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ORIGINAL PAPER

Calculation of high-dimensional probability density functions of stochastically excited nonlinear mechanical systems

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Abstract Technical systems are subjected to a variety of excitations that cannot generally be described in deterministic ways. External disturbances like wind gusts or road roughness as well as uncertainties in system parameters can be described by random variables, with statistical parameters identified through measurements, for instance.

For general systems the statistical characteristics such as the probability density function (pdf) may be difficult to calculate. In addition to numerical simulation methods (Monte Carlo Simulations, MCS) there are differential equations for the pdf that can be solved to obtain such characteristics, most prominently the Fokker–Planck equation (FPE).

A variety of different approaches for solving FPEs for nonlinear systems have been investigated in the last decades. Most of these are limited to considerably low dimensions to avoid high numerical costs due to the "curse of dimension". Problems of higher dimension, such as $d = 6$, have been solved only rarely.

In this paper we present results for stationary pdfs of nonlinear mechanical systems with dimensions up to $d = 10$ using a Galerkin method, which expands approximative solutions (weighting functions) into orthogonal polynomials.

Keywords Fokker–Planck equation · Nonlinear systems · Multiple degree-of-freedom systems · Stochastic systems

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1 Introduction

The FPE is a linear homogeneous partial differential equation for the pdf $p(t, \mathbf{x})$ with variable coefficients:

$$\frac{\partial p(t,\mathbf{x})}{\partial t} + \sum_{i=1}^{d} \frac{\partial}{\partial x_i}\big[f_i(t,\mathbf{x})p(t,\mathbf{x})\big] - \frac{1}{2}\sum_{i=1}^{d}\sum_{j=1}^{d} \frac{\partial^2}{\partial x_i \partial x_j}\big[B_{ij}(t,\mathbf{x})p(t,\mathbf{x})\big] = 0. \tag{1}$$

The coefficients are obtained from the system of stochastic differential equations

$$\begin{aligned} &\mathrm{d}\mathbf{X}_t = \mathbf{f}(t,\mathbf{X}_t)\,\mathrm{d}t + \mathbf{G}(t,\mathbf{X}_t)\,\mathrm{d}\mathbf{W}_t \\ &\mathbf{X}_t \in \mathbb{R}^d;\ \mathbf{W}_t \in \mathbb{R}^m \\ &\mathbf{f} \text{ drift vector;} \\ &\mathbf{B} = \mathbf{G}\mathbf{G}^T \text{ diffusion matrix,} \end{aligned} \tag{2}$$

with the stochastic vector process $\mathbf{X}_t$ and the increment of the vector Wiener Process $\mathrm{d}\mathbf{W}_t$.

We consider mechanical systems, where the equations describe the system behavior under white noise excitation and may include additional filter equations. The system properties as well as the filter equations may be nonlinear, the excitation additive (external excitation) or multiplicative (parameter excitation). In this paper we will consider additive excitations only.

The dimension of the FPE (1) equals d for the stationary case corresponding to p as a function of all the state variables and $d + 1$ for the nonstationary case, with time t as additional variable. As a result, even for considerably small mechanical systems high-dimensional FPEs need to be solved.

Exact FPE solutions exist only for very restricted classes of problems. Linear systems under Gaussian excitation have Gaussian response pdfs. The well-known problem of the Duffing oscillator with additive white noise is an example of a nonlinear problem where the exact stationary response pdf is known. For a vast majority of general nonlinear systems, however, exact solutions are not available, which implies the need for numerical solution methods.

A general approach of numerically computing pdfs regardless of the nature of the mechanical problem lies in the numerical integration of the stochastic differential equations, known as Monte Carlo Simulation.

In 2006, Naess stated that "despite the common belief that a response pdf of any nonlinear system can be found numerically, this is far from reality" [1] and recent publications suggest that this still holds true.

