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Synchronization of stochastic chaotic neural networks with reaction-diffusion terms

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Abstract In this paper, we are concerned with the synchronization problem of a class of stochastic reaction-diffusion neural networks with time-varying delays and Dirichlet boundary conditions. By using the Lyapunov–Krasovskii functional method, feedback control approach and stochastic analysis technology, delay-dependent synchronization conditions including the information of reaction-diffusion terms are presented, which are expressed in terms of linear matrix inequalities (LMIs). The feedback controllers can be constructed by solving the derived LMIs. Finally, illustrative examples are given to show the effectiveness of the proposed technique.

Keywords Synchronization · Reaction-diffusion · Chaotic neural networks · LMIs

1 Introduction

Synchronization of chaotic systems has attracted considerable attention over the past decades due to their extensive applications in secure communication, parallel recognition, image processing, and other engineering areas [3, 26, 40, 47]. A great deal of effective approaches such as drive-response method [12, 21, 42], adaptive control method [24, 31, 33], impulsive control method [15, 46], sliding mode control method [8, 9, 13] have been proposed to synchronize chaotic systems. Recently, it has been found that neural networks can exhibit some complicated dynamics such as periodic oscillations and even chaotic attractors if time delays and parameters of networks are chosen appropriately [6, 22]. Consequently, a great number of results on synchronization issue of chaotic neural networks have been reported in the literature; see, e.g., [1, 2, 14, 16, 17, 24, 29, 30, 43, 48, 49] and the references therein. There are several significant synchronization definitions for stochastic systems, such as mean square asymptotic synchronization and almost sure synchronization [32]. Usually, a system is said to be mean square asymptotically synchronized if the error converges to zero in quadratic mean, while a system is said to be almost surely synchronized if the error converges to zero with probability one.

In the real world, there are lots of reaction-diffusion phenomena in nature and engineering fields. Strictly speaking, diffusion effects cannot be avoided in the neural networks model when electrons are moving in asymmetric electromagnetic fields [18]. Therefore, it is reasonable and significant to take reaction-diffusion effect into full account in the research of neural networks. Many results on stability analysis of reaction-

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diffusion neural networks have been presented in the literature; see, for example, [5, 19, 23, 25, 27, 35, 36, 38, 39] and the references therein.

Recently, the problem of synchronization control of reaction-diffusion neural networks has been addressed in the literature. In [20, 37], the problem of asymptotic synchronization of delayed reaction-diffusion neural networks was studied with feedback control approach; adaptive exponential synchronization of reaction-diffusion neural networks was discussed in [11]. When Dirichlet boundary conditions of diffusion equation were considered, sufficient synchronization conditions were presented in [7, 34, 41]. The synchronization problem was investigated under the impulsive control in [7], where the proposed criterion was based on the p-norm. By using the Lyapunov functional method, results on exponential synchronization of reaction-diffusion neural networks were provided in [34]. When fuzzy logic in the structure of networks appears, the synchronization criteria were presented in [41], in which the effect of reaction-diffusion was considered. It is worth noting that all of the above reported conditions are derived by certain inequality technique and expressed in the form of algebraic inequalities, which makes their checking somewhat difficult and inconvenient by the developed algorithms. Moreover, all of these are delay-independent, which are generally more conservative than the delay-dependent ones. On the other hand, the synaptic transmission in real nervous systems is a noisy process [10, 28, 44], and stochastic disturbances play important roles in chaos synchronization [31]. Therefore, it is of both practical and theoretical importance to study the problem of synchronization of stochastic delayed reaction-diffusion neural networks with Dirichlet boundary conditions.

In this paper, we consider the problem of synchronization of a class of stochastic reaction-diffusion neural networks with time-varying delays and Dirichlet boundary conditions. The main purpose of this paper is to provide mean square asymptotic synchronization criteria and almost sure synchronization criteria for the considered chaotic neural networks. Based on the Lyapunov–Krasovskii functional method, feedback control approach, partial differential equation concept, and stochastic analysis technology, we present delay-dependent synchronization conditions including the information of reaction-diffusion terms. The derived synchronization schemes are expressed in terms of LMIs, which can be checked easily by Matlab LMI Toolbox. Finally, the effectiveness of the developed methods is illustrated by simulation examples.

Notation Throughout this paper, for real symmetric matrices X and Y, the notation $X \geq Y$ (respectively, $X > Y$) means that the matrix $X - Y$ is positive semidefinite (respectively, positive definite). $\lambda_{\min}(X)$ denotes the minimum eigenvalue of matrix X. I is an identity matrix with appropriate dimension. $\mathcal{R}^n$ denotes the n-dimensional Euclidean space, and the notation $|\cdot|$ refers to the Euclidean vector norm. The notation M^T represents the transpose of the matrix M. The symmetric terms in a symmetric matrix are denoted by $*$. $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathcal{P})$ denotes a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$, where Ω is a sample space, $\mathcal{F}$ is the σ-algebra of subset of the sample space and $\mathcal{P}$ is the probability measure on $\mathcal{F}$. Ψ is an open bounded domain with smooth boundary $\partial\Psi$ and $\operatorname{mes}\Psi > 0$ denotes the measure of Ψ, dS is the element of $\partial\Psi$, $\bar{n}$ is the outer normal vector of $\partial\Psi$. $C_0^n(\Psi)$ represents the space of derivable of n-order real functions with compact support on Ψ. Let $\mathcal{L}^2(\mathcal{R}\times\Psi,\ \mathcal{R}^n)$ denote the space of real Lebesgue measurable functions on $\mathcal{R}\times\Psi$. Define the norm $\|y(t,x)\|_2 = (\sum_{i=1}^n \|y_i(t,x)\|^2)^{1/2}$, $\|y_i(t,x)\| = (\int_\Psi |y_i(t,x)|^2 dx)^{1/2}$ for any $y(t,x) \in \mathcal{L}^2(\mathcal{R}\times\Psi,\ \mathcal{R}^n)$. Denote by $\mathcal{L}^p_{\mathcal{F}_0}([-\bar{\tau},0]\times\Psi,\mathcal{R}^n)$ the family of all $\mathcal{F}_0$-measurable $C([-\bar{\tau},0]\times\Psi,\mathcal{R}^n)$-valued random variables $\xi = \{\xi(\theta) : -\bar{\tau} \leq \theta \leq 0\}$ such that $\sup_{-\bar{\tau}\leq\theta\leq 0} \mathcal{E}\|\xi(\theta)\|_2^2 < \infty$ where $\mathcal{E}\{\cdot\}$ is the expectation operator with respect to some probability measure $\mathcal{P}$. Matrices, if not explicitly stated, are assumed to have compatible dimensions.

2 Problem formulation

We consider the following reaction-diffusion delayed neural network:

$$\begin{aligned}\frac{du_i(t,x)}{dt} &= \sum_{k=1}^{l} \frac{\partial}{\partial x_k}\left(D_{ik}\frac{\partial u_i(t,x)}{\partial x_k}\right) - c_i u_i(t,x) \\ &\quad + \sum_{j=1}^{n} a_{ij} f_j\big(u_j(t,x)\big) \\ &\quad + \sum_{j=1}^{n} b_{ij} f_j\big(u_j(t-\tau(t),x)\big) + J_i, \\ &i = 1,2,\ldots,n \end{aligned} \tag{1}$$

or, in a compact form:

$$\begin{aligned}\frac{du(t,x)}{dt} &= \sum_{k=1}^{l}\frac{\partial}{\partial x_k}\left(D_k\frac{\partial u(t,x)}{\partial x_k}\right) - Cu(t,x)\\ &\quad + Af\big(u(t,x)\big)\\ &\quad + Bf\big(u\big(t-\tau(t),x\big)\big) + J, \qquad (2)\end{aligned}$$

where

$$\begin{aligned}u(t,x) &= \big[u_1(t,x), u_2(t,x), \ldots, u_n(t,x)\big]^T,\\ f\big(u(t,x)\big) &= \big[f_1\big(u_1(t,x)\big), f_2\big(u_2(t,x)\big), \ldots,\\ &\qquad f_n\big(u_n(t,x)\big)\big]^T,\\ f\big(u\big(t-\tau(t),x\big)\big) &= \big[f_1\big(u_1\big(t-\tau(t),x\big)\big), , \ldots,\\ &\qquad f_n\big(u_n\big(t-\tau(t),x\big)\big)\big]^T,\\ C &= \operatorname{diag}\{c_1, c_2, \ldots, c_n\},\\ D_k &= \operatorname{diag}\{D_{1k}, D_{2k}, \ldots, D_{nk}\},\\ J &= \operatorname{diag}\{J_1, J_2, \ldots, J_n\},\end{aligned}$$

$x = (x_1, x_2, \ldots, x_l)^T \in \Psi \subset \mathcal{R}^l$, $\Psi = \{x \mid |x_k| \le L_k\}$, L_k is a constant, $k = 1, 2, \ldots, l$; $u_i(t,x)$ is the state of the ith neuron; $f_i(u_i(t,x))$ denotes the activation function of the ith neuron; J_i denotes the external input on the ith neuron; $c_i > 0$ represents the rate with which the ith unit will reset its potential to the resting state in isolation when disconnected from the networks and external inputs; $A = (a_{ij})_{n\times n}$ is the connection weigh matrix; $B = (b_{ij})_{n\times n}$ is the delayed connection weigh matrix; $D_{ik} = D_{ik}(t,x) > 0$ denotes the transmission diffusion operator along the ith neuron; $\tau(t)$ represents the transmission delay that satisfies

$$0 < \tau(t) \le \bar{\tau}, \qquad \dot{\tau}(t) \le \mu < 1.$$

The boundary condition and initial condition for system (2) are given by

$$\begin{aligned}u(t,x) &= 0, \quad (t,x) \in [-\bar{\tau}, \infty] \times \partial\Psi,\\ u(s,x) &= \phi(s,x), \quad (s,x) \in [-\bar{\tau}, 0] \times \Psi,\end{aligned}$$

where $\phi(s,x) = [\phi_1(s,x), \phi_2(s,x), \ldots, \phi_n(s,x)]^T$.

