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Motion equations in redundant coordinates with application to inverse dynamics of constrained mechanical systems

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Abstract The basis for any model-based control of dynamical systems is a numerically efficient formulation of the motion equations, preferably expressed in terms of a minimal set of independent coordinates. To this end the coordinates of a constrained system are commonly split into a set of dependent and independent ones. The drawback of such coordinate partitioning is that the splitting is not globally valid since an atlas of local charts is required to globally parameterize the configuration space. Therefore different formulations in redundant coordinates have been proposed. They usually involve the inverse of the mass matrix and are computationally rather complex. In this paper an efficient formulation of the motion equations in redundant coordinates is presented for general non-holonomic systems that is valid in any regular configuration. This gives rise to a globally valid system of redundant differential equations. It is tailored for solving the inverse dynamics problem, and an explicit inverse dynamics solution is presented for general full-actuated systems. Moreover, the proposed formulation gives rise to a non-redundant system of motion equations for non-redundantly full-actuated systems that do not exhibit input singularities.

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1 Introduction

Let the configuration of the mechanical system be represented by the coordinates $q^a, a = 1, \dots, n$ summarized in $q \in \mathbb{V}^n$, which is called the representing point in $\mathbb{V}^n$. The system's energy is given by the Lagrangian $L(\dot{q}, q) = T(\dot{q}, q) - U(q)$, T being the kinetic (co)energy[1] and U the potential energy. Dissipative forces are omitted for simplicity but without loosing generality. The system is subject to r_g independent geometric constraints $g^\kappa(q) = 0$ and r_v independent completely non-holonomic constraints $f^\lambda{}_a(q)\dot{q}^a = 0$, where only scleronomic constraints are assumed (summation over repeated indices is assumed throughout). They give rise to a system of $r = r_g + r_v$ Pfaffian constraints

$$B^{a_1}{}_a(q)\dot{q}^a = 0, \; |\{a_1\}| = r \tag{1}$$

[1] In this paper it is assumed that T is quadratic in $\dot{q}$, which covers multibody systems and most classical electromechanical systems. Non-quadratic T lead to fundamental differences in the control and the geometric interpretation of the system dynamics (then the configuration space is no longer a Riemann but a Finsler space) [18].

with coefficient matrix

$$B = \begin{pmatrix} \frac{\partial g^{\kappa}}{\partial q^{a}} \\ f^{\lambda}{}_{a} \end{pmatrix}.$$

Assuming ideal constraints the dynamics of a controlled mechanical system is governed by the Lagrangian equations of first kind

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^{a}} - \frac{\partial L}{\partial q^{a}} + B^{a_1}{}_{a}\lambda_{a_1} = M^{i}{}_{a}(q)u_i, \quad (2)$$

where $u_i, i = 1, \ldots, m$ are the control forces and M is the $n \times m$ control matrix. Such control systems are usually referred to as control-affine Euler–Lagrange systems [34]. A good overview of different methods for enforcing constraints in numerical formulations can be found in [20].

With kinetic energy $T(\dot{q}, q) = \frac{1}{2}\dot{q}^{T}G(q)\dot{q}$, given in terms of the generalized mass matrix G, the Lagrangian motion equations can be written in matrix form

$$G(q)\ddot{q} + C(\dot{q}, q)\dot{q} + Q(q) + B^{T}(q)\lambda = Mu(t), \quad (3)$$

where $C(\dot{q}, q)\dot{q}$ represents Coriolis and centrifugal forces, and $Q(q)$ are potential forces due to $U(q)$.

Model-based control of constrained mechanical systems requires computationally efficient formulations of the governing equations of motions allowing for real-time solution of the inverse dynamics problem. Formulating the motion equations in terms of a minimal set of generalized coordinates is still the method of choice for real-time applications. Now the inverse dynamics problem is to determine the control forces u for a given trajectory $q(t)$ of the system, i.e. input to the inverse dynamics is the state $(q, \dot{q})$ and its time derivative whereas the Lagrange multipliers λ are not of interest. Various approaches have been proposed to eliminate λ, and to reduce the number of motion equations to the degree of freedom for systems with holonomic and linear non-holonomic constraints [3, 19, 21, 35, 39], and for non-linear non-holonomic systems [17]. The numerical formulations of these approaches have been cultivated in several contributions using coordinate partitioning methods [6, 11, 30, 33, 40] and pseudoinverse solutions [38].

The reduction of the number of equations, and the elimination of Lagrange multipliers, requires selection of independent position and velocity coordinates. More precisely, in the general case of non-holonomic constraints, the generalized position and velocity coordinates are partitioned separately. This is usually done explicitly by coordinate partitioning methods or implicitly by decomposing the constraint Jacobian in the current configuration. Apparently the crucial point in the coordinate partitioning method is the selection of a proper set of independent coordinates. For most practical multibody systems there is no such selection that is globally valid. To cope with this shortcoming multibody simulation codes using coordinate partitioning are equipped with strategies to switch between different sets of independent coordinates. However, since to each particular selection corresponds one specific system of motion equations, such a switching requires a complete change of the dynamic model. Moreover, the switching condition must be monitored numerically, which adds computational burden to the code. On the other hand a decomposition, e.g. using a singular value decomposition (SVD), in every evaluation step is also computationally very expensive.

Solving the inverse dynamics does not require reduction the number of equations. Moreover, the Lagrange multipliers can be eliminated without introduction of independent coordinates and without reducing the number of equations. This formulation is still computationally complex since it requires inversion of the generalized mass matrix.

A projection method, in which the constrained motion equations are projected to the cotangent space of the configuration space, was developed in [9], where orthogonality is measured with respect to the generalized mass matrix. This projection does not reduce the number of equations. It rather yields a redundant system, which makes its application to forward dynamics simulation difficult.

In this paper another formulation in terms of dependent coordinates is presented that does not involve inversion of the mass matrix. It is computationally efficient and tailored for the inverse dynamics. This inverse dynamics solution yields a singularity-free non-redundant system of motion equations if the system is non-redundantly actuated and does not exhibit input singularities. Hence this formulation may also be applied to the forward dynamics problem. A formulation similar to that presented in this paper, which does not involve the mass matrix in the projection, was proposed in [1] where the forward dynamics problem was addressed.

In Sect. 2 the minimal coordinate formulation is recalled, and the coordinate partitioning method is briefly discussed. A formulation in terms of redundant coordinates is presented in Sect. 3. The actual inverse dynamics problem is addressed in Sect. 4, and different actuation schemes are distinguished for completeness. An explicit solution is presented for redundantly full-actuated systems. The proposed formulation is applied in Sect. 5 to two simple non-holonomic systems first. It is shown that the method can be used to derive a minimal set of independent motion equations for such nun-redundantly actuated mechanisms. Further numerical results are presented for a 2-DOF redundantly actuated parallel manipulator.