There is a variety of approaches for the numerical solution of FPEs, but most of these are practicable only for considerably low dimensions, as stated before. Methods for the discretization of the state space include Finite Element and Finite Difference methods, such as in [2] and [3], but these methods bear the disadvantage that, generally, a d-dimensional infinite space has to be handled for the pdf. Semi-analytical methods include the path integral method (see for example [1, 4, 5]) and the cell mapping method [6]. The results reported in the literature for all of these methods have in common that they provide solutions to problems with dimension of up to four.

More recent approaches for the numerical solution of FPEs combine the discretization of either the state space or the time-domain with continuous methods. Feuersänger [7] presents a numerical sparse-grid method capable of solving problems of moderately higher dimension. FPEs of dimension up to $d = 6$ are solved approximately on finite rectangular domains. [8] uses a partition-of-unity finite element approach to solve problems of dimension four, applying higher-order polynomials to locally achieve higher resolutions.

The approach described in this paper is based on a Galerkin method that uses orthogonal polynomials to expand approximative weighting functions. The general procedure has been outlined in different works, such as [9–11], for example. In [12] results for the nonlinear dynamics of quarter car models with two degrees of freedom have been presented demonstrating the relevance to technical applications.

The fundamental advantage of this method is that in many applications some information about the shape of the pdf is available. This knowledge can be included in the weighting functions, so that the numerical costs can be decreased dramatically, allowing the solution of problems with higher dimension.

In the first part of the paper the general approach is demonstrated with the classical problem of the single-degree-of-freedom (SDOF) Duffing oscillator. The procedure is then extended to a similar problem with $d = 10$ to demonstrate its capabilities for problems of higher dimensions. Alternative types of weighting functions are presented for problems with strongly coupled state variables. As a last example a different type of nonlinearity describing the behavior of a quarter car model is considered. Throughout the paper, results from direct Monte Carlo Simulations will be used to compare the approximative results whenever exact reference solutions are not available. Ergodicity is assumed for all systems when generating MCS distribution estimates.

2 Introduction to the method

The general procedure of the presented method can be separated into three principal steps that have to be performed consecutively:

Step 1 In a first step we need to find an appropriate approximative solution, which will be used to generate the weighting functions for the Galerkin method. In many cases linearization and the calculation of corresponding Gaussian stationary pdfs leads to satisfactory approximative solutions, however, in particular cases it will be advantageous to use weighting functions that already include knowledge about the nonlinear behavior, as far as these are available.

Step 2 The second step of the procedure consists in calculating expansion polynomials according to orthogonality requirements based on the given weighting functions.

Step 3 Introducing the ansatz, which includes expansion polynomials and weighting functions, into the FPE leads to a residual R, which is a function of the expansion coefficients in our ansatz yet to be determined. Applying a Galerkin procedure to the residual leads to a linear homogeneous system of equations, which has to be solved to compute these coefficients.

In order to more comprehensively demonstrate the procedure we will outline these three steps in more detail, considering a comparably simple mechanical SDOF-problem.

The Duffing oscillator with additive white noise is an appropriate test problem as it provides an exact solution for comparison purposes. It is described by the following differential equation:

$$\ddot{X}_t + \gamma \dot{X}_t + \omega^2\big(1 + \varepsilon X_t^2\big)X_t = \sigma\omega\xi_t, \tag{3}$$

with linear damping γ, natural angular frequency of the linear system ω, excitation intensity σ and nonlinearity parameter ε.