Based on the drive-response concept for synchronization of coupled chaotic systems, the corresponding response system of (2) is constructed as

$$\begin{aligned}d\tilde{u}(t,x) &= \Bigg[\sum_{k=1}^{l}\frac{\partial}{\partial x_k}\left(D_k\frac{\partial \tilde{u}(t,x)}{\partial x_k}\right) - C\tilde{u}(t,x)\\ &\quad + Af\big(\tilde{u}(t,x)\big) + Bf\big(\tilde{u}\big(t-\tau(t),x\big)\big)\\ &\quad + J + V(t,x)\Bigg]dt\\ &\quad + \sigma\big(t, \tilde{u}(t,x) - u(t,x), \tilde{u}\big(t-\tau(t),x\big)\\ &\qquad - u\big(t-\tau(t),x\big)\big)\, d\omega(t), \qquad (3)\end{aligned}$$

where $V(t,x)$ indicates the control input, which will be appropriately designed. σ is a matrix valued function; $\omega(t)$ is a vector-form Brownian motion on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\ge 0}, \mathcal{P})$, which is assumed to satisfy

$$\mathcal{E}\big\{d\omega(t)\big\} = 0, \qquad \mathcal{E}\big\{d\omega^2(t)\big\} = dt.$$

The boundary condition and initial condition for system (3) are given as

$$\begin{aligned}\tilde{u}(t,x) &= 0, \quad (t,x) \in [-\bar{\tau}, \infty] \times \partial\Psi,\\ \tilde{u}(s,x) &= \varphi(s,x), \quad (s,x) \in [-\bar{\tau}, 0] \times \Psi,\end{aligned}$$

where $\varphi(s,x) = [\varphi_1(s,x), \varphi_2(s,x), \ldots, \varphi_n(s,x)]^T$.

Consider a delayed state feedback controller

$$\begin{aligned}V(t,x) &= K_1\big[\tilde{u}(t,x) - u(t,x)\big]\\ &\quad + K_2\big[\tilde{u}\big(t-\tau(t),x\big) - u\big(t-\tau(t),x\big)\big],\end{aligned}$$

where K_1 and K_2 are the controller gains to be determined. Define the synchronization error as $e(t,x) = \tilde{u}(t,x) - u(t,x)$; thus, the error dynamical system between (2) and (3) is given by

$$\begin{aligned}de(t,x) &= \Bigg[\sum_{k=1}^{l}\frac{\partial}{\partial x_k}\left(D_k\frac{\partial e(t,x)}{\partial x_k}\right) - Ce(t,x)\\ &\quad + Ag\big(e(t,x)\big) + Bg\big(e\big(t-\tau(t),x\big)\big)\\ &\quad + K_1e(t,x) + K_2e\big(t-\tau(t),x\big)\Bigg]dt\\ &\quad + \sigma\big(t, e(t,x), e\big(t-\tau(t),x\big)\big)\, d\omega(t), \qquad (4)\end{aligned}$$

where

$$\begin{aligned}g\big(e(t,x)\big) &= f\big(\tilde{u}(t,x)\big) - f\big(u(t,x)\big)\\ &= f\big(e(t,x) + u(t,x)\big) - f\big(u(t,x)\big).\end{aligned}$$

Throughout this paper, the following assumptions, definitions and lemma are needed to derive our main results.

Assumption 1 [45] *For any $\alpha, \beta \in \mathcal{R}$, $\alpha \neq \beta$*

$$\gamma_j \leq \frac{f_j(\alpha) - f_j(\beta)}{\alpha - \beta} \leq \nu_j, \quad j = 1, 2, \ldots, n,$$

where γ_j and ν_j are known constant scalars.

Assumption 2 *There exist positive constants ρ_1 and ρ_2 such that*

$$\operatorname{trace}\big(\sigma^T(t)\sigma(t)\big) \leq \rho_1 e^T(t,x)e(t,x) + \rho_2 e^T\big(t-\tau(t),x\big)e\big(t-\tau(t),x\big).$$

Definition 1 The system of (2) and (3) is said to be asymptotically synchronized in the mean square, if for any given $\varphi,\phi \in \mathcal{L}^p_{\mathcal{F}_0}([-\bar{\tau}, 0] \times \Psi, \mathcal{R}^n)$,

$$\lim_{t\to\infty} \mathcal{E}\big\|e(t,x;\varphi-\phi)\big\|_2^2 = 0.$$

Definition 2 The system of (2) and (3) is said to be almost surely (a.s.) synchronized, if for any given $\varphi,\phi \in \mathcal{L}^p_{\mathcal{F}_0}([-\bar{\tau}, 0] \times \Psi, \mathcal{R}^n)$ the following formula holds:

$$\lim_{t\to\infty} e(t,x;\varphi-\phi) = 0.$$

Lemma 1 (Friedrichs inequality [4]) *For $u \in C_0^1(\Psi)$, and $\Psi \subset \Psi_1 \subset \mathcal{R}^n$ $\Psi_1 = \{x \mid |x_k| \leq \delta\}$, $k = 1, 2, \ldots, n$, we have*

$$\int_\Psi u^2(x)\,dx \leq \frac{\delta^2}{n}\int_\Psi \sum_{k=1}^n \left(\frac{\partial u}{\partial x_k}\right)^2 dx.$$

The objective of this paper is to establish synchronization conditions for the reaction-diffusion neural network (2) and (3) with Dirichlet boundary conditions.

3 Main results

For convenience, we use the following notations:

$$\Lambda_1 = \operatorname{diag}\{\gamma_1, \gamma_2, \ldots, \gamma_n\},$$

$$\Lambda_2 = \operatorname{diag}\{\nu_1, \nu_2, \ldots, \nu_n\},$$

$$D_L = \operatorname{diag}\left\{\sum_{k=1}^l \frac{D_{1k}}{L_k^2}, \sum_{k=1}^l \frac{D_{2k}}{L_k^2}, \ldots, \sum_{k=1}^l \frac{D_{nk}}{L_k^2}\right\}.$$

Now we present our main results as follows.

Theorem 1 *For given scalars $\bar{\tau} > 0$, $\mu < 1$, under Assumptions 1 and 2, the two coupled reaction-diffusion neural networks (2) and (3) are asymptotically synchronized in the mean square, if there exist a scalar $\lambda > 0$, matrices $Q_1 > 0$, $Q_2 > 0$, $Q_3 > 0$, $R > 0$, $T > 0$, diagonal matrices $P > 0$, $H_1 > 0$, $H_2 > 0$, such that the following LMIs hold:*

$$P - \lambda I \leq 0, \tag{5}$$

$$\Xi = \begin{bmatrix} \Xi_{11} & PK_2 & 0 & PA + \frac{H_1(\Lambda_1+\Lambda_2)}{2} & PB & T \\ * & \Xi_{22} & 0 & 0 & \frac{H_2(\Lambda_1+\Lambda_2)}{2} & 0 \\ * & * & -Q_2 & 0 & 0 & -T \\ * & * & * & Q_3 - H_1 & 0 & 0 \\ * & * & * & * & (\mu-1)Q_3 - H_2 & 0 \\ * & * & * & * & * & -\frac{1}{\bar{\tau}}R \end{bmatrix} < 0, \tag{6}$$

where

$$\Xi_{11} = -2PC - 2PD_L + PK_1 + K_1^T P^T + Q_1 + Q_2 + \bar{\tau}R + \lambda\rho_1 I - \Lambda_1 H_1 \Lambda_2,$$

$$\Xi_{22} = -(1-\mu)Q_1 - \Lambda_1 H_2 \Lambda_2 + \lambda\rho_2 I,$$

$$P = \operatorname{diag}\{P_1, P_2, \ldots, P_n\}.$$

Proof Define a Lyapunov–Krasovskii functional candidate for system (4) as

$$V\big(t,e(t,x)\big) = \int_\Psi e^T(t,x)Pe(t,x)\,dx$$

$$+\int_{\Psi}\int_{t-\tau(t)}^{t} e^{T}(s,x)Q_1 e(s,x)\,ds\,dx$$
$$+\int_{\Psi}\int_{t-\bar{\tau}}^{t} e^{T}(s,x)Q_2 e(s,x)\,ds\,dx$$
$$+\int_{\Psi}\int_{t-\tau(t)}^{t} g^{T}\big(e(s,x)\big)Q_3 g\big(e(s,x)\big)\,ds\,dx$$
$$+\int_{\Psi}\int_{-\bar{\tau}}^{0}\int_{t+\theta}^{t} e^{T}(s,x)Re(s,x)\,ds\,d\theta\,dx$$
$$+\int_{\Psi}\left[\int_{t-\bar{\tau}}^{t} e^{T}(s,x)\,dsT\int_{t-\bar{\tau}}^{t} e(s,x)ds\right]dx. \quad (7)$$