2 Motion equations in minimal coordinates

Not all $q \in \mathbb{V}^n$ are admissible configurations. The configuration space (c-space) of the system is defined by the geometric constraints:

$$V := \{q | g^\kappa(q) = 0\}. \tag{4}$$

This variety is locally a smooth submanifold of $\mathbb{V}^n$. The motion of the system is represented by the motion of its representing point q in this c-space. The *local degree of freedom* (DOF) is $\delta_{\mathrm{loc}} := \dim V = n - r_g$—the number of independent finite motions. If only geometric constraints are present, the δ-dimensional tangent space to V at q, denoted $T_q V$, reveals possible velocities at q. The system dynamics would thus evolve on the $2\delta_{\mathrm{loc}}$-dimensional tangent bundle TV (the state space). Since the system is subject to additional non-holonomic constraints, the instantaneous motions, i.e. the velocities, are further restricted. At any point q the vector space of admissible velocities is

$$\overline{T_q V} := \{\dot{q} | B(q)\dot{q}^a = 0\} \subset T_q V. \tag{5}$$

Its dimension is the *differential DOF at* q: $\delta_{\mathrm{diff}} := \dim \overline{T_q V} = n - r < \delta_{\mathrm{loc}} = \dim V$—the number of independent generalized velocities [22, 25]. Consequently, the motion of the system evolves on the non-holonomic tangent bundle $\overline{TV}$ of dimension $\delta_{\mathrm{loc}} + \delta_{\mathrm{diff}} = 2\delta_{\mathrm{loc}} - r_v$; a subvariety of TV.

Now the elimination of λ is achieved by projecting the motion equations (2) to the non-holonomic tangent bundle. To this end a subset of δ_{diff} generalized velocities, $\dot{q}^{a_2}$, is selected as local coordinates on $\overline{T_q V}$. This induces the *coordinate partitioning* $\dot{q} = (\dot{q}^{a_1}, \dot{q}^{a_2}), |\{a_2\}| = \delta_{\mathrm{diff}}$. If the constraints (1) are non-holonomic, this partitioning cannot be transferred to q, since then $\delta_{\mathrm{loc}} > \delta_{\mathrm{diff}}$. Only for holonomic systems q^{a_2} would represent local coordinates on V. The constraints (1) can now be written in matrix form, denoting $\dot{q}_1 = (\dot{q}^{a_1})$ and $\dot{q}_2 = (\dot{q}^{a_2})$,

$$B_1\dot{q}_1 + B_2\dot{q}_2 = 0. \tag{6}$$

Since $\dot{q}_2$ are locally valid independent velocity coordinates, B_1 is full rank r, and the constraints can be resolved as

$$\dot{q}^a = F^a{}_{a_2}\dot{q}^{a_2} \tag{7}$$

where

$$F = \begin{pmatrix} -B_1^{-1}B_2 \\ I \end{pmatrix} \tag{8}$$

is an orthogonal complement to B, i.e. $BF \equiv 0$. Hence premultiplication of (3) with F^T eliminates λ. Introducing (7) and $\ddot{q} = F\ddot{q}_2 + \dot{F}\dot{q}_2$ into (3) yields

$$\overline{G}(q)\ddot{q}_2 + \overline{C}(\dot{q}, q)\dot{q}_2 + \overline{Q}(q) = \overline{M}(q)u(t) \tag{9}$$

with

$$\overline{G} := F^T G F, \quad \overline{C} := F^T (CF + G\dot{F})$$
$$\overline{Q} := F^T Q, \quad \overline{M} := F^T M.$$

This is a system of δ_{diff} independent motion equations without Lagrange multipliers that is often referred to as the *minimal coordinate formulation*. Notice, however, that only the number of generalized velocities and accelerations is reduced, but (9) involves all generalized coordinates that must be determined according to the geometric constraints. The equations (9) can be traced back to the work of Voronets [39], at least, and are a special kind of Maggi's equations [21].

Remark 1 The computation of the orthogonal complement (8) requires $r^2(n-r)$ multiplications, $r(n-r)(r-1)$ additions, and the inversion of the dense $r \times r$ matrix B_1. The problematic step in this construction is the actual selection of independent velocity coordinates. Since the c-space V is non-Euclidean, any such choice is only locally valid in general. This is a severe limitation of the formulation (9). Moreover, F becomes numerically ill-conditioned closed to such parameter singularities. Coordinate partitioning methods have been introduced at the outset of computational multibody dynamics to tackle the increasing complexity by reducing the size of the numerical problem and eliminating the Lagrange multipliers [2, 14, 31, 33, 40]. The critical point is how the independent coordinates are actually chosen. Advanced multibody simulation codes, using a coordinate partitioning, are usually equipped with a strategy for switching between different independent coordinates. The latter requires monitoring the numerical condition of F, which increases the numerical complexity. Coordinate partitioning has been used in relative and absolute coordinate formulations [3], and also in conjunction with the natural coordinates approach [11, 13]. Currently, absolute or natural coordinates formulation including Lagrange multipliers are preferred over the minimal coordinates formulation (coordinate partitioning), mainly because of the density of the system matrices, but also due to the lack of a globally valid coordinate partitioning. Also the augmented Lagrangian formulation [5], which needs no minimal coordinates, is deemed more robust and efficient for forward dynamics simulation. Nevertheless formulations without Lagrange multipliers are desirable for use in model-based control. Further methods to reduce the constraint violation during numerical time integration have been the subject of intensive research [4, 6, 37].

Remark 2 Interestingly the equations (9) are seeing a renaissance in analytical mechanics [8] and for model-based control [7]. They can also be derived via a constrained variational principle. Inserting (7) into the Lagrangian yields the so-called *constrained Lagrangian*

$$\overline{L}(q,\dot{q}_2) := L(q, F\dot{q}_2). \tag{10}$$

The equations of motion can then be expressed as the (Voronets) equations

$$\frac{d}{dt}\frac{\partial \overline{L}}{\partial \dot{q}^{a_2}} - \frac{\partial \overline{L}}{\partial q^{a_2}} + \frac{\partial \overline{L}}{\partial \dot{q}^{a_1}} A^{a_1}_{b_2 a_2}\dot{q}^{b_2} = \overline{M}^i{}_{a_2} u_i \tag{11}$$

with $\frac{\partial}{\partial q^{a_2}} = F^a{}_{a_2}\frac{\partial}{\partial q^a}$, where the terms

$$\begin{aligned} A^{a_1}_{b_2 a_2} &= F^c{}_{a_2}\frac{\partial}{\partial q^c}F^{a_1}{}_{b_2} - F^c{}_{b_2}\frac{\partial}{\partial q^c}F^{a_1}{}_{a_2} \\ &= \frac{\partial}{\partial q^{a_2}}F^{a_1}{}_{b_2} - \frac{\partial}{\partial q^{b_2}}F^{a_1}{}_{a_2} \\ &\quad + F^{c_1}{}_{a_2}\frac{\partial}{\partial q^{c_1}}F^{a_1}{}_{b_2} - F^{c_1}{}_{b_2}\frac{\partial}{\partial q^{c_1}}F^{a_1}{}_{a_2} \end{aligned} \tag{12}$$

indicate that the constraints (1) are non-holonomic, since $A^{a_1}_{b_2 a_2} \equiv 0$ if and only if (1) is holonomic. For this reason A has been called the *object of non-holonomy* [22]. It appears as a central object in non-holonomic mechanics. Expressing (11) explicitly in matrix form, with kinetic the energy $T(\dot{q},q) = \frac{1}{2}\dot{q}^T G(q)\dot{q}$, gives (9). The terms (12) can also be expressed as Lie-bracket of the vector fields F_{a_2} on V:

$$\begin{aligned} [F_{a_2}, F_{b_2}]^a &= F^c{}_{a_2}\frac{\partial}{\partial q^c}F^a{}_{b_2} - F^c{}_{b_2}\frac{\partial}{\partial q^c}F^a{}_{a_2} \\ &= A^{a_1}_{b_2 a_2}\delta^a_{a_1}. \end{aligned}$$

In the non-holonomic setting, writing the constraints (1) in the form

$$\dot{q}_1 + B_1^{-1}B_2\dot{q}_2 = 0, \tag{13}$$

$B_1^{-1}B_2$ can be interpreted as Ehresmann connection on the non-holonomic tangent bundle $\overline{TV}$, and A is the corresponding curvature [7, 8].