Introducing $X_{1,t} = X_t$ and $X_{2,t} = \omega\dot{X}_t$ we get

$$\mathrm{d}X_{1,t} = \omega X_{2,t}\,\mathrm{d}t, \tag{4}$$

$$\mathrm{d}X_{2,t} = \big(-\gamma X_{2,t} - \omega\big(1 + \varepsilon X_{1,t}^2\big)X_{1,t}\big)\,\mathrm{d}t + \sigma\,\mathrm{d}W_t. \tag{5}$$

Step 1 In order to find a first approximative solution we consider the linearized system with $\varepsilon = 0$

$$\begin{aligned}\mathrm{d}X_{1,t} &= \omega X_{2,t}\,\mathrm{d}t,\\ \mathrm{d}X_{2,t} &= (-\gamma\dot{X}_t - \omega X_t)\,\mathrm{d}t + \sigma\,\mathrm{d}W_t,\end{aligned} \tag{6}$$

and the shape of the solution is known to be a zero-mean Gaussian distribution (e.g. [13]):

$$p(x_1, x_2) = c\exp\big(\alpha_{11}x_1^2 + \alpha_{12}x_1x_2 + \alpha_{22}x_2^2\big). \tag{7}$$

Here the constant c is used to scale $p(x_1, x_2)$ so that

$$\int_\infty^\infty \int_\infty^\infty p(x_1, x_2)\,\mathrm{d}x_1\,\mathrm{d}x_2 = 1. \tag{8}$$

In this simple case the unknown coefficients α_{ij} can be found by inserting (7) into the FPE (1) and we get

$$\begin{aligned}\alpha_{11} &= -\frac{\gamma}{\sigma^2},\\ \alpha_{12} &= 0,\\ \alpha_{22} &= -\frac{\gamma}{\sigma^2},\\ c &= \frac{\gamma}{\pi\sigma^2}.\end{aligned} \tag{9}$$

We see that in this case the state variables x_1 and x_2 are independent and we can write

$$\begin{aligned}p(x_1, x_2) &= c\exp\left(-\frac{\gamma}{\sigma^2}x_1^2\right)\exp\left(-\frac{\gamma}{\sigma^2}x_2^2\right)\\ &= cp_{x_1}(x_1)p_{x_2}(x_2).\end{aligned} \tag{10}$$

Here, $p_{x_1}(x_1)$ and $p_{x_2}(x_2)$ are chosen as weighting functions $G^{(1)}(x_1)$ and $G^{(2)}(x_2)$ for the Galerkin method to find solutions of the nonlinear problem.

Step 2 The weighting functions are used to generate a basis of polynomials in x_1 and x_2

$$P_k^{(1)}(x_1) = \sum_{i=0}^{k} a_{ki}^{(1)}x_1^i \quad \text{and} \tag{11}$$

$$P_k^{(2)}(x_2) = \sum_{i=0}^{k} a_{ki}^{(2)}x_2^i. \tag{12}$$

The coefficients of the polynomials are determined via the orthogonality conditions:

$$\begin{aligned}&\int_{-\infty}^{\infty} P_k^{(j)}(x_j)P_l^{(j)}(x_j)G^{(j)}(x_j)\,\mathrm{d}x_j\\ &\quad= \begin{cases}0, & k \neq l,\\ f_k, & k = l,\end{cases}\end{aligned} \tag{13}$$

and form a basis of the space of generalized Hermite-polynomials with arbitrary f_k.

Finding the polynomial coefficients $a_{kl}^{(j)}$ leads to a linear system of the following form ($k = 2$):

$$\begin{pmatrix} m_0^{(j)} & m_1^{(j)} & m_2^{(j)}\\ m_1^{(j)} & m_2^{(j)} & m_3^{(j)}\\ m_2^{(j)} & m_3^{(j)} & m_4^{(j)}\end{pmatrix}\begin{pmatrix} a_{20}^{(j)}\\ a_{21}^{(j)}\\ a_{22}^{(j)}\end{pmatrix} = \begin{pmatrix} 0\\ 0\\ \frac{f_2^{(j)}}{a_{22}^{(j)}}\end{pmatrix}. \tag{14}$$

The $m_k^{(j)}$s are the (Taylor-)moments with respect to the weighting function:

$$m_k^{(j)} = \int_{-\infty}^{\infty} x_j^k G^{(j)}(x_j)\,\mathrm{d}x_j. \tag{15}$$

With given weighting functions and orthogonal polynomial bases we then form an expansion for the approximative pdf:

$$\tilde{p}(x_1, x_2) = \sum_{i_1=0}^{n_1}\sum_{i_2=0}^{n_2} C_{i_1 i_2} P_{i_1}^{(1)}(x_1) P_{i_2}^{(2)}(x_2) \times G^{(1)}(x_1) G^{(2)}(x_2), \tag{16}$$

where n_1 and n_2 define the order of the expansion, which will determine the accuracy of the approximation.