By using *Itô* differential formula, we obtain

$$\begin{aligned}\mathcal{L}V\big(t,e(t,x)\big) = {} & 2\int_{\Psi} e^{T}(t,x)P\Bigg[\sum_{k=1}^{l}\frac{\partial}{\partial x_k} \\ & \times\left(D_k\frac{\partial e(t,x)}{\partial x_k}\right) - Ce(t,x) \\ & + Ag\big(e(t,x)\big) + Bg\big(e(t-\tau(t),x)\big) \\ & + K_1 e(t,x) + K_2 e\big(t-\tau(t),x\big)\Bigg] \\ & + \operatorname{trace}\big(\sigma^{T}(t)P\sigma(t)\big) \\ & + \int_{\Psi} e^{T}(t,x)Q_1 e(t,x)\,dx \\ & - \big(1-\dot{\tau}(t)\big)\int_{\Psi} e^{T}\big(t-\tau(t),x\big) \\ & \times Q_1 e\big(t-\tau(t),x\big)\,dx \\ & + \int_{\Psi} e^{T}(t,x)Q_2 e(t,x)\,dx \\ & - \int_{\Psi} e^{T}\big(t-\bar{\tau},x\big)Q_2 e\big(t-\bar{\tau},x\big)dx \\ & + \int_{\Psi} g^{T}\big(e(t,x)\big)Q_3 g\big(e(t,x)\big)\,dx \\ & - \big(1-\dot{\tau}(t)\big)\int_{\Psi} g^{T}\big(e\big(t-\tau(t),x\big)\big) \\ & \times Q_3 g\big(e\big(t-\tau(t),x\big)\big)\,dx \\ & + \bar{\tau}\int_{\Psi} e^{T}(t,x)Re(t,x)\,dx \\ & - \int_{\Psi}\int_{t-\bar{\tau}}^{t} e^{T}(s,x)Re(s,x)\,ds\,dx \\ & + 2\int_{\Psi}\Bigg[\big(e^{T}(t,x) - e^{T}(t-\bar{\tau},x)\big) \\ & \times T\int_{t-\bar{\tau}}^{t} e(s,x)\,ds\Bigg]dx. \end{aligned} \quad (8)$$

From Green formula and Dirichlet boundary conditions, we have

$$\begin{aligned}&\int_{\Psi} e_i(t,x)P_i\sum_{k=1}^{l}\frac{\partial}{\partial x_k}\left(D_{ik}\frac{\partial e_i(t,x)}{\partial x_k}\right)dx \\ &= \int_{\partial\Psi}\left(e_i(t,x)P_i D_{ik}\frac{\partial e_i(t,x)}{\partial x_k}\right)_{k=1}^{l}\cdot\bar{n}\,dS \\ &\quad - \int_{\Psi} P_i\sum_{k=1}^{l} D_{ik}\left(\frac{\partial e_i(t,x)}{\partial x_k}\right)^2 dx \\ &= -\int_{\Psi} P_i\sum_{k=1}^{l} D_{ik}\left(\frac{\partial e_i(t,x)}{\partial x_k}\right)^2 dx, \end{aligned} \quad (9)$$

where

$$\begin{aligned}&\left(e_i(t,x)P_i D_{ik}\frac{\partial e_i(t,x)}{\partial x_k}\right)_{k=1}^{l} \\ &= \Bigg(e_i(t,x)P_i D_{i1}\frac{\partial e_i(t,x)}{\partial x_1},\ldots, \\ &\qquad e_i(t,x)P_i D_{il}\frac{\partial e_i(t,x)}{\partial x_l}\Bigg)^{T}.\end{aligned}$$

By Lemma 1, it can be seen that

$$\begin{aligned}&-\int_{\Psi} P_i\sum_{k=1}^{l} D_{ik}\left(\frac{\partial e_i(t,x)}{\partial x_k}\right)^2 dx \\ &\le -\int_{\Psi}\sum_{k=1}^{l} P_i\frac{D_{ik}}{L_k^2}e_i^2(t,x)\,dx. \end{aligned} \quad (10)$$

Thus,

$$\begin{aligned}&2\int_{\Psi} e^{T}(t,x)P\sum_{k=1}^{l}\frac{\partial}{\partial x_k}\left(D_k\frac{\partial e(t,x)}{\partial x_k}\right)dx \\ &= 2\int_{\Psi}\sum_{i=1}^{n} e_i(t,x)P_i\sum_{k=1}^{l}\frac{\partial}{\partial x_k}\left(D_{ik}\frac{\partial e_i(t,x)}{\partial x_k}\right)dx\end{aligned}$$

$$\leq -2\int_\Psi \sum_{i=1}^{n} P_i \sum_{k=1}^{l} \frac{D_{ik}}{L_k^2} e_i^2(t,x)\,dx$$
$$= -2\int_\Psi e^T(t,x) P D_L e(t,x)\,dx. \tag{11}$$

From Assumption 2 and (5), we have

$$\operatorname{trace}\big(\sigma^T(t)P\sigma(t)\big) \leq \lambda\rho_1 e^T(t,x)e(t,x) + \lambda\rho_2 e^T\big(t-\tau(t),x\big) \times e\big(t-\tau(t),x\big). \tag{12}$$

It can be deduced from Assumption 1 that, for diagonal matrices $H_1 > 0$, $H_2 > 0$, the following inequalities hold:

$$0 \geq e^T(t,x)\Lambda_1 H_1 \Lambda_2 e(t,x) - e^T(t,x)H_1(\Lambda_1+\Lambda_2) \times g\big(e(t,x)\big) + g^T\big(e(t,x)\big)H_1 g\big(e(t,x)\big), \tag{13}$$

$$0 \geq e^T\big(t-\tau(t),x\big)\Lambda_1 H_2 \Lambda_2 e\big(t-\tau(t),x\big) - e^T\big(t-\tau(t),x\big)H_2(\Lambda_1+\Lambda_2)g\big(e\big(t-\tau(t),x\big)\big) + g^T\big(e\big(t-\tau(t),x\big)\big)H_2 g\big(e\big(t-\tau(t),x\big)\big). \tag{14}$$

In addition, it is easy to get the following inequality from Jensen inequality:

$$-\int_\Psi \int_{t-\bar\tau}^{t} e^T(s,x)Re(s,x)\,ds\,dx \leq -\frac{1}{\bar\tau}\int_\Psi \bigg[\int_{t-\bar\tau}^{t} e(s,x)\,ds\bigg]^T R\bigg[\int_{t-\bar\tau}^{t} e(s,x)\,ds\bigg]dx. \tag{15}$$

