrix}, \tag{24}$$

which has rank 3, hence the array is synchronizable.

The variables r are given by (14), with

$$\Phi=\begin{bmatrix}1 & 0 & -\frac{2}{7} & -\frac{5}{7}\\ 0 & 1 & -\frac{2}{7} & -\frac{5}{7}\\ 0 & 0 & 1 & -1\end{bmatrix}.$$

Even though we can obtain an approximated model of these systems, in particular the terms composing the vector $F(\cdot)$, we assume these terms are completely unknown. This consideration will serve to show the robustness performance of the proposed technique. Hence we propose a control input $\mathcal{U}$ of the following form:

$$\mathcal{U}=-K_p r-K_v\dot{r}-K_s\operatorname{sign}(r), \tag{25}$$

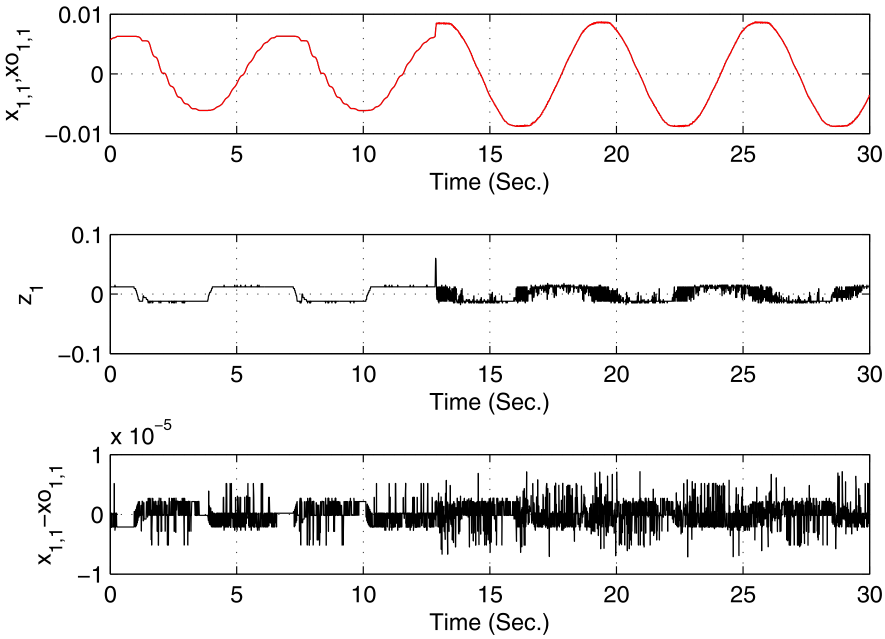


Fig. 6 Deriver performance of system Σ_1

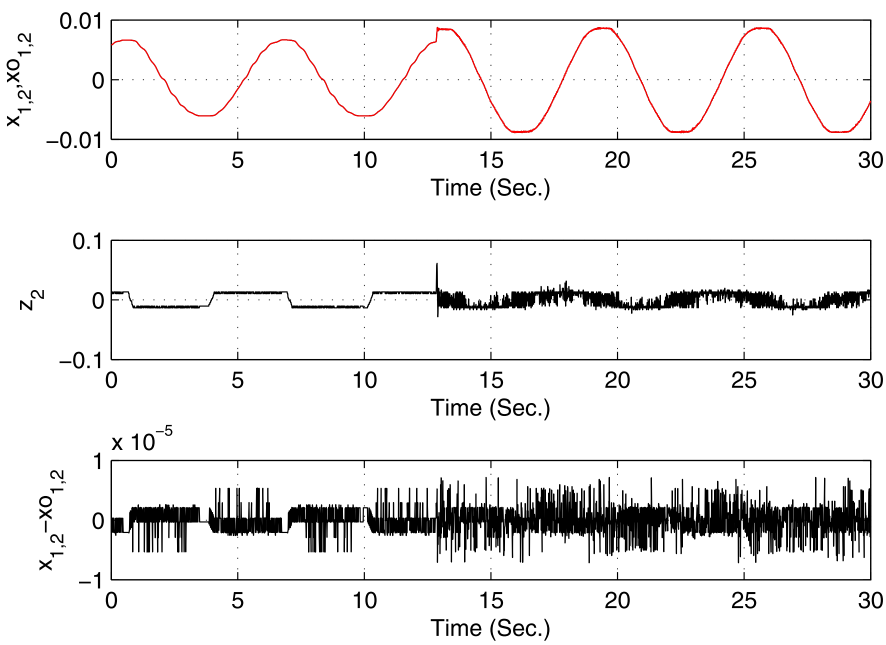


Fig. 7 Performance of deriver for system Σ_2

with K_p, K_v, and K_s definite positive matrices. This leads to solve the equation

$$K_p r + K_v \dot{r} + K_s \operatorname{sign}(r) = -\Phi v \tag{26}$$

from which we must choose a congruent set of coupling signals v.