Inserting (16) into the FPE (1) leads to differentiation of the polynomials as well as multiplication with the state variables x_i. In the case of linear weighting functions (and all exponential functions with polynomial argument) these operations yield expressions that can be expressed analytically in terms of the polynomial basis.

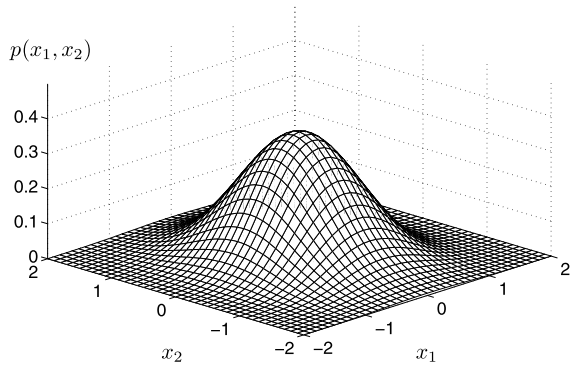
Step 3 The residual

$$R(C_{00}, C_{01}, \ldots, C_{n_1 n_2}) = R(\mathbf{C}) \tag{17}$$

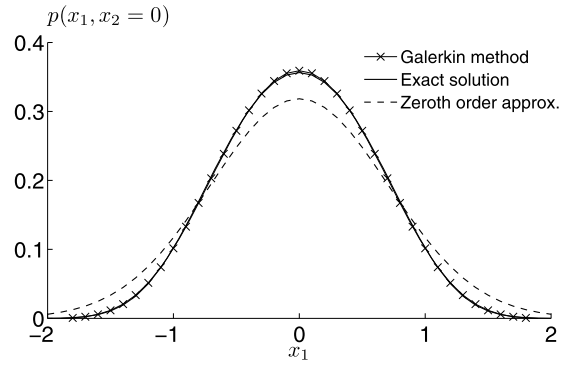
of the FPE (1) can thus be expressed by products of the expansion polynomials and the weighting functions. The basic principle of the presented Galerkin method lies in the projection of the residual with respect to the expansion polynomials (11) and (12) via integration over the entire scope of the FPE. In the sense of a Galerkin method we require that each of these projections is equal to zero, i.e.,

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R(\mathbf{C}) P_{h_1}^{(1)}(x_1) P_{h_2}^{(2)}(x_2)\,\mathrm{d}x_1\,\mathrm{d}x_2 = 0, \quad \text{for all } h_1, h_2. \tag{18}$$

With the residual expressed as a combination of the polynomial bases (11) and (12), we see that due to the orthogonality requirements (13) we are left with only those terms where the polynomial orders coincide. This leads to a homogeneous linear system of equations for the coefficient vector $\mathbf{C}$. The expansion of the residual into the orthogonal polynomials is such that in each equation only polynomials of order close to the considered expansion polynomial order appear. This leads to a sparse matrix problem with non-zero elements only close to the main diagonal and thus it leads to efficient numerical solution methods.



(a) 2-dimensional pdf



(b) 2D pdf at $x_2 = 0$

Fig. 1 pdf for Duffing oscillator (3)

In the case of the Duffing oscillator (4) the exact solution of the stationary pdf is known to be [9]

$$p(x_1, x_2) = c\exp\left(-\frac{\gamma}{\sigma^2}\left(1 + \frac{1}{2}\varepsilon x_1^2\right)x_1^2 - \frac{\gamma}{\sigma^2}x_2^2\right). \tag{19}$$

This allows to compare results that we obtain through the Galerkin method with the exact pdf. Figure 1(a) shows the calculated pdf for $\gamma = \omega = \sigma = 1$, $\varepsilon = 0.5$. For a better visualization of the results, we cut the 2-dimensional pdf at $x_2 = 0$ (Fig. 1(b)). Also, the zeroth approximation, which is the solution of the linear problem with $\varepsilon = 0$, is depicted.