Remark 3 (Non-ideal constraints) The generalized constraint forces $K := B^{a_1}{}_a \lambda_{a_1}$ in (3) are only correct if the constraints are ideal, i.e. perform no work and the constraint reactions can be modeled by the Lagrange multipliers. In case of non-ideal constraints, represented by some constraint forces K on the left-hand side of (3), the non-vanishing term $\overline{K} := F^T K$, which embodies the working constraint forces, would remain in the Voronets equations (9). Many practical systems are subject to non-ideal constraints with general $K(q,\dot{q},\ddot{q})$. For instance frictional contact can be modeled as $K = B^{a_1}{}_a\lambda_{a_1} + K_f(\dot{q},\lambda)$ where K_f may be non-linear in its arguments. Then λ cannot be eliminated from (3) by projection. Moreover $K(q,\dot{q},\ddot{q})$ may be non-linear in its arguments so that (9) is an implicit system in $\ddot{q}$.

3 Projected motion equations in redundant coordinates

It is known that the Lagrange multipliers λ can be eliminated from (3) without reducing the number of equations [31, 32]. This proceeds by solving (3) for $\ddot{q}$, inserting this in $B\ddot{q} + \dot{B}\dot{q} = 0$, and solving for λ, leading to

$$\lambda = \left(BGB^T\right)^{-1}\left(\dot{B}\dot{q} + BG^{-1}(Mu - C\dot{q} - Q)\right). \tag{14}$$

Replacing the Lagrange multipliers in (3) yields

$$G\ddot{q}+C\dot{q}+Q+B^T\left(BGB^T\right)^{-1}\times\left(\dot{B}\dot{q}+BG^{-1}(Mu-C\dot{q}-Q)\right)=Mu. \quad (15)$$

These are n motion equations in the n generalized coordinates. This formulation is applicable for the forward dynamics, as it can be evaluated in an admissible state $(q,\dot{q})\in\overline{TV}$, and be solved for the generalized accelerations $\ddot{q}$. It does not yield a simple solution for the control forces u, however. From a computational point of view (15) is rather expensive due to the appearance of the inverse mass matrix.

Equations (15) are clearly redundant due to the imposed constraints. Using $\dot{B}\dot{q}=-B\ddot{q}$, the expression for the Lagrange multipliers (14) can be rewritten as

$$\lambda=-\left(B_G^+\right)^T(G\ddot{q}+C\dot{q}+Q-Mu), \quad (16)$$

where

$$B_G^+:=G^{-1}B^T\left(BG^{-1}B^T\right)^{-1} \quad (17)$$

is the weighted right pseudoinverse of B with respect to the metric G, i.e. $BB_G^+=I_r$. The pseudoinverse allows for two interpretations.

1. The right pseudoinverse gives a solution $\ddot{q}$ of $B\ddot{q}+\dot{B}\dot{q}=0$ such that the Gibbs–Appell function $S=\frac{1}{2}\ddot{q}^TG\ddot{q}$ is minimized.
2. $(B_G^+)^T$ is also the left pseudoinverse of B^T that yields a solution λ of (3) such that $J=\frac{1}{2}\|G\ddot{q}+C\dot{q}+B^T\lambda+Q-Mu\|$ is minimized. Clearly, if the state and the applied forces Mu are compatible, then $J=0$, which is the desired case (compare to Gauss' principle of least constraints). Otherwise, if $J\neq 0$, i.e. the system is not in a dynamic equilibrium, λ is such that the error in the dynamic balance (the generalized constraint forces) is minimized.

Using the symmetry of G, (15) can be rewritten as

$$N_{B,G}^T(G\ddot{q}+C\dot{q}+Q-Mu)=0, \quad (18)$$

where

$$N_{B,G}:=I_n-B_G^+B \quad (19)$$

is a projector to the null-space (kernel) of B since $BN_{B,G}\equiv 0$. The null-space of B has dimension $\operatorname{rank}N_{B,G}=n-\operatorname{rank}B=n-r=\delta_{\text{diff}}$. Thus only δ_{diff} equations in (18) are independent. This is clear since substitution of the reactions λ introduces the dependencies according to the constraints. One could stop at this point since (18) provides motion equations free of Lagrange multipliers, but from a computational point of view the pseudoinverse (17) is rather expensive since it involves the inverse of the mass matrix G.

Remark 4 Udwadia and Kalaba [38] presented a formulation, without Lagrange multipliers, in terms of n coordinates. This formulation, making use of the principal square root $B^{\frac{1}{2}}$, is

$$G\ddot{q}=f+G^{\frac{1}{2}}D^+\left(b-BG^{-1}f\right) \quad (20)$$

with $D:=G^{\frac{1}{2}}B$, $b:=-\dot{B}\dot{q}$, and $f:=Mu-C\dot{q}-Q$. Because of $G^{\frac{1}{2}}D^+=B^T(BGB^T)^{-1}=GB_G^+$ this is identical to (15) and thus equivalent to (18). Its actual evaluation requires determination of $G^{\frac{1}{2}}$ and G^{-1}, which is numerically even more expensive than (18).

The projector to the null-space of B is not unique, and moreover any positive definite matrix can be used as weight in (19). In particular with the identity matrix as weight, $N_B=N_B^T=I_n-B^+B$, where $B^+:=B^T(BB^T)^{-1}$. Therewith the final form of the projected motion equations is

$$N_B(G\ddot{q}+C\dot{q}+Q)=\widetilde{M}u \quad (21)$$

with the $n\times m$ control matrix $\widetilde{M}:=N_BM$.

In contrast to (9) the projected equations do not require explicit selection of independent generalized velocities. However, (18), respectively, (21) is a redundant system of equations and not directly applicable to the forward dynamics simulation. Nevertheless if $\widetilde{M}$ is full rank on V (21) may be transformed to a system with regular coefficient matrix as shown for the examples in Sects. 5.1 and 5.2.

Remark 5 A similar formulation was derived by Aghili [1]. In that publication a projected form of the motion equations was derived that can be written in the form

$$G_c\ddot{q}=N_Bf-G\dot{N}_B\dot{q}, \quad (22)$$

where $G_c:=G+N_BG-GN_B$, and $f:=\widetilde{M}u-C\dot{q}-Q$. Since G_c is positive definite and symmetric (22) is directly applicable to the forward dynamics simulation. It requires the time derivative of the null-space projector, however.

Remark 6 The evaluation of the null-space projector N_B requires $rn(n+2r)$ multiplications, $n^2r+2r^2n-rn-r^2$ additions, and the inversion of the $r\times r$ matrix BB^T. The latter is always regular, as long as the system does not encounter c-space singularities, where V is not a smooth manifold.