For example, let us consider that the gains in the controller to be $k_{p1} = 21$, $k_{v1} = 1$, $k_{s1} = 0.3$, $k_{p2} = 20$, $k_{v2} = 1$, $k_{s2} = 0.05$, $k_{p3} = 10$, $k_{v3} = 1$, and $k_{s3} = 0.4$. In this experiment the nodes evolve disconnected from time $t = 0$ to $t \approx 12.8$ seconds. At this time the coupling signals are activated.

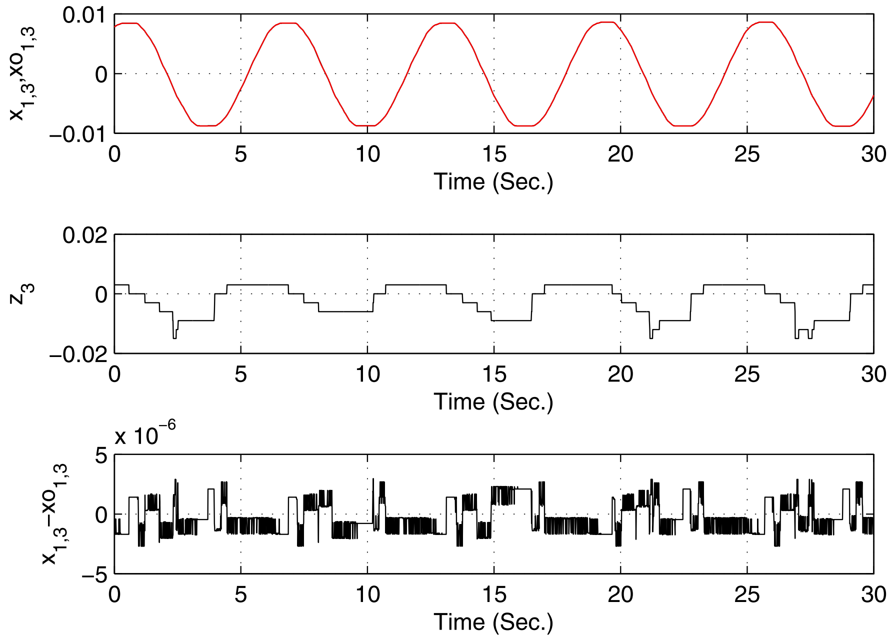


Fig. 8 Performance of the deriver for system Σ_3

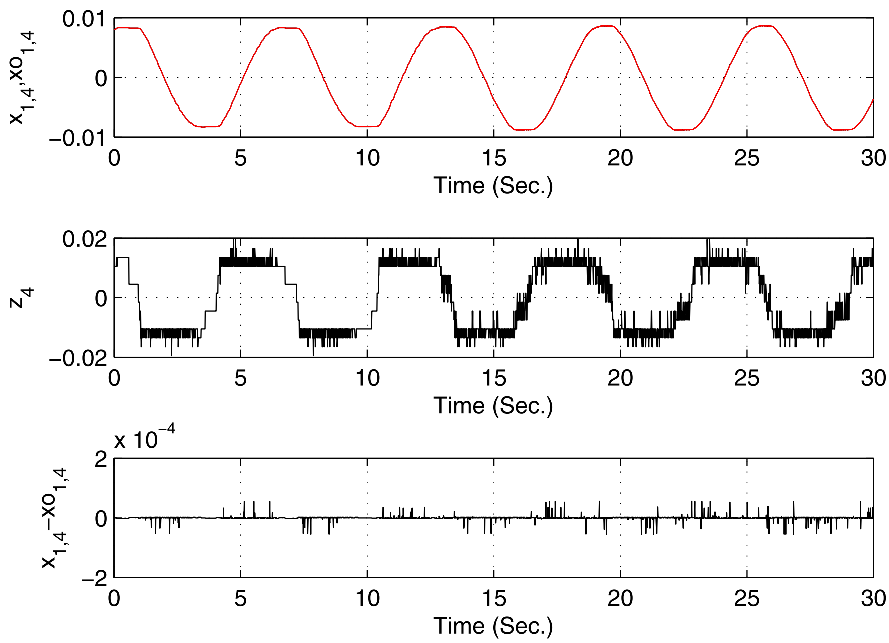


Fig. 9 Performance of deriver for system Σ_4

Figures 6, 7, 8 and 9 show the performance of the derivers for each node. These figures show the outputs x_{i1} and the signals $\hat{x}_i$ and z_i. The difference between the signals x_{ii} and $\hat{x}_i$ have a maximum value of 7×10^{-5} which is small enough to conclude a good estimation of the velocity z_i.