The problem can easily be extended to a problem with an additional cubical damping term (governed by

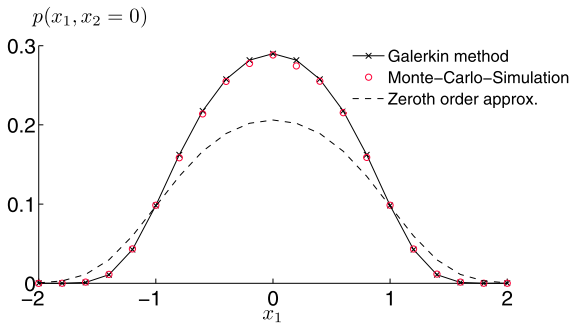


Fig. 2 pdf for nonlinear oscillator (20) with $\varepsilon = 0.5$, $\beta = 0.25$ at $x_2 = 0$

a second nonlinearity parameter β):

$$\begin{aligned} \mathrm{d}X_{1,t} &= \omega X_{2,t}\,\mathrm{d}t, \\ \mathrm{d}X_{2,t} &= \left(-\gamma\left(1+\beta\dot{X}_t^2\right)\dot{X}_t - \omega^2\left(1+\varepsilon X_t^2\right)X_t\right)\mathrm{d}t \\ &\quad + \sigma\,\mathrm{d}W_t. \end{aligned} \tag{20}$$

In this particular case, in order to obtain better results, we can use our knowledge of the solution to the Duffing oscillator in our zeroth order approximation. Instead of the weighting functions given by (10) we set

$$G^{(1)}(x_1) = \exp\left(-\frac{\gamma}{\sigma^2}\left(1+\frac{1}{2}\varepsilon x_1^2\right)x_1^2\right), \tag{21}$$

$$G^{(2)}(x_2) = \exp\left(-\frac{\gamma}{\sigma^2}x_2^2\right), \tag{22}$$

so that the nonlinearities in the restoring term are already incorporated in our approach. In contrast to the previous case we do not have an exact solution for (20) and instead use results from Monte Carlo simulations as reference.

Figure 2 shows the two results for the pdf, again we use a cut at $x_2 = 0$.

3 Higher-dimensional problems with polynomial nonlinearities

The problem of a nonlinear oscillator, as described above, can easily be extended to problems with higher dimension by coupling a number of such oscillators via spring and/or damper elements, as shown in Fig. 3. In the following we consider a problem with five such coupled oscillators, leading to a 10-dimensional FPE,

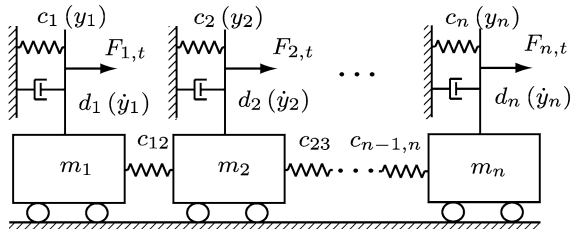


Fig. 3 Mechanical model of oscillator with n degrees of freedom

which represents a problem dimension significantly higher than dimensions commonly tackled with other numerical solution techniques in literature.

Step 1 For higher-dimensional problems with arbitrary system parameters, finding solutions to the linearized problem can be more difficult than in the previous case. We know that the solution pdf is Gaussian and can be written as

$$\begin{aligned} &p(x_1, x_2, \ldots, x_d) \\ &\quad = c\exp\left(\sum_{i=1}^{d}\sum_{j=1}^{d}\alpha_{ij}x_i x_j\right), \end{aligned} \tag{23}$$

however, we still have to determine the coefficients α_{ij}. Inserting (23) into the FPE and solving for α_{ij} leads to large systems of nonlinear equations, which, depending on the system parameters, may be very hard to solve. For more complex systems, all state variables may be coupled, which means that essentially all α_{ij} are nonzero.