Remark 7 The right pseudoinverse B^+ exists only if B has full rank r. The independence of the constraints (1) is not always guaranteed. In case of redundant constraints B^+ is the generalized inverse determined, e.g. via a SVD. A major source of redundant constraints are permanently redundant cut-joint constraints in MBS with kinematic loops that have been addressed in [23, 27, 36, 41].

Remark 8 Taking B^+ is a purely algebraic operation that does not respect the physical units, unlike B_G^+. Hence the projector and (21) are generally not consistent in terms of physical units, unlike (9). From a numerical point of view this aspect may only be relevant when the equations involve elements with different scales.

4 Inverse dynamics in redundant coordinates

4.1 The control problem

The inverse dynamics problem is to solve the motion equations for the control forces u, given a desired trajectory $q(t)$. Depending on the number of control forces the solution may not be unique. On the other hand the system dynamics may by unaffected by some of the control forces. To be precise different actuation schemes must be distinguished. W.l.o.g. local coordinates can be used and the control system be formulated upon (9).

The constrained mechanical system represents a control-affine control system

$$\dot{x}=f(x)+\sum_{i=1}^{m}g^i(x)u_i \tag{23}$$

on the state space $\overline{TV}$ with state vector $x:=(q,\dot{q})$, where

$$f:=\begin{pmatrix}F\dot{q}_2\\ -F\overline{G}^{-1}(\overline{C}\dot{q}_2+\overline{Q})+\dot{F}\dot{q}_2\end{pmatrix} \tag{24}$$

is the drift vector field, and the columns $g^i, i=1,\ldots,m$ of

$$g:=\begin{pmatrix}0\\ F\overline{G}^{-1}\overline{M}\end{pmatrix} \tag{25}$$

are the control vector fields that determine how the control forces affect the system's state. For non-holonomic systems accessibility and controllability must be addressed separately for position and state [10]. The latter describe the effect of the control forces over a finite period of time. The actuation on the other hand determines the immediate effect of control forces in a given state of the PKM. Apparently the degree of actuation has to do with the number of independent control vector fields, but also with the vector space spanned by g_i.

Now the *degree of actuation* (DOA) can be defined as the number of independent input vector fields in the control system (23). With regular $\overline{G}$ and F, the DOA is

$$\alpha(q):=\operatorname{rank}\overline{M}(q). \tag{26}$$

If $\alpha(q)<\delta_{\mathrm{diff}}$, the system is called *underactuated*, and if $\alpha(q)=\delta_{\mathrm{diff}}$, it is called *full-actuated* at q. The system is called *redundantly actuated* at q, if $m-\alpha(q)>0$ and *non-redundantly actuated* at q, if $m=\alpha(q)$. Apparently a system can be redundantly underactuated. Notice further that a non-holonomically constrained system can be redundantly actuated even if the number of control forces equals the local DOF since $\delta_{\mathrm{loc}}>\delta_{\mathrm{diff}}$.

Actuation is a pointwise property in contrast to controllability, which is a local property. Configurations q where the DOA changes, i.e. when α is not constant in a neighborhood of q, are called *input singularities* [25, 42].

4.2 Inverse dynamics of full-actuated systems

Equations (21) are tailored for the inverse dynamics as they do not involve Lagrange multipliers (like the minimal coordinate formulation (9)), and they are valid in all regular configurations (unlike (9)) since B^+, and thus the null-space projector, exists along any regular trajectory. Their advantage over the formulation (15) and (20) is that only a single matrix must be inverted in order to solve for u. The inverse dynamics solution for u is

$$u = \widetilde{M}^+ N_B \big(G(q)\ddot{q} + C(q,\dot{q})\dot{q} + Q(q,\dot{q},t)\big). \qquad (27)$$

The crucial final step is to express the pseudoinverse of $\widetilde{M}$ explicitly taking into account the properties of the projected control matrix. The rank of $\widetilde{M}$ depends on the DOA, but it cannot exceed δ_{diff}. Assuming full actuation, then $m \geq \delta_{\text{diff}}$, and $\widetilde{M}$ has rank δ_{diff}, which may only fail at input singularities. If $\delta_{\text{diff}} = m$ (non-redundant actuation), $\widetilde{M}^+ = (\widetilde{M}^T\widetilde{M})^{-1}\widetilde{M}^T$ is simply the left pseudoinverse. It remains to treat the situation when $m > \delta_{\text{diff}}$.

$\widetilde{M}$ is a $n \times m$ matrix with rank $\delta_{\text{diff}} \leq m < n$. Now for any matrix exists a pseudoinverse. Its actual computation is numerically expensive if the standard SVD procedure is applied. However, the assumption that the system is full-actuated, i.e. rank $\widetilde{M} = \delta_{\text{diff}}$, implies that there is always a full rank $n \times \delta_{\text{diff}}$ submatrix $\widetilde{M}_1$ such that the control matrix can be partitioned as $\widetilde{M} = (\widetilde{M}_1, \widetilde{M}_2)$ with $n \times (m - \delta_{\text{diff}})$ matrix $\widetilde{M}_2$. It is shown in [26] that the pseudoinverse is then determined by

$$\widetilde{M}^+ = \begin{pmatrix} \widetilde{M}_1^+\big(I_n - \widetilde{M}_2\big(I_{m-\delta} + R^T R\big)^{-1} R_1^T \widetilde{M}_1^+\big) \\ \big(I_{m-\delta} + R^T R\big)^{-1} R^T \widetilde{M}_1^+ \end{pmatrix} \qquad (28)$$

abbreviating $R = \widetilde{M}_1^+\widetilde{M}_2$, and $\widetilde{M}_1^+ = (\widetilde{M}_1^T\widetilde{M}_1)^{-1}\widetilde{M}_1^T$.

In summary in the formulation (21) the control matrix $\widetilde{M}$ is partitioned instead of the set of the generalized velocity coordinates. This is computationally advantageous since the motion equations are unaltered and not affected by the lack of a globally valid coordinate partitioning. That is, instead of monitoring the conditioning of the orthogonal complement (7) (to detect switching points for independent velocity coordinates $\dot{q}_2$) the submatrix $\widetilde{M}_1$ is considered and $\widetilde{M}$ is repartitioned. The left-hand side of (21) is globally valid in all configurations, except at c-space singularities where the rank of B drops.

Remark 9 In (21) the dynamic balance (Lagrange equations) is projected to $\overline{T_q V}^*$ and the $\dot{q}$ and $\ddot{q}$ are assumed to satisfy the constraints. If this is not guaranteed, e.g. when using measured values $q_0(t)$ that may not evolve in $\overline{TV}$, the velocity $\dot{q}_0$ must be projected to $\overline{T_q V}$ via $\dot{q} = N_B\dot{q}_0$, and the accelerations via $\dot{q} = N_B\ddot{q}_0 + \dot{N}_B\dot{q}_0$. This leads to

$$\widetilde{G}(q)\ddot{q} + \widetilde{C}(\dot{q},q)\dot{q} + \widetilde{Q}(q) = \widetilde{M}(q)u(t) \qquad (29)$$

with

$$\widetilde{G} := N_B^T G N_B, \quad \widetilde{C} := N_B^T\big(CN_B + G\dot{N}_B\big)$$
$$\widetilde{Q} := N_B^T Q. \qquad (30)$$

This is formally similar to (9), but (29) is a system of n equations of which only δ_{diff} are independent. The time derivative is readily found to be $\dot{N}_B = -(B^+\dot{B}N_B) - (B^+\dot{B}N_B)^T$, which is obviously symmetric.