Figure 10 shows the outputs of the nodes after and before the coupling signals are applied. The initial error amplitude among the four system signals is between 2×10^{-3} and 4×10^{-3} units. This difference decreases to 0.2×10^{-3} and 0.1×10^{-3} units near 10 seconds after the coupling signals are activated, see

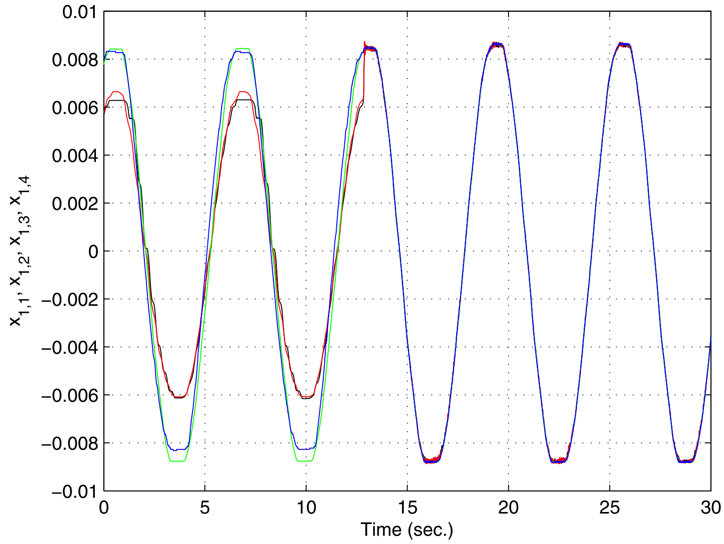


Fig. 10 Behavior of the output of each node before and after applying the coupling signals

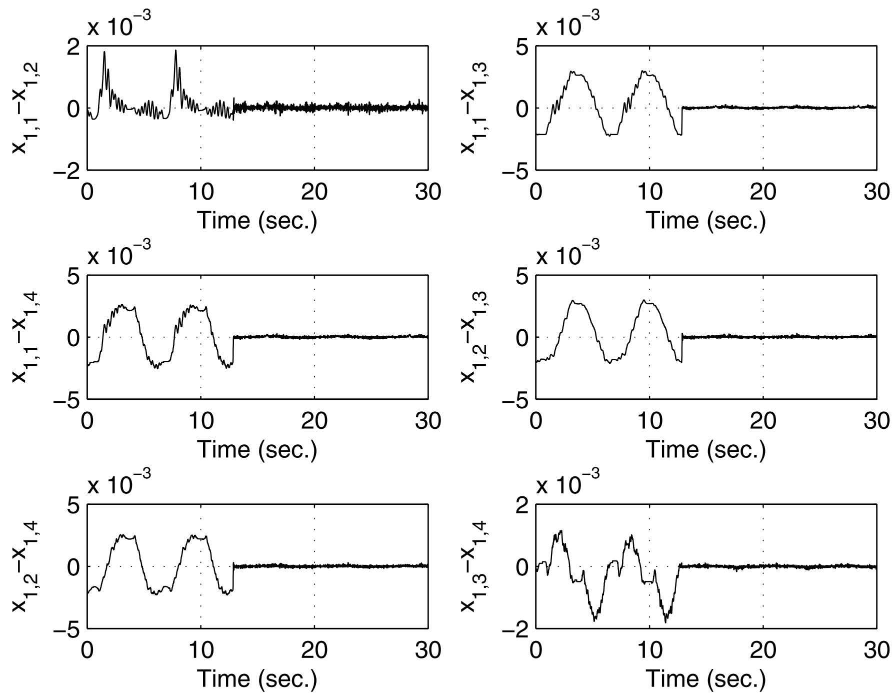


Fig. 11 Output errors behavior of the systems array

Fig. 11. This effect can also be observed in the variables r, shown in Fig. 12, where one can appreciate that the variables remain in a small neighborhood of the origin, which produces a small synchronization error.