Combining (8)–(15) results in

$$\begin{aligned}
&\mathcal{L}V\big(t,e(t,x)\big)\\
&\leq \int_\Psi e^T(t,x)\big[-2PD_L - 2PC + PK_1 + K_1^T P^T + Q_1 + Q_2 + \bar\tau R + \lambda\rho_1 I - \Lambda_1 H_1 \Lambda_2\big]e(t,x)dx\\
&\quad + \int_\Psi e^T(t,x)\big[2PA + H_1(\Lambda_1+\Lambda_2)\big] \times g\big(e(t,x)\big)\,dx\\
&\quad + 2\int_\Psi e^T(t,x)PBg\big(e\big(t-\tau(t),x\big)\big)\,dx\\
&\quad + 2\int_\Psi e^T(t,x)PK_2 e\big(t-\tau(t),x\big)\,dx\\
&\quad + \int_\Psi e^T\big(t-\tau(t),x\big)H_2(\Lambda_1+\Lambda_2) \times g\big(e\big(t-\tau(t),x\big)\big)\,dx\\
&\quad + \int_\Psi e^T\big(t-\tau(t),x\big)\big[\lambda\rho_2 I - (1-\mu)Q_1 - \Lambda_1 H_2 \Lambda_2\big]e\big(t-\tau(t),x\big)\,dx\\
&\quad - \int_\Psi e^T(t-\bar\tau,x)Q_2 e(t-\bar\tau,x)\,dx\\
&\quad + \int_\Psi g^T\big(e(t,x)\big)(Q_3 - H_1)g\big(e(t,x)\big)\,dx\\
&\quad + 2\int_\Psi e^T(t,x)T\int_{t-\bar\tau}^{t} e(s,x)\,ds\,dx\\
&\quad - 2\int_\Psi e^T(t-\bar\tau,x)T\int_{t-\bar\tau}^{t} e(s,x)\,ds\,dx\\
&\quad - \frac{1}{\bar\tau}\int_\Psi \bigg[\int_{t-\bar\tau}^{t} e(s,x)\,ds\bigg]^T R\bigg[\int_{t-\bar\tau}^{t} e(s,x)\,ds\bigg]dx\\
&= \int_\Psi \xi^T(t,x)\Xi\xi(t,x)\,dx\\
&\leq -\lambda_{\min}(-\Xi)\|\xi(t,x)\|_2^2\\
&< 0,
\end{aligned}$$

for $\xi(t,x) \neq 0$, where

$$\xi^T(t,x) = \big[e^T(t,x)\ e^T\big(t-\tau(t),x\big)\ e^T(t-\bar\tau,x)\ g^T\big(e(t,x)\big)\ g^T\big(e\big(t-\tau(t),x\big)\big)\ \textstyle\int_{t-\bar\tau}^{t} e^T(s,x)ds\big].$$

Thus, it is easy to see that system of (2) and (3) is asymptotically synchronized in the mean square through Lyapunov–Krasovskii theory. This completes the proof. □

Based on Theorem 1, we are now ready to give the parameterization of the controller gains in the following theorem.

Theorem 2 *For given scalars $\bar\tau > 0$, $\mu < 1$, under Assumptions 1 and 2, the two coupled reaction-diffusion neural networks (2) and (3) are asymptotically synchronized in the mean square, if there exist a scalar $\lambda > 0$, matrices $Q_1 > 0$, $Q_2 > 0$, $Q_3 > 0$, $R > 0$, $T > 0$, X_1, X_2, diagonal matrices $P > 0$, $H_1 > 0$, $H_2 > 0$, such that the following LMIs hold:*

$$P - \lambda I \leq 0, \tag{16}$$

$$\Xi = \begin{bmatrix} \Xi_{11} & X_2 & 0 & PA + \frac{H_1(\Lambda_1+\Lambda_2)}{2} & PB & T \\ * & \Xi_{22} & 0 & 0 & \frac{H_2(\Lambda_1+\Lambda_2)}{2} & 0 \\ * & * & -Q_2 & 0 & 0 & -T \\ * & * & * & Q_3 - H_1 & 0 & 0 \\ * & * & * & * & (\mu-1)Q_3 - H_2 & 0 \\ * & * & * & * & * & -\frac{1}{\bar{\tau}}R \end{bmatrix} < 0, \tag{17}$$

where

$$\Xi_{11} = -2PC - 2PD_L + X_1 + X_1^T + Q_1 + Q_2 + \bar{\tau}R + \lambda\rho_1 I - \Lambda_1 H_1 \Lambda_2,$$

$$\Xi_{22} = -(1-\mu)Q_1 - \Lambda_1 H_2 \Lambda_2 + \lambda\rho_2 I,$$

$$P = \mathrm{diag}\{P_1, P_2, \ldots, P_n\}.$$

Proof Let $K_1 = P^{-1}X_1$, $K_2 = P^{-1}X_2$ in Theorem 1, then we can obtain the desired result immediately. □

Remark 1 Based on the Lyapunov stability theory for stochastic systems, analysis method for partial differential equation and the drive-response concept, we have provided theoretical results in Theorems 1 and 2 on asymptotic synchronization in the mean square of stochastic delayed neural networks with reaction-diffusion term. The results are expressed by a set of LMIs, which can be solved readily using Matlab LMI Toolbox.

Remark 2 If we set $\sigma(t, e(t,x), e(t-\tau(t),x)) = 0$, the synchronization problem for reaction-diffusion neural networks without stochastic perturbation has been studied in [7, 11, 20, 34, 37, 41]. However, the results of these are not presented in terms of LMIs, which makes their checking by the developed algorithms somewhat difficult and inconvenient. Furthermore, the sign of elements in connection weight is not considered in these provided criteria, which may lead to conservatism to some extent.

Remark 3 It is worth noting that almost all results about dynamics analysis or synchronization problem about reaction-diffusion system are delay-independent in the literature due to the difficulty in dealing with the reaction-diffusion term. In our results, the delay-dependent conditions are derived in virtue of the appropriate Lyapunov–Krasovskii functional.

The following theorem represents the almost sure synchronization result.

Theorem 3 *For given scalars $\bar{\tau} > 0$, $\mu < 1$, under Assumptions 1 and 2, the reaction-diffusion neural networks* (2) *and* (3) *are almost surely synchronized, if there exist a scalar $\lambda > 0$, matrices $Q_1 > 0$, $Q_2 > 0$, $Q_3 > 0$, $R > 0$, $T > 0$, $W > 0$, diagonal matrices $P > 0$, $H_1 > 0$, $H_2 > 0$, such that the following LMIs hold*:

$$P - \lambda I \leq 0, \tag{18}$$

$$\Pi = \begin{bmatrix} \Pi_{11} & PK_2 & 0 & PA + \frac{H_1(\Lambda_1+\Lambda_2)}{2} & PB & T \\ * & \Pi_{22} & 0 & 0 & \frac{H_2(\Lambda_1+\Lambda_2)}{2} & 0 \\ * & * & -Q_2 & 0 & 0 & -T \\ * & * & * & Q_3 - H_1 & 0 & 0 \\ * & * & * & * & (\mu-1)Q_3 - H_2 & 0 \\ * & * & * & * & * & -\frac{1}{\bar{\tau}}R \end{bmatrix} < 0, \tag{19}$$

where

$$\Pi_{11} = -2PC - 2PD_L + PK_1 + K_1^T P^T + Q_1 + Q_2 + \bar{\tau} R + \lambda\rho_1 I - \Lambda_1 H_1 \Lambda_2 + W,$$
$$\Pi_{22} = -(1-\mu)Q_1 - \Lambda_1 H_2 \Lambda_2 + \lambda\rho_2 I - W,$$
$$P = \mathrm{diag}\{P_1, P_2, \ldots, P_n\}.$$

Proof Define a Lyapunov–Krasovskii functional candidate for system (4) as

$$\begin{aligned}
V\big(t, e(t,x)\big) &= \int_\Psi e^T(t,x) P e(t,x)\,dx \\
&\quad + \int_\Psi \int_{t-\tau(t)}^t e^T(s,x) Q_1 e(s,x)\,ds\,dx \\
&\quad + \int_\Psi \int_{t-\bar{\tau}}^t e^T(s,x) Q_2 e(s,x)\,dsdx \\
&\quad + \int_\Psi \int_{t-\tau(t)}^t g^T\big(e(s,x)\big) Q_3 g\big(e(s,x)\big)\,ds\,dx \\
&\quad + \int_\Psi \int_{-\bar{\tau}}^0 \int_{t+\theta}^t e^T(s,x) R e(s,x)\,ds\,d\theta\,dx \\
&\quad + \int_\Psi \left[\int_{t-\bar{\tau}}^t e^T(s,x)\,ds T \int_{t-\bar{\tau}}^t e(s,x)\,ds\right] dx.
\end{aligned} \tag{20}$$