Remark 10 If rank $M = \delta_{\text{diff}} = m$ for all $q \in V$, i.e. there are no input singularities, then (27) is a system of δ_{diff} independent equations governing the system's dynamics. Clearly if the system's motion is globally uniquely determined by the actuator motion, the actuator coordinates represent globally valid minimal coordinates. The latter are not always contained in q, however. The formulation provides thus a method for deriving globally valid motion equations even if no globally valid coordinate partitioning of q exists but the motion is globally determined be the actuator coordinates. This corresponds to a projection to the cotangent bundle of the actuator coordinate manifold (Ref. Sects. 5.1 and 5.2).

In particular if the actuator coordinates are included in q, it can be split as $q = (q_{\text{p}}, q_{\text{a}})$, with q_{p} and q_{a} being the vector of passive and actuated coordinates, respectively. Then q_{a} represent globally valid (possibly dependent) parameters, and $M = \binom{0}{I_m}$. Then the motion equations can be formulated in terms of q_{a} as presented in [28]. As special case, when $m = \delta_{\text{diff}}$, this leads to the minimal coordinate formulation (9) with $q_2 := q_{\text{a}}$.

5 Examples and application

5.1 Actuated skate

Figure 1 shows a simplified model of an actuated skate. For illustration purposes the skate is only allowed to move on a horizontal plane without tilting

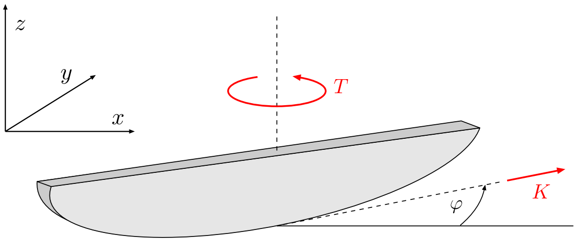

Fig. 1 Simple model of an actuated skate

and rolling. The configuration is parameterized with $q = (\varphi, x, y)$ where x, y describe the position of the contact point, and φ the rotation angle about the vertical axis through the contact point. The body is actuated by the drive force K in direction of motion and the steering torque T about the vertical axis. The skate is subject to $r_v = 1$ non-holonomic knife edge constraint $\dot{x}\sin\varphi - \dot{y}\cos\varphi = 0$, with corresponding coefficient matrix in (1)

$$B = \begin{pmatrix} 0 & \sin\varphi & -\cos\varphi \end{pmatrix}. \tag{31}$$

There are no position constraints ($r_g = 0$), hence the system has DOF $\delta_{\mathrm{loc}} = 3$ and $\delta_{\mathrm{diff}} = 2$. Since there are $m = \delta_{\mathrm{diff}}$ generalized actuation forces the system is non-redundantly full-actuated.

Denote the mass of the skate with m_s, and with Θ_s the inertia moment about the vertical axis through the contact point. Then the three equations of motion of the skate are

$$\begin{aligned} \Theta_s\ddot{\varphi} &= T \\ m_s\ddot{x} - m_s\dot{y}\dot{\varphi} + \lambda_1\sin\varphi &= K\cos\varphi \\ m_s\ddot{y} + m_s\dot{x}\dot{\varphi} - \lambda_2\cos\varphi &= K\sin\varphi \end{aligned} \tag{32}$$

with Lagrange multipliers λ_1, λ_2. Written in the form (3) the control matrix is

$$M = \begin{pmatrix} 1 & 0 \\ 0 & \cos\varphi \\ 0 & \sin\varphi \end{pmatrix}. \tag{33}$$

The inverse dynamics problem consists in finding the drive force K and steering torque T for a given trajectory.

The pseudoinverse of B is $B^+ \equiv B^T$, and the null-space projector

$$N_B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos^2\varphi & \sin\varphi\cos\varphi \\ 0 & \sin\varphi\cos\varphi & \sin^2\varphi \end{pmatrix}. \tag{34}$$

The projector has constant rank 2. The projected control matrix is $\widetilde{M} = N_B M = M$, and the projected system (21) of 2 independent equations is

$$\begin{pmatrix} \Theta\ddot{\varphi} \\ m_s\cos\varphi(\cos\varphi(\ddot{x} - \dot{y}\dot{\varphi}) + \sin\varphi(\ddot{y} + \dot{x}\dot{\varphi})) \\ m_s\sin\varphi(\cos\varphi(\ddot{x} - \dot{y}\dot{\varphi}) + \sin\varphi(\ddot{y} + \dot{x}\dot{\varphi})) \end{pmatrix} = \widetilde{M}\begin{pmatrix} T \\ K \end{pmatrix}. \tag{35}$$

This is a globally valid system of 3 independent motion equations. Since rank $\widetilde{M} = \delta_{\mathrm{diff}} = m = 2$ the pseudoinverse of $\widetilde{M}$ is

$$\widetilde{M}^+ = \left(\widetilde{M}^T\widetilde{M}\right)^{-1}\widetilde{M}^T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi & \sin\varphi \end{pmatrix} \tag{36}$$

and the inverse dynamics solution follows by premultiplication of (35) with $\widetilde{M}^+$

$$\begin{pmatrix} T \\ F \end{pmatrix} = \begin{pmatrix} \Theta\ddot{\varphi} \\ m_s(\cos\varphi(\ddot{x} - \dot{y}\dot{\varphi}) + \sin\varphi(\ddot{y} + \dot{x}\dot{\varphi})) \end{pmatrix}. \tag{37}$$

This is an everywhere regular system of two motion equations. Due to the non-redundant actuation ($m = \delta_{\mathrm{diff}}$) and the absence of input singularities (always rank $\widetilde{M} = \delta_{\mathrm{diff}}$) they represent a set of independent equations governing the dynamics. That is, they are equally applicable to solve the forward and inverse dynamics problem (remark 10). In fact, the transformation $\widetilde{M}^+ N_B$ corresponds to a projection to the manifold of actuator coordinates (the convective tangential coordinate along the forward motion and the steering angle). This would actually represent globally valid coordinates on $\overline{TV}$, and the described method is a way to make use of this fact without explicit reference to these coordinates.

For comparison consider the minimal coordinate formulation using independent velocities $\dot{q}_2 = (\dot{\varphi}, \dot{x})$. Then, with $\dot{q} = (\dot{\varphi}, \dot{x}, \dot{y})$ the orthogonal complement (8) is

$$F = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & \tan\varphi \end{pmatrix}. \tag{38}$$

Clearly this is not valid for $\varphi = \frac{1}{2}k\pi, k \in \mathbb{Z}$. The same would be observed at $\varphi = k\pi$ if $\dot{x}$ is replaced by $\dot{y}$.