The synchronization error is due to various factors like dry friction, dead zone and backlash of the mechanisms, produced by the coupling between the motors and the mechanical parts, achieved with gears and belts. Even though the effect of this kind of perturbations was not considered in the analysis and, in consequence, in the design of the control signals, the performance of the synchronization is acceptable.

The coupling signals shown in Fig. 13 have values between 0.2 and 2 volts. These signals exhibit high frequency components of small amplitude that did not

Fig. 12 Behavior of the auxiliary signals r_i

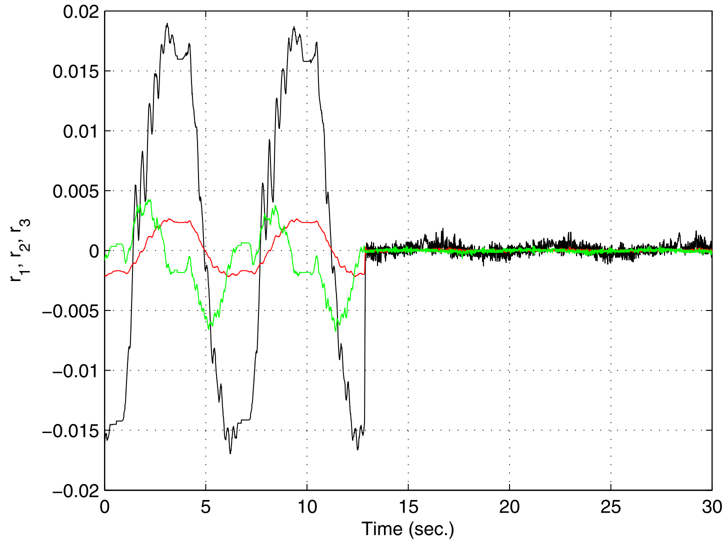
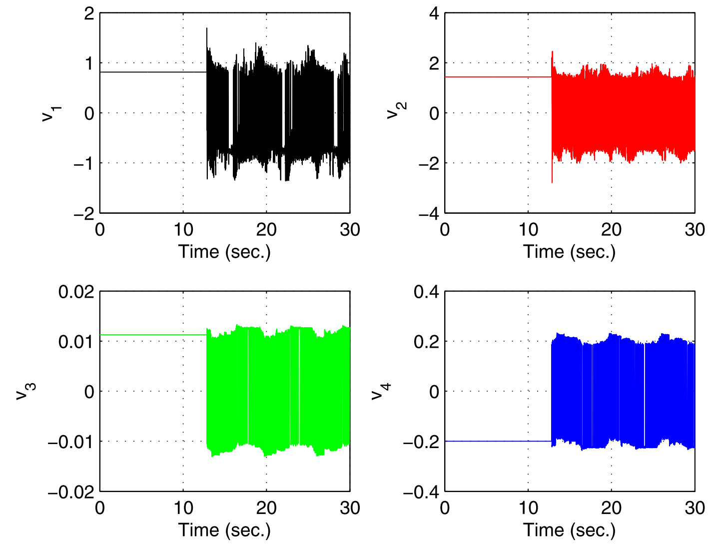


Fig. 13 Coupling signals



produce any experimental problem in the implementation.

6 Conclusions

The synchronization technique proposed in this paper can be applied to different kinds of arrays of second-order dynamical systems, like master–slave, rings, and trees. Furthermore, by means of the individual priority in the connection sense, the unidirectional or bidirectional coupling among the nodes conforming the array can be established. This technique does not impose a symmetric structure of the connectivity matrix, and uses a robust, finite-time, convergent observer that eliminates the need of using a velocity sensor.

The application of the proposed synchronization method to an array of four physical systems showed a good robustness, even when the node dynamics were not known and under the presence of non-smooth dynamics like dry friction, dead zone, and backlash. The use of other control techniques that consider these dynamics will surely improve the closed-loop system performance.

The proposed synchronization technique was designed for 1-DOF mechanical systems; however, its extension to higher DOF systems seems very possible and is under investigation.

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