By using *Itô* differential formula and after some derivation, we obtain

$$\begin{aligned}
\mathcal{L}V\big(t, e(t,x)\big) &\le \int_\Psi e^T(t,x)\big[-2PD_L - 2PC \\
&\quad + 2PK_1 + Q_1 + Q_2 + \bar{\tau} R + \lambda\rho_1 I \\
&\quad - \Lambda_1 H_1 \Lambda_2\big] e(t,x)\,dx \\
&\quad + \int_\Psi e^T(t,x)\big[2PA + H_1(\Lambda_1 + \Lambda_2)\big] \\
&\quad \times g\big(e(t,x)\big)\,dx \\
&\quad + 2\int_\Psi e^T(t,x) PBg\big(e\big(t-\tau(t),x\big)\big)\,dx \\
&\quad + 2\int_\Psi e^T(t,x) PK_2 e\big(t-\tau(t),x\big)\,dx \\
&\quad + \int_\Psi e^T\big(t-\tau(t),x\big) H_2(\Lambda_1 + \Lambda_2) \\
&\quad \times g\big(e\big(t-\tau(t),x\big)\big)\,dx \\
&\quad + \int_\Psi e^T\big(t-\tau(t),x\big)\big[\lambda\rho_2 I - (1-\mu)Q_1 \\
&\quad - \Lambda_1 H_2 \Lambda_2\big] e\big(t-\tau(t),x\big)\,dx \\
&\quad - \int_\Psi e^T(t-\bar{\tau},x) Q_2 e(t-\bar{\tau},x)\,dx \\
&\quad + \int_\Psi g^T\big(e(t,x)\big)(Q_3 - H_1) g\big(e(t,x)\big)\,dx \\
&\quad + 2\int_\Psi \left[\big(e^T(t,x) - e^T(t-\bar{\tau},x)\big) T \right. \\
&\quad \left. \times \int_{t-\bar{\tau}}^t e(s,x)\,ds\right] dx \\
&\quad - \frac{1}{\bar{\tau}} \int_\Psi \left[\int_{t-\bar{\tau}}^t e(s,x)\,ds\right]^T R \left[\int_{t-\bar{\tau}}^t e(s,x)\,ds\right] dx \\
&\quad + \int_\Psi e^T(t,x) W e(t,x)\,dx \\
&\quad - \int_\Psi e^T(t,x) W e(t,x)\,dx \\
&\quad + \int_\Psi e^T\big(t-\tau(t),x\big) W e\big(t-\tau(t),x\big)\,dx \\
&\quad - \int_\Psi e^T\big(t-\tau(t),x\big) W e\big(t-\tau(t),x\big)\,dx \\
&= \int_\Psi \xi^T(t,x) \Pi \xi(t,x)\,dx \\
&\quad - \int_\Psi e^T(t,x) W e(t,x)\,dx \\
&\quad + \int_\Psi e^T\big(t-\tau(t),x\big) W e\big(t-\tau(t),x\big)\,dx \\
&\le -\eta\left(\int_\Psi e^T(t,x) e(t,x)\,dx \right. \\
&\quad \left. + \int_\Psi e^T\big(t-\tau(t),x\big) e\big(t-\tau(t),x\big)\,dx\right) \\
&\quad - \int_\Psi e^T(t,x) W e(t,x)\,dx \\
&\quad + \int_\Psi e^T\big(t-\tau(t),x\big) W e\big(t-\tau(t),x\big)\,dx \\
&= -\int_\Psi e^T(t,x)(W + \eta I) e(t,x)\,dx \\
&\quad + \int_\Psi e^T\big(t-\tau(t),x\big)(W - \eta I) e\big(t-\tau(t),x\big)\,dx \\
&= -w_1\big(e(t,x)\big) + w_2\big(e\big(t-\tau(t),x\big)\big),
\end{aligned}$$

where $\eta = \lambda_{\min}(-\Pi) > 0$ and

$$\xi^T(t,x) = [e^T(t,x)\ e^T(t-\tau(t),x)\ e^T(t-\bar{\tau},x)\ g^T(e(t,x))\ g^T(e(t-\tau(t),x))\ \int_{t-\bar{\tau}}^{t} e^T(s,x)\,ds].$$

It is obvious that $w_1(e(t,x)) > w_2(e(t,x))$ for any $e(t,x) \neq 0$. Therefore, by LaSalle-type invariant principle of stochastic differential equation [14, 49], we can see that the system of (2) and (3) can be almost surely synchronized. The proof is completed. □

By the analysis result in Theorem 3, the controller gains K_1 and K_2 can be obtained readily.

Theorem 4 *For given scalars $\bar{\tau} > 0$, $\mu < 1$, under Assumptions 1 and 2, the two coupled reaction-diffusion neural networks (2) and (3) are almost surely synchronized, if there exist a scalar $\lambda > 0$, matrices $Q_1 > 0$, $Q_2 > 0$, $Q_3 > 0$, $R > 0$, $T > 0$, $W > 0$, X_1, X_2, diagonal matrices $P > 0$, $H_1 > 0$, $H_2 > 0$, such that the following LMIs hold*:

$$P - \lambda I \le 0, \tag{21}$$

$$\Pi = \begin{bmatrix} \Pi_{11} & X_2 & 0 & PA + \frac{H_1(\Lambda_1+\Lambda_2)}{2} & PB & T \\ * & \Pi_{22} & 0 & 0 & \frac{H_2(\Lambda_1+\Lambda_2)}{2} & 0 \\ * & * & -Q_2 & 0 & 0 & -T \\ * & * & * & Q_3 - H_1 & 0 & 0 \\ * & * & * & * & (\mu-1)Q_3 - H_2 & 0 \\ * & * & * & * & * & -\frac{1}{\bar{\tau}}R \end{bmatrix} < 0, \tag{22}$$

where

$$\Pi_{11} = -2PC - 2PD_L + X_1 + X_1^T + Q_1 + Q_2 + \bar{\tau}R + \lambda\rho_1 I - \Lambda_1 H_1 \Lambda_2 + W,$$

$$\Pi_{22} = -(1-\mu)Q_1 - \Lambda_1 H_2 \Lambda_2 + \lambda\rho_2 I - W,$$

$$P = \mathrm{diag}\{P_1, P_2, \ldots, P_n\}.$$

Proof Let $K_1 = P^{-1}X_1$, $K_2 = P^{-1}X_2$ in Theorem 3, then we can get the desired result readily. □

Remark 4 To the best of our knowledge, the almost sure synchronization problem for neural networks with reaction-diffusion still has not been investigated fully in the literature. By virtue of the LaSalle-type invariant principle of stochastic differential equation, almost sure synchronization conditions for stochastic reaction-diffusion neural networks are proposed in Theorems 3 and 4. The criteria are delay-dependent and expressed in terms of LMIs.

4 Illustrative examples

In this section, we shall give some examples to demonstrate the effectiveness of the proposed approach in the paper.

Example 1 Consider the following reaction-diffusion neural networks:

$$\frac{du(t,x)}{dt} = \frac{\partial}{\partial x}\left(D\frac{\partial u(t,x)}{\partial x}\right) - Cu(t,x) + Af\big(u(t,x)\big) + Bf\big(u\big(t-\tau(t),x\big)\big), \tag{23}$$

where

$$C = \begin{bmatrix} 1.0 & 0.0 \\ 0.0 & 1.0 \end{bmatrix}, \qquad A = \begin{bmatrix} 2.0 & -0.1 \\ -5.0 & 2.8 \end{bmatrix},$$

$$B = \begin{bmatrix} -1.6 & -0.1 \\ -0.3 & -2.5 \end{bmatrix}, \qquad D = \begin{bmatrix} 0.1 & 0.0 \\ 0.0 & 0.1 \end{bmatrix},$$

and

$$x \in [-2, 2], \quad f(\alpha) = \tanh(\alpha), \quad \tau(t) = 1.$$

The corresponding response system can be given as

$$d\tilde{u}(t,x) = \left[\frac{\partial}{\partial x}\left(D\frac{\partial \tilde{u}(t,x)}{\partial x}\right) - C\tilde{u}(t,x) + Af\big(\tilde{u}(t,x)\big) + Bf\big(\tilde{u}(t-\tau(t),x)\big) + V(t,x)\right]dt$$

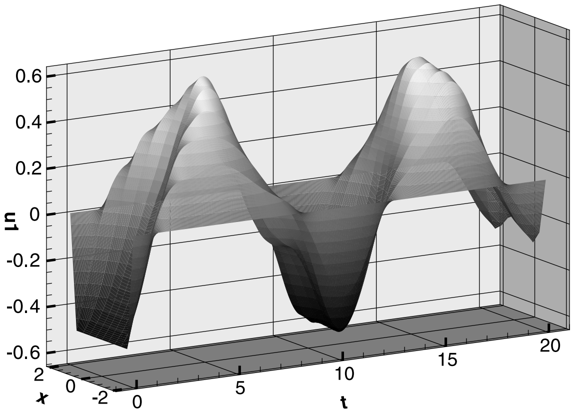

Fig. 1 Chaotic behaviors of $u_1(t,x)$ of system (23)

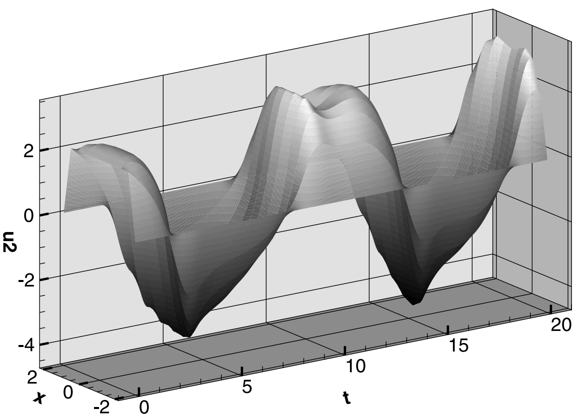

Fig. 2 Chaotic behaviors of $u_2(t,x)$ of system (23)

$$+\sigma\big(t, \tilde{u}(t,x) - u(t,x),\\ \tilde{u}\big(t-\tau(t),x\big) - u(t-\tau(t),x)\big)\,d\omega(t), \qquad (24)$$

where

$$\sigma\big(t, e(t,x), e\big(t-\tau(t),x\big)\big) = \begin{pmatrix} e_1(t,x) & 0 \\ 0 & e_2(t-\tau(t),x) \end{pmatrix}.$$

Thus, we can set $\rho_1 = \rho_2 = 1$. The initial conditions are $u(t,x) = [-0.5, 2.1]^T$, $\tilde{u}(t,x) = [0.5, -0.5]^T$, and the boundary conditions are set as Dirichlet boundary conditions. The simulation results of system (23) are provided in Figs. 1–2. The chaotic behavior on the section can be seen in Figs. 3 and 4, where x is set as -1.5 and 0.5, respectively.