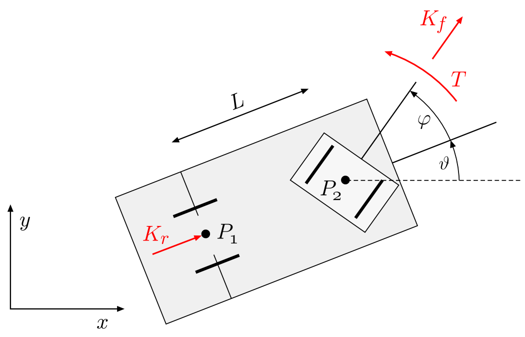

Fig. 2 Simple model of a car. K_f is the driving force at the front wheels, and K_r is that at the rear wheels

5.2 Front driven car

The second example is the simplified car model in Fig. 2 consisting of the chassis body (including rear wheels) and the front wheelset. The two bodies are connected by a revolute joint. The front wheels generate a traction force K_f in rolling direction of the wheels (K_r is zero when front driven). For simplicity the wheels are not considered explicitly, and the kinematic rolling condition is replaced by sliding condition. The car is thus actuated by the drive force K and the steering torque T. Restricted to planar motions the system has $n = 4$ coordinates $q = (x, y, \vartheta, \varphi)$ where x, y are position coordinates of the point P_1 on the rear axis. The system is subject to the two non-holonomic edge constraints

$$\begin{aligned} s_\vartheta \dot{x} - c_\vartheta \dot{y} &= 0 \\ s_{\vartheta+\varphi}\dot{x} - c_{\vartheta+\varphi}\dot{y} - Lc_\varphi \dot{\vartheta} &= 0 \end{aligned} \tag{39}$$

where $s_a = \sin a$ and $c_a = \cos a$. They can be written as (1) with

$$B = \begin{pmatrix} s_\vartheta & -c_\vartheta & 0 & 0 \\ s_{\vartheta+\varphi} & -c_{\vartheta+\varphi} & -Lc_\varphi & 0 \end{pmatrix}. \tag{40}$$

This matrix has always full rank 2 so that $\delta_{\mathrm{diff}} = 2$. There are no geometric constraints. The problem of parameterization of the motion in terms of independent coordinates was discussed in [12] in context of manipulator control.

Denoting with m_1 and m_2 the mass of chassis and front wheelset, respectively, with Θ_1 the inertia moment of the chassis about the point P_1, and with Θ_2 that of the wheelset about P_2. The equations of motion are in matrix form (3)

$$\begin{pmatrix} m_1+m_2 & 0 & -Lm_2 s_\vartheta & 0 \\ 0 & m_1+m_2 & Lm_2 c_\vartheta & 0 \\ -Lm_2 s_\vartheta & Lm_2 c_\vartheta & L^2 m_2 + \Theta_1 & 0 \\ 0 & 0 & 0 & \Theta_2 \end{pmatrix} \begin{pmatrix} \ddot{x}_1 \\ \ddot{y}_1 \\ \ddot{\vartheta} \\ \ddot{\varphi} \end{pmatrix} + \begin{pmatrix} 0 & 0 & -Lm_2 c_\vartheta \dot{\vartheta} & 0 \\ 0 & 0 & -Lm_2 s_\vartheta \dot{\vartheta} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \dot{x}_1 \\ \dot{y}_1 \\ \dot{\vartheta} \\ \dot{\varphi} \end{pmatrix} + B^T \lambda = Mu \tag{41}$$

with $\lambda = (\lambda_1, \lambda_2)$. The control matrix for the front driven car is

$$M = \begin{pmatrix} c_{\vartheta+\varphi} & 0 \\ s_{\vartheta+\varphi} & 0 \\ Ls_\varphi & 0 \\ 0 & 1 \end{pmatrix}, \tag{42}$$

and $u = (K_f, T)$ comprises the traction force and steering torque. The car is non-redundantly full-actuated ($\delta_{\mathrm{diff}} = m$). The pseudoinverse $B^+ = B^T \times (BB^T)^{-1}$ exists for all q, and the null-space projector to B is

$$N_B = \frac{1}{a} \begin{pmatrix} 2L^2 c_\vartheta^2 c_\varphi^2 & L^2 c_\varphi^2 s_{2\vartheta} & Lc_\vartheta s_{2\varphi} & 0 \\ L^2 c_\varphi^2 s_{2\vartheta} & 2L^2 c_\varphi^2 s_\vartheta^2 & Ls_\vartheta s_{2\varphi} & 0 \\ Lc_\vartheta s_{2\varphi} & Ls_\vartheta s_{2\varphi} & 2s_\varphi^2 & 0 \\ 0 & 0 & 0 & a \end{pmatrix} \tag{43}$$

with $a = L^2 + (L^2 - 1)c_{2\phi} + 1$. N_B has constant rank $\delta_{\mathrm{diff}} = m = 2$. Hence (21) is a globally valid system of $n = 4$ equations of which δ_{diff} are independent. The pseudoinverse of the full rank $n \times m$ matrix $\widetilde{M} = N_B M$ is

$$\widetilde{M}^+ = \begin{pmatrix} c_\vartheta c_\varphi & s_\vartheta c_\varphi & s_\varphi / L & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{44}$$

which has constant rank 2. Premultiplication of (41) with $\widetilde{M}^+$ yields a system of two independent globally valid motion equations (27).

In the minimal coordinate formulation (9) either $\dot{q}_2 = (\dot{x}, \dot{\varphi})$ or $\dot{q}_2 = (\dot{y}, \dot{\varphi})$ could be used as independent coordinates. In the first case (8) is

$$F = \begin{pmatrix} 1 & 0 \\ -\tan\vartheta & 0 \\ -\frac{\sin(\vartheta+\varphi)}{L\cos\varphi} & \frac{\cos(\vartheta+\varphi)\tan\vartheta}{L\cos\varphi} \\ 0 & 1 \end{pmatrix}. \tag{45}$$

For $\vartheta = \frac{k}{2}\pi$ and $\varphi = \frac{k}{2}\pi$ this F is not defined. For the second choice of $\dot{q}_2$ this happens at $\vartheta = \frac{k}{k}\pi$ and $\varphi = \frac{k}{2}\pi$. Thus there is no globally valid partitioning of $\dot{q}$, and the system (9) exhibits parameterization singularities that have no physical significance.

5.3 Rear driven car

The above car equipped with a rear drive can be modeled by a drive forces K_r perpendicular to the rear axle as in Fig. 2 (now K_f is zero). The corresponding control matrix is

$$M = \begin{pmatrix} c_\vartheta & 0 \\ s_\vartheta & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{46}$$

and $u = (K_r, T)$. Now the right pseudoinverse of $\widetilde{M} = N_B M$ is

$$\widetilde{M}^+ = \begin{pmatrix} \cos\vartheta & \sin\vartheta & \tan\varphi/L & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{47}$$

This pseudoinverse does not exist at $\varphi = \frac{k}{2}\pi$, which happens when the steering wheels are perpendicular to the direction of forward motion dictated by the constraints of the rear wheels. These configurations are the input singularities of the rear driven car, i.e. they are observable in reality. Now the model cannot be reduced to a globally valid system of two equations by premultiplication with $\widetilde{M}^+$. Yet the motion equations (41) are valid independently of such singularities and globally governs the dynamics of a system with input singularities.

5.4 Inverse dynamics of redundantly full-actuated parallel manipulators

Parallel kinematics machine (PKM) form a large class holonomically constrained mechanical control systems. A major problem of PKM is the abundance of input singularities within the workspace. Redundant actuation has been proposed to tackle this phenomenon [24]. A redundantly actuated PKM (RA-PKM) with DOF $\delta = \delta_{\text{loc}} = \delta_{\text{diff}}$ is equipped with $m > \delta$ redundant actuators. The benefit is clear by noting that if a non-redundantly actuated PKM, controlled by δ actuators, would exhibit an input singularity, one of the $m - \delta$ redundant actuators of the RA-PKM can replace those that are instantaneously not influencing the PKM motion.