An alternative and more practicable way of computing the Gaussian pdf for high-dimensional problems is to consider the covariance matrix $\mathbf{K}$. In the linear case, instead of (2) we use

$$\mathrm{d}\mathbf{X}_t = \mathbf{A}\mathbf{X}_t\,\mathrm{d}t + \mathbf{G}\,\mathrm{d}\mathbf{W}_t \tag{24}$$

for a time-invariant system of linear stochastic differential equations, with linear system matrices $\mathbf{A}$ and $\mathbf{G}$.

In this case, the covariance matrix $\mathbf{K}$ is the solution to the (Lyapunov) matrix equation [13]

$$\mathbf{A}\mathbf{K} + \mathbf{K}\mathbf{A}^T = -\mathbf{G}\mathbf{G}^T. \tag{25}$$

There are efficient numerical methods of solving matrix equations of this type for the covariance matrix $\mathbf{K}$ [14].

The coefficients α_{ij} in (23) can then be obtained from the equivalent formulation

$$p(x_1, x_2, \ldots, x_d) = \frac{1}{(2\pi)^{\frac{d}{2}} |K|^{\frac{1}{2}}} \exp\left(-\frac{1}{2}\mathbf{x}^T K^{-1}\mathbf{x}\right). \quad (26)$$

For general distributions like (23) we cannot separate the weighting functions multiplicatively for the different state variables as we did in (10). In order to obtain an expansion with decoupled weighting functions and polynomials, we calculate marginal probability density functions:

$$p_{x_i}(x_i) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} p(x_1, \ldots, x_d) \times \mathrm{d}x_1 \cdots \mathrm{d}x_{i-1}\, \mathrm{d}x_{i+1} \cdots \mathrm{d}x_d \quad (27)$$

for each of the variables x_j. The expansion is then similar to the 2D-case:

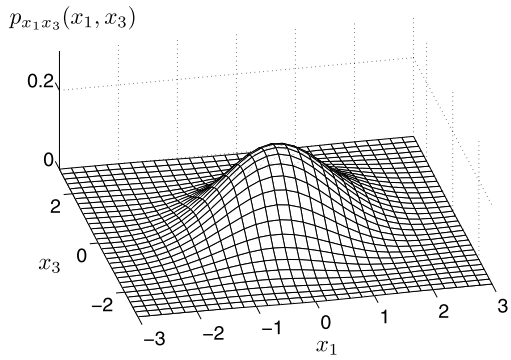
$$\tilde{p}(x_1, \ldots, x_d) = \sum_{i_1=0}^{n_1} \cdots \sum_{i_d=0}^{n_d} C_{i_1 \cdots i_d} P_{i_1}^{(1)}(x_1) \cdots P_{i_d}^{(d)}(x_d) \times G^{(1)}(x_1) \cdots G^{(d)}(x_d). \quad (28)$$

Steps 2 and 3 of the Galerkin method are carried out in full analogy to the 2D-case.

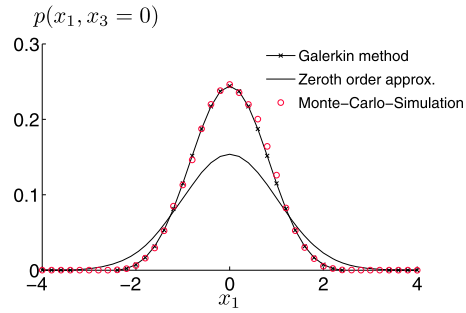
Results have been computed for a variety of sets of system parameters as well as different excitation patterns. In the following we present results for cubical restoring and damping terms, with excitation processes only at the outermost bodies. This represents a rather disadvantageous case for the approach with decoupled weighting functions as it implies stronger coupling between adjacent bodies.