By using the Matlab LMI control Toolbox to solve the LMIs in Theorem 2, we obtain a set of feasible

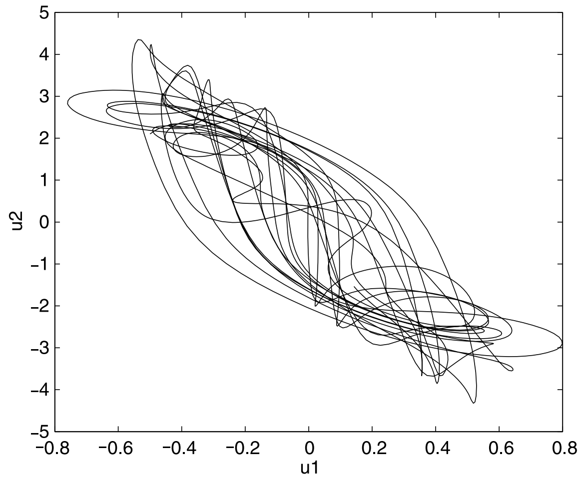

Fig. 3 Chaotic behaviors of system (23) when $x = -1.5$

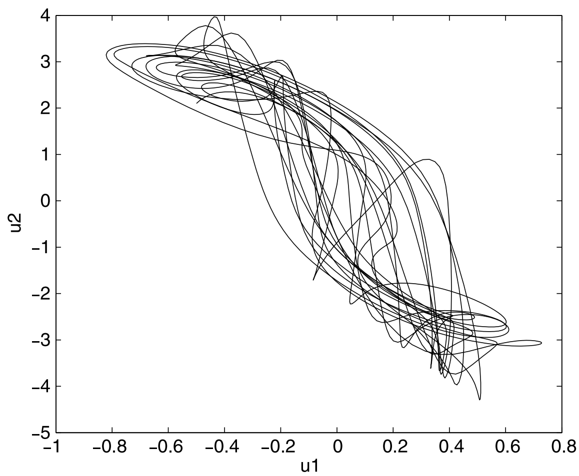

Fig. 4 Chaotic behaviors of system (23) when $x = 0.5$

solutions as

$$P = \begin{bmatrix} 13.3861 & 0 \\ 0 & 5.1928 \end{bmatrix},$$

$$Q_1 = \begin{bmatrix} 135.3701 & -0.2855 \\ -0.2855 & 135.6596 \end{bmatrix},$$

$$Q_2 = \begin{bmatrix} 85.0808 & 0.1991 \\ 0.1991 & 85.4081 \end{bmatrix},$$

$$Q_3 = \begin{bmatrix} 38.0962 & 2.4755 \\ 2.4755 & 41.2688 \end{bmatrix},$$

$$R = \begin{bmatrix} 91.0809 & 0.2434 \\ 0.2434 & 91.4243 \end{bmatrix},$$

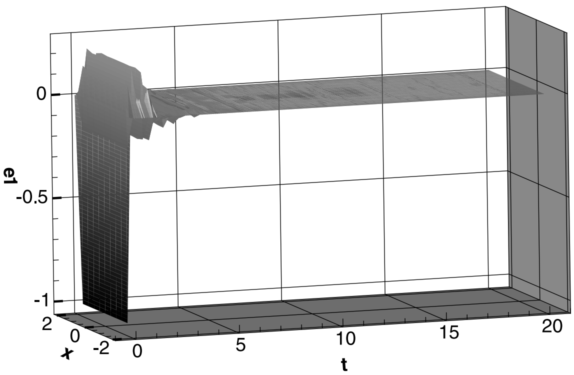

Fig. 5 Dynamical behavior of synchronization error $e_1(t,x)$ of Example 1

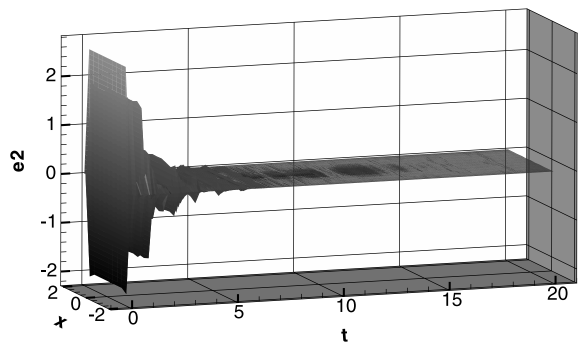

Fig. 6 Dynamical behavior of synchronization error $e_2(t,x)$ of Example 1

$$T=\begin{bmatrix}31.0997 & -0.1817\\ -0.1817 & 31.0819\end{bmatrix},$$

$$H_1=\begin{bmatrix}33.8932 & 0\\ 0 & 35.4738\end{bmatrix},$$

$$H_2=\begin{bmatrix}61.3025 & 0\\ 0 & 61.7208\end{bmatrix},$$

$$X_1=\begin{bmatrix}-268.9010 & 269.4968\\ -245.4324 & -265.1668\end{bmatrix},$$

$$X_2=\begin{bmatrix}7.1808 & 0.3631\\ -0.0322 & 4.5636\end{bmatrix}, \qquad \lambda=68.5803,$$

$$K_1=\begin{bmatrix}-20.0881 & 20.1326\\ -47.2644 & -51.0648\end{bmatrix},$$

$$K_2=\begin{bmatrix}0.5364 & 0.0271\\ -0.0062 & 0.8788\end{bmatrix}.$$

Therefore, the system of (23) and (24) with Dirichlet boundary conditions and parameters given in this example is asymptotically synchronized with the control gains K_1 and K_2. It is obvious that the information of reaction-diffusion terms plays an important part in synchronization. The dynamical behavior of the error system can be seen in Figs. 5 and 6.

Example 2 Consider the following reaction-diffusion neural networks with Dirichlet boundary conditions:

$$\frac{du(t,x)}{dt}=\frac{\partial}{\partial x}\left(D\frac{\partial u(t,x)}{\partial x}\right)-Cu(t,x)+Af\big(u(t,x)\big)+Bf\big(u\big(t-\tau(t),x\big)\big), \tag{25}$$

and

$$\begin{aligned}d\tilde{u}(t,x)=&\bigg[\frac{\partial}{\partial x}\left(D\frac{\partial \tilde{u}(t,x)}{\partial x}\right)-C\tilde{u}(t,x)\\&+Af\big(\tilde{u}(t,x)\big)+Bf\big(\tilde{u}(t-\tau(t),x)\big)\\&+V(t,x)\bigg]dt\\&+\sigma\big(t,\tilde{u}(t,x)-u(t,x),\\&\quad\tilde{u}\big(t-\tau(t),x\big)-u\big(t-\tau(t),x\big)\big)\,d\omega(t),\end{aligned} \tag{26}$$

where $x\in[-2,2]$, $f(\alpha)=\tanh(\alpha)$, $\tau(t)=1$, $\rho_1=\rho_2=1$, and

$$C=\begin{bmatrix}1.0 & 0.0\\ 0.0 & 1.0\end{bmatrix}, \qquad A=\begin{bmatrix}2.0 & -0.1\\ -5.0 & 3.2\end{bmatrix},$$

$$B=\begin{bmatrix}-1.5 & -0.1\\ -0.2 & -2.5\end{bmatrix}, \qquad D=\begin{bmatrix}0.1 & 0.0\\ 0.0 & 0.1\end{bmatrix}.$$

The simulation results of system (25) with the initial conditions $u(t,x)=[0.4,1.1]^T$ and $\tilde{u}(t,x)=[-0.2,-0.5]^T$ are given in Figs. 7 and 8. The chaotic behavior can be seen in Figs. 9 and 10, where x is set as -1.5 and 0.5, respectively.