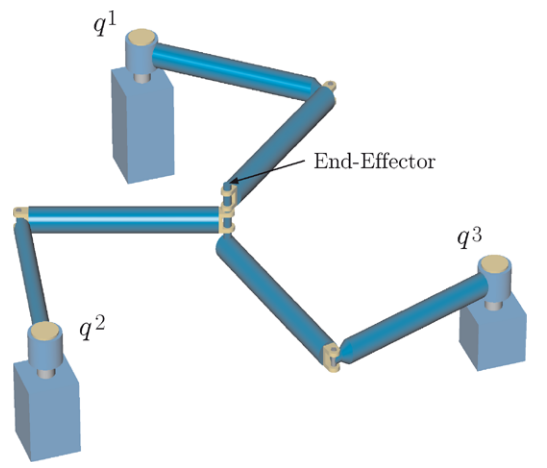

Fig. 3 Multibody model of the planar 2R̲RR/R̲R RA-PKM

A PKM is a holonomically constrained system, and its motion equations are naturally expressed in terms of actuator coordinates as they are measurable. In the minimal coordinate formulation δ out of the m actuator coordinates are selected as independent coordinates. Then clearly these minimal coordinates fail in input singularities of the non-redundantly actuated PKM with these δ actuators. As an example consider the planar RA-PKM in Fig. 3. Details of this prototype and its real-time control are reported in [16]. It has the DOF $\delta = 2$, and could be controlled by actuating two of the base revolute joints. This would lead to the input singularities shown in Fig. 4. Figure 4(a) shows the input singularities when joints 1 and 2 are actuated and the end-effector is controlled along a circular path. They occur whenever the two middle links are parallel. In these configurations the motion orthogonal to the middle links cannot be controlled by the two joints, thus the motion equations (9) expressed in terms of these two joint angles are not feasible. Clearly this can be avoided using the coordinates of joint 1 and 3 instead. Then, however, the configurations in Fig. 4(b) are singular. Thanks to the redundant actuation the RA-PKM does not possess these input singularities, and there is always a local parameterization in terms of δ actuator coordinates. However, to cover the entire motion range of the RA-PKM it is necessary to switch between different minimal (actuator) coordinates as proposed in [15]. The redundant formulation (21), respectively, (29) does not need local coordinates.

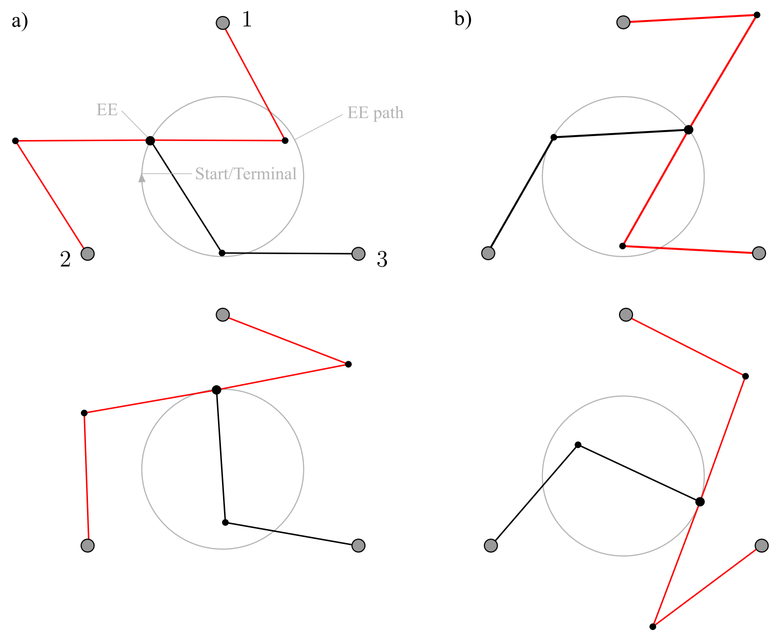

Fig. 4 Different input singularities if the 2RR/RRR PKM is non-redundantly actuated. The mechanism shown in red color is the equivalent non-redundantly actuated mechanisms being instantaneously in an input singularity

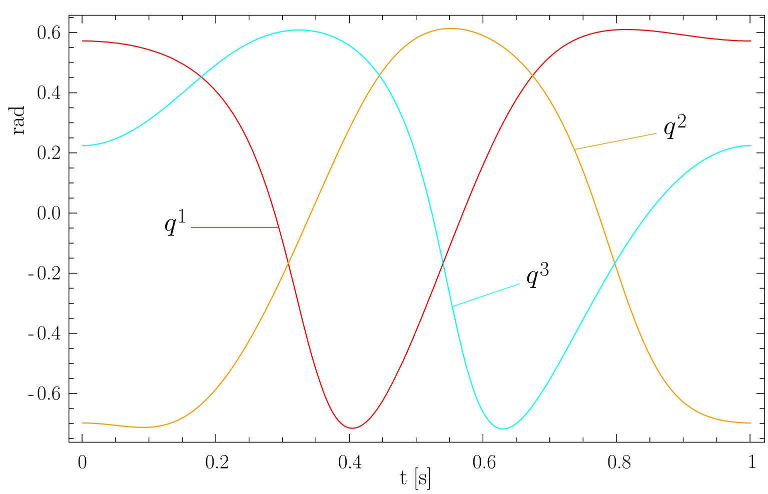

Fig. 5 Actuator trajectories for the circular path in Fig. 4

For illustration the PKM is steered along a circular end-effector (EE) path as shown in Fig. 4. The corresponding actuator trajectories are shown in Fig. 5. If the motion equations (9) are expressed in terms of the joint angles of the first and second drive, with q^1 and q^2, respectively, as independent generalized coordinates $q_2 = (q^1, q^2)$, the model is not valid at the poses in Fig. 4(a) (and any pose with the middle links aligned parallel) where its configuration cannot be determined uniquely by q^1 and q^2. In these configurations q^1, q^2 fail as minimal coordinates, and matrix B_1 becomes singular so that the orthogonal complement F in (8) does not exists. Then the actuator forces u deduced from (9) tend to infinity—an artifact due to the minimal coordinate model.

Application of (27) together with (28) requires to monitor the numerical conditioning of the 2×2 matrix $\widetilde{M}_1 \widetilde{M}_1^T$, which is easily implemented. $\delta = 2$ independent coordinates, and the corresponding submatrix $\widetilde{M}_1$, is selected from the $m = 3$ actuator coordinates. Only the combinations $q_2^{(1)} = (q^1, q^2)$, $q_2^{(2)} = (q^1, q^3)$ and $q_2^{(3)} = (q^2, q^3)$ are considered. Figure 6 shows the determinant for different choices of independent coordinates when tracing the circular EE path. For each selection the determinant becomes