Consider the problem of almost sure synchronization; a set of feasible solutions can be obtained by using the Matlab LMI control Toolbox to solve the LMIs in Theorem 4:

$$P=\begin{bmatrix}15.8687 & 0\\ 0 & 5.8611\end{bmatrix},$$

$$Q_1=\begin{bmatrix}96.8400 & -0.0836\\ -0.0836 & 96.9576\end{bmatrix}$$

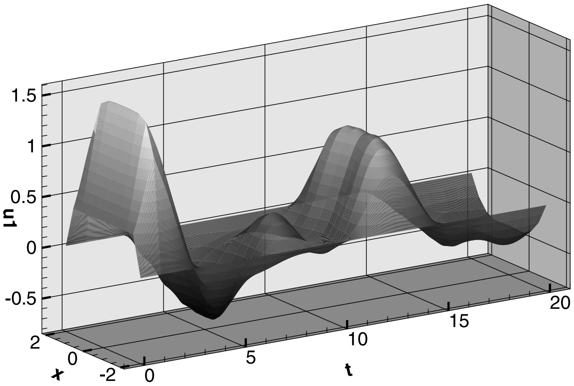

Fig. 7 Chaotic behaviors of $u_1(t, x)$ of system (25)

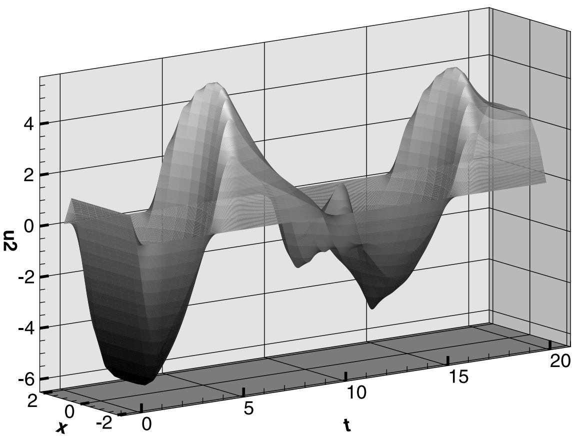

Fig. 8 Chaotic behaviors of $u_2(t, x)$ of system (25)

$$Q_2 = \begin{bmatrix} 97.0509 & 0.3342 \\ 0.3342 & 97.4978 \end{bmatrix},$$

$$Q_3 = \begin{bmatrix} 42.2129 & 2.4316 \\ 2.4316 & 45.7156 \end{bmatrix},$$

$$R = \begin{bmatrix} 104.3716 & 0.1388 \\ 0.1388 & 104.4742 \end{bmatrix},$$

$$T = \begin{bmatrix} 35.3721 & -0.1593 \\ -0.1593 & 35.4853 \end{bmatrix},$$

$$H_1 = \begin{bmatrix} 41.3979 & 0 \\ 0 & 43.1330 \end{bmatrix},$$

$$H_2 = \begin{bmatrix} 70.1457 & 0 \\ 0 & 70.0578 \end{bmatrix},$$

$$W = \begin{bmatrix} 96.8400 & -0.0836 \\ -0.0836 & 96.9576 \end{bmatrix},$$

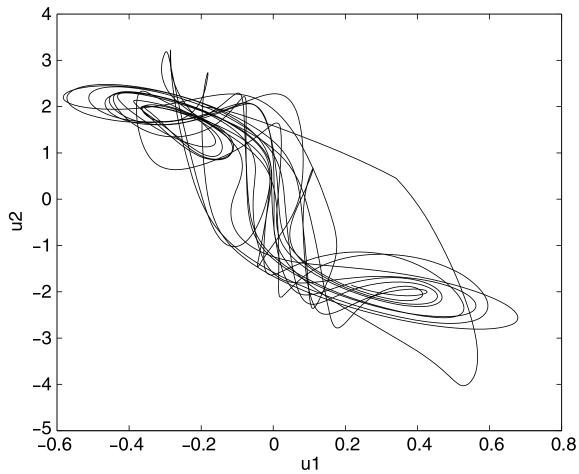

Fig. 9 Chaotic behaviors of system (25) when $x = -1.5$

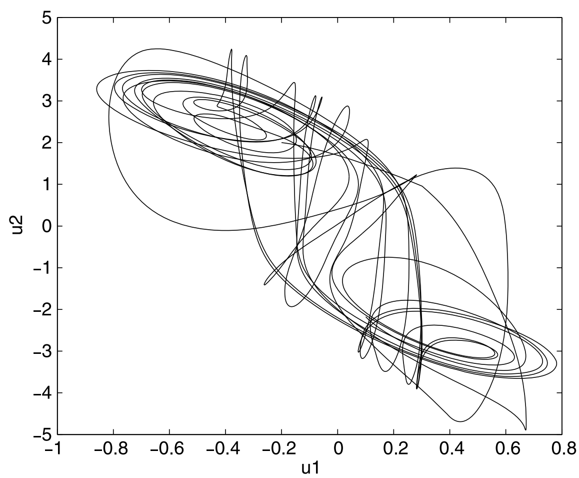

Fig. 10 Chaotic behaviors of system (25) when $x = 0.5$

$$X_1 = \begin{bmatrix} -334.7252 & -55.2896 \\ 84.0929 & -331.3464 \end{bmatrix},$$

$$X_2 = \begin{bmatrix} 7.9380 & 0.3609 \\ 0.3344 & 4.5805 \end{bmatrix}, \qquad \lambda = 94.5999,$$

$$K_1 = \begin{bmatrix} -21.0934 & -3.4842 \\ 14.3476 & -56.5329 \end{bmatrix},$$

$$K_2 = \begin{bmatrix} 0.5002 & 0.0227 \\ 0.0571 & 0.7815 \end{bmatrix}.$$

Thus, from Theorem 4, the system of (25) and (26) is almost surely synchronized with the control gains K_1 and K_2. The dynamical behavior of the synchronization error can be seen in Figs. 11 and 12.

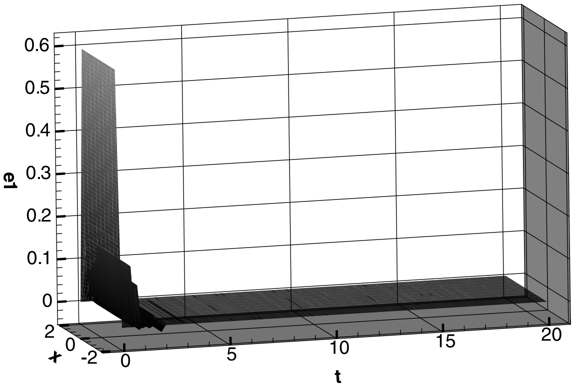

Fig. 11 Dynamical behavior of synchronization error $e_1(t, x)$ of Example 2

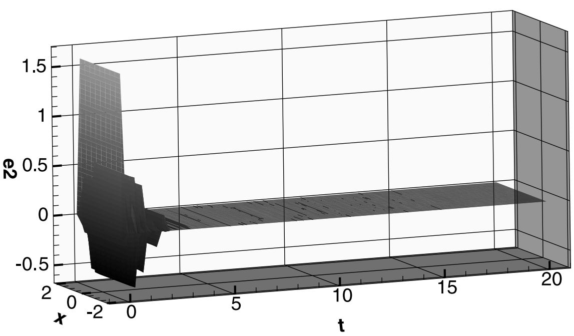

Fig. 12 Dynamical behavior of synchronization error $e_2(t, x)$ of Example 2

Remark 5 If the diffusion coefficients $D = 0$, system (23) and (25) become ordinary differential equations and the chaotic attractor has been given in existing literature [14, 16, 31, 42, 49]. In our examples, to demonstrate the chaotic behavior of reaction-diffusion chaos systems, we show the chaotic attractor on certain sections, for instance, $x = -1.5$ and $x = 0.5$. It is worth noting that such an approach has not been utilized to date. There are some quantitative methods to verify chaos such as the Lyapunov exponent and Kolmogorov entropy, which will be investigated in our future work.

5 Conclusions

In this paper, we have considered the synchronization problem for delayed stochastic neural networks with reaction-diffusion terms. Delay-dependent criteria have been obtained guaranteeing asymptotic synchronization in the mean square and almost sure synchronization of the considered systems, respectively. These conditions are given in terms of LMIs. The effectiveness of the proposed approach has been demonstrated via simulation examples.