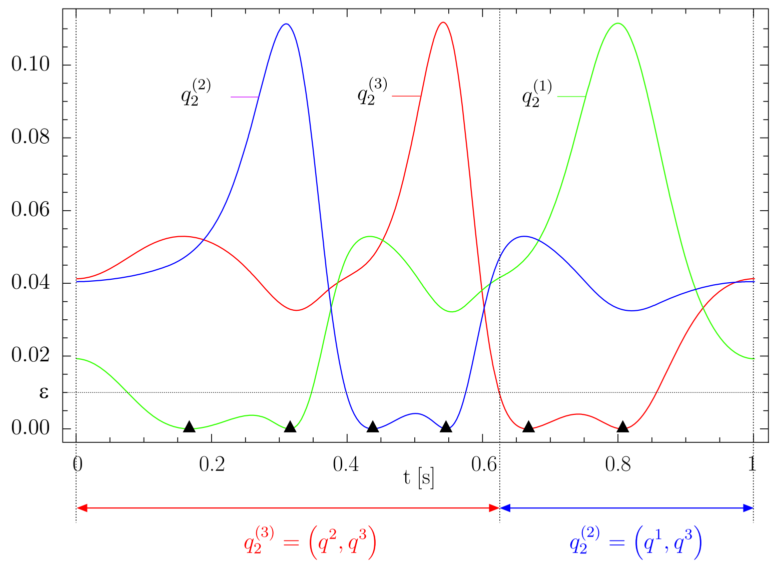

Fig. 6 Determinant of $\widetilde{M}_1^T \widetilde{M}_1$ for different selections of independent coordinates q_2. The points ▲ indicate the singularities of the respective parameterization

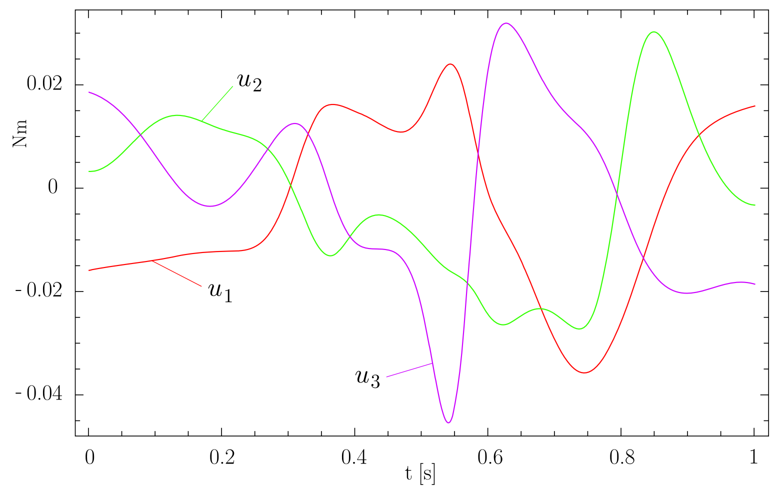

Fig. 7 Actuator torques obtained from the inverse dynamics (27) when tracing the circular EE path in Fig. 4

zero in the two configurations where the equivalent non-redundantly actuated PKM passes an input singularity. As a simple criterion, whenever the determinant of $\widetilde{M}_1$ drops below a specified threshold ε another combination with maximum determinant is selected. The motion starts with $q_2 := (q^2, q^3)$, as shown in Fig. 6.

The selection of minimal coordinates is only necessary for the pseudoinverse of $\widetilde{M}$. Equations (21) are not affected and globally valid. The control torques u_1, u_1, and u_3 for the circular EE-trajectory calculated with (27) are smooth as shown in Fig. 7.

The presented inverse dynamics formulation can directly be incorporated in standard model-based control schemes such as computed torque and augmented PD controllers [29]. Crucial for the latter is the skew symmetry of $\widetilde{\dot{G}} - 2\widetilde{C}$.

6 Summary

A formulation of the motion equations in terms of redundant coordinates without Lagrange multipliers for constrained mechanical systems is presented. This formulation is applicable to holonomic and non-holonomic systems. The use of redundant coordinates ensures a globally valid parameterization. That is, in contrast to minimal coordinate formulations, relying on the partitioning of generalized velocity coordinates, the Lagrange multipliers are eliminated without reducing the number of equations, leading to a redundant system of motion equations that are globally valid. This formulation is tailored for efficiently solving the inverse dynamics problem. Due to the redundancy it cannot be used directly for forward dynamics simulation. However, for non-redundantly actuated systems it can give rise to a system of independent motion equations induced by the inverse dynamics solution. It is shown for two simple non-holonomic examples how the proposed method can be applied to solve the inverse and forward dynamics problems of non-redundantly actuated control systems. Numerical results are also reported for the inverse dynamics of a redundantly actuated 2-DOF parallel manipulator.

Future work will focus on the integration of the presented inverse dynamics solution in model-based controllers. Such control schemes would not exhibit parameterization singularities like those based on the minimal coordiantes formulations. The motion equations are derived in terms of the holonomic velocity coordinates $\dot{q}$, but can be easily extended to non-holonomic velocities ω linearly related by $\omega = D\dot{q}$.

Appendix: List of symbols

V	configuration space defined by geometric constraints
$\overline{T_q V}$	(non-holonomic) tangent space defined by kinematic constraints
$\overline{T_q V}^*$	(non-holonomic) cotangent space defined by kinematic constraints
δ_{loc}	local degree of freedom ($\delta_{\mathrm{loc}} = \dim V$)
δ_{diff}	differential degree of freedom ($\delta_{\mathrm{diff}}(q) = \dim \overline{T_q V}$)
n	number of generalized coordinates of unconstrained system
r	number of kinematic constraints
m	number of generalized actuator forces
$\{a\}$	index set of the n generalized coordinates of unconstrained system
$\{a_1\}$	index set of r dependent generalized velocities of constrained system
$\{a_2\}$	index set of δ_{diff} independent generalized velocities of constrained system
$\dot{q}_1 \equiv (\dot{q}^{a_1})$	vector of r dependent generalized velocities
$\dot{q}_2 \equiv (\dot{q}^{a_2})$	vector of δ_{diff} independent generalized velocities
u_i	generalized control forces, $i = 1, \dots, m$
B	$r \times n$ coefficient matrix of the system of Pfaffian constraints
F	$n \times \delta_{\mathrm{diff}}$ orthogonal complement to B ($BF = 0$)
B_G^+	G-weighted pseudoinverse of matrix B
$N_{B,G}$	$n \times n$ projection matrix to null-space of B defined by B_G^+ ($BN_{B,G}$)
M	$n \times m$ control matrix relating m actuation forces to n generalized forces
$\overline{M}$	$\delta_{\mathrm{diff}} \times m$ control matrix relating m actuation forces to δ_{diff} independent generalized forces
$\widetilde{M}$	$n \times m$ control matrix relating m actuation forces to n generalized forces satisfying the constraints
G	$n \times n$ generalized mass matrix of unconstrained system
C	$n \times n$ Coriolis and centrifugal matrix of unconstrained system
Q	n vector of generalized forces of unconstrained system
$\overline{G}$	$\delta_{\mathrm{diff}} \times \delta_{\mathrm{diff}}$ generalized mass matrix in δ_{diff} minimal coordinates
$\overline{C}$	$\delta_{\mathrm{diff}} \times \delta_{\mathrm{diff}}$ Coriolis and centrifugal matrix in δ_{diff} minimal coordinates
$\overline{Q}$	δ_{diff} vector of generalized forces in δ_{diff} minimal coordinates
$\widetilde{G}$	$n \times n$ generalized mass matrix projected to null-space of B
$\widetilde{C}$	$n \times n$ Coriolis and centrifugal matrix projected to null-space of B
$\widetilde{Q}$	n vector of generalized forces projected to null-space of B
I_k	$k \times k$ identity matrix

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