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References

1. Cao, J., Lu, J.: Adaptive synchronization of neural networks with or without time-varying delay. Chaos **16**, 013133 (2006)
2. Cao, J., Li, P., Wang, W.: Global synchronization in arrays of delayed neural networks with constant and delayed coupling. Phys. Lett. A **353**, 318–325 (2006)
3. Cao, J., Wang, Z., Sun, Y.: Synchronization in an array of linearly stochastically coupled networks with time delays. Physica A **385**, 718–728 (2008)
4. Evans, L.C.: Partial Differential Equations. American Mathematical Society, Providence (1998)
5. Gan, Q., Xu, R., Yang, P.: Stability analysis of stochastic fuzzy cellular neural networks with time-varying delays and reaction-diffusion terms. Neural Process. Lett. **32**, 45–57 (2010)
6. Gilli, M.: Strange attractors in delayed cellular neural networks. IEEE Trans. Circuits Syst. I **40**, 849–853 (1993)
7. Hu, C., Jiang, H., Teng, Z.: Impulsive control and synchronization for delayed neural networks with reaction-diffusion terms. IEEE Trans. Neural Netw. **21**, 67–81 (2010)
8. Huang, H., Feng, G.: Synchronization of nonidentical chaotic neural networks with time delays. Neural Netw. **22**, 1841–1845 (2009)
9. Huang, H., Feng, G., Sun, Y.: Robust synchronization of chaotic systems subject to parameter uncertainties. Chaos **19**, 033128 (2009)
10. Kwon, O.M., Lee, S.M., Park, J.H.: Improved delay-dependent exponential stability for uncertain stochastic neural networks with time-varying delays. Phys. Lett. A **374**, 1232–1241 (2010)
11. Li, S., Yang, H., Lou, X.: Adaptive exponential synchronization of delayed neural networks with reaction-diffusion terms. Chaos Solitons Fractals **40**, 930–939 (2009)
12. Li, T., Song, A., Fei, S., Guo, Y.: Synchronization control of chaotic neural networks with time-varying and distributed delays. Nonlinear Anal. **71**, 2372–2384 (2009)
13. Li, W., Chang, K.: Robust synchronization of drive-response chaotic systems via adaptive sliding mode control. Chaos Solitons Fractals **39**, 2086–2092 (2009)
14. Li, X., Cao, J.: Adaptive synchronization for delayed neural networks with stochastic perturbation. J. Franklin Inst. **345**, 779–791 (2008)
15. Li, X., Fu, X.: Synchronization of chaotic delayed neural networks with impulsive and stochastic perturbations. Commun. Nonlinear Sci. Numer. Simul. **16**, 885–894 (2011)

16. Li, X., Ding, C., Zhu, Q.: Synchronization of stochastic perturbed chaotic neural networks with mixed delays. J. Franklin Inst. **347**, 1266–1280 (2010)
17. Liang, J., Wang, Z., Liu, X.: Exponential synchronization of stochastic delayed discrete-time complex networks. Nonlinear Dyn. **53**, 153–165 (2008)
18. Liao, X., Fu, Y., Gao, J., Zhao, X.: Stability of Hopfield neural networks with reaction-diffusion terms. Acta Electron. Sin. **28**, 78–81 (2000)
19. Liu, Z., Peng, J.: Delay-independent stability of stochastic reaction-diffusion neural networks with Dirichlet boundary conditions. Neural Comput. Appl. **19**, 151–158 (2010)
20. Lou, X., Cui, B.: Asymptotic synchronization of a class of neural networks with reaction-diffusion terms and time-varying delays. Comput. Math. Appl. **52**, 897–904 (2006)
21. Lou, X., Cui, B.: Synchronization of neural networks based on parameter identification and via output or state coupling. J. Comput. Appl. Math. **222**, 440–457 (2008)
22. Lu, H.: Chaotic attractors in delayed neural networks. Phys. Lett. A **298**, 109–116 (2002)
23. Lu, J.: Robust global exponential stability for interval reaction-diffusion Hopfield neural networks with distributed delays. IEEE Trans. Circuits Syst. II **54**, 1115–1119 (2007)
24. Lu, J., Cao, J.: Adaptive synchronization of uncertain dynamical networks with delayed coupling. Nonlinear Dyn. **53**, 107–115 (2008)
25. Luo, Q., Deng, F., Bao, J., Zhao, B., Fu, Y.: Stabilization of stochastic Hopfield neural network with distributed parameters. Sci. China Ser. F **47**, 752–762 (2004)
26. Ojalvo, J.G., Roy, R.: Spatiotemporal communication with synchronized optical chaos. Phys. Rev. Lett. **86**, 5204–5207 (2001)
27. Pan, J., Liu, X., Zhong, S.: Stability criteria for impulsive reaction-diffusion Cohen–Grossberg neural networks with time-varying delays. Math. Comput. Model. **51**, 1037–1050 (2010)
28. Park, J.H., Kwon, O.M.: Analysis on global stability of stochastic neural networks of neutral type. Mod. Phys. Lett. B **22**, 3159–3170 (2008)
29. Park, J.H., Kwon, O.M.: Synchronization of neural networks of neutral type with stochastic perturbation. Mod. Phys. Lett. B **23**, 1743–1751 (2009)
30. Song, Q.: Design of controller on synchronization of chaotic neural networks with mixed time-varying delays. Neurocomputing **72**, 3288–3295 (2009)
31. Sun, Y., Cao, J.: Adaptive lag synchronization of unknown chaotic delayed neural networks with noise perturbation. Phys. Lett. A **364**, 277–285 (2007)
32. Tanelli, M., Picasso, B., Bolzern, P., Colaneri, P.: Almost sure stabilization of uncertain continuous-time Markov jump linear systems. IEEE Trans. Autom. Control **55**, 195–201 (2010)
33. Tang, Y., Qiu, R., Fang, J., Miao, Q., Xia, M.: Adaptive lag synchronization in unknown stochastic chaotic neural networks with discrete and distributed time-varying delays. Phys. Lett. A **372**, 4425–4433 (2008)
34. Wang, K., Teng, Z., Jiang, H.: Global exponential synchronization in delayed reaction-diffusion cellular neural networks with the Dirichlet boundary conditions. Math. Comput. Model. **52**, 12–24 (2010)
35. Wang, L., Zhang, F., Wang, Y.: Stochastic exponential stability of the delayed reaction-diffusion recurrent neural networks with Markovian jumping parameters. Phys. Lett. A **372**, 3201–3209 (2008)
36. Wang, L., Zhang, Y., Zhang, Z., Wang, Y.: LMI-based approach for global exponential robust stability for reaction-diffusion uncertain neural networks with time-varying delay. Chaos Solitons Fractals **41**, 900–905 (2009)
37. Wang, Y., Cao, J.: Synchronization of a class of delayed neural networks with reaction-diffusion terms. Phys. Lett. A **369**, 201–211 (2007)
38. Wang, Z.S., Zhang, H.: Global asymptotic stability of reaction-diffusion Cohen–Grossberg neural networks with continuously distributed delays. IEEE Trans. Neural Netw. **21**, 39–49 (2010)
39. Wang, Z.S., Zhang, H., Li, P.: An LMI approach to stability analysis of reaction-diffusion Cohen–Grossberg neural networks concerning Dirichlet boundary conditions and distributed delays. IEEE Trans. Syst. Man Cybern., Part B, Cybern. **40**, 1596–1606 (2010)
40. Yang, T., Chua, L.O.: Impulsive stabilization for control and synchronization of chaotic system: Theory and application to secure communication. IEEE Trans. Circuits Syst. I **44**, 976–988 (1997)
41. Yu, F., Jiang, H.: Global exponential synchronization of fuzzy cellular neural networks with delays and reaction-diffusion terms. Neurocomputing **74**, 509–515 (2011)
42. Yu, W., Cao, J.: Synchronization control of stochastic delayed neural networks. Physica A **373**, 252–260 (2007)
43. Yu, W., Cao, J., Lu, W.: Synchronization control of switched linearly coupled neural networks with delay. Neurocomputing **73**, 858–866 (2010)
44. Zhang, B., Xu, S., Zong, G., Zou, Y.: Delay-dependent exponential stability for uncertain stochastic Hopfield neural networks with time-varying delays. IEEE Trans. Circuits Syst. I **56**, 1241–1247 (2009)
45. Zhang, B., Xu, S., Zou, Y.: Improved delay-dependent exponential stability criteria for discrete-time recurrent neural networks with time-varying delays. Neurocomputing **72**, 321–330 (2008)
46. Zhang, H., Ma, T., Huang, G., Wang, Z.: Robust global exponential synchronization of uncertain chaotic delayed neural networks via dual-stage impulsive control. IEEE Trans. Syst. Man Cybern., Part B, Cybern. **40**, 831–843 (2010)
47. Zhang, Y., He, Z.: A secure communication scheme based on cellular neural networks. In: Proceedings of the IEEE International Conference on Intelligent Process Systems vol. 1, pp. 521–524 (1997)
48. Zhang, Y., Xu, S., Chu, Y., Lu, J.: Robust global synchronization of complex networks with neutral-type delayed nodes. Appl. Math. Comput. **216**, 768–778 (2010)
49. Zhu, Q., Cao, J.: Adaptive synchronization under almost every initial data for stochastic neural networks with time-varying delays and distributed delays. Commun. Nonlinear Sci. Numer. Simul. **16**, 2139–2159 (2011)