Again, we consider marginal pdfs, in this case for the displacements of the first two bodies, the state variables x_1 and x_3. The 2-dimensional marginal pdf $p_{x_1x_3}(x_1, x_3)$ is depicted in Fig. 4(a), while Fig. 4(b) shows a cut through $p_{x_1x_3}$ at $x_3 = 0$.

In order to decrease the computational costs, it is reasonable to not expand all 10 state variables to the same order, but to expand to lower orders those variables that are not of specific interest in favor of those that are. In this case these were the variables x_1 and x_3 with expansion order $n_j = 4$, while x_2 and x_4 were expanded to order $n_j = 2$ and the remaining states to either $n_j = 1$ or 0.



(a) x_1-, x_3-marginal pdfs for nonlinear 10D oscillator (figure 3)



(b) Marginal pdf at $x_3 = 0$

Fig. 4 Marginal pdf $p_{x_1x_3}$ for 10D problem

4 Nonlinear systems with strongly coupled state variables

For systems with strong coupling between some of the state variables, it seems unreasonable to expect good results from an expansion that is based on decoupled weighting functions. From the solution of the linearized system we have information about the correlation of the state variables that gets lost when calculating marginal probability density functions. Instead we want to demonstrate how the use of coupled weighting functions can solve this problem, although the generation of associated multivariate polynomials becomes more elaborate.

Let us consider two bodies similar to problem (3), which are coupled by a spring element as shown in Fig. 5, leading to a two-degree-of-freedom (2-DOF) system. The use of coupled weighting functions as presented in the following is generally feasible also for higher-dimensional problems, however, the presented procedure becomes more complicated. Thus,

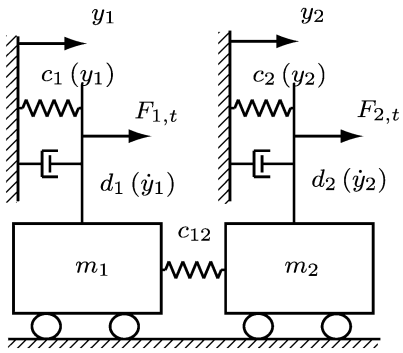


Fig. 5 Mechanical model of 2-DOF oscillator

the 2-DOF problem is an appropriate example to demonstrate the general properties of multivariate weighting functions and polynomials.

Similar to the previous cases we use the first-order formulation:

$$x_1 = y_1, \qquad x_2 = \dot{y}_1,$$
$$x_3 = y_2, \qquad x_4 = \dot{y}_2,$$

and, for cubical restoring elements, we obtain the following system of equations:

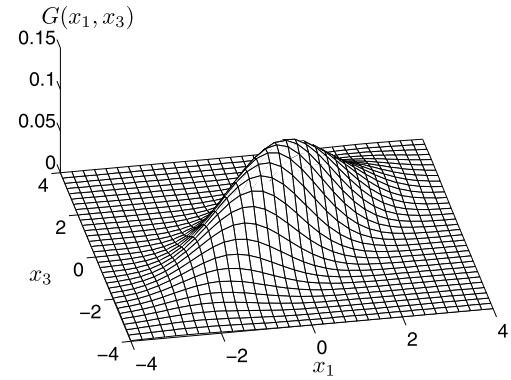
$$\mathrm{d}X_{1,t} = \omega_1 X_{2,t}\,\mathrm{d}t, \tag{29}$$
$$\mathrm{d}X_{2,t} = \big(-\gamma X_{2,t} - \omega_1\big(1+\varepsilon X_{1,t}^2\big)X_{1,t} + \omega_{12}X_{3,t}\big)\,\mathrm{d}t + \sigma_1\,\mathrm{d}W_{2,t}, \tag{30}$$
$$\mathrm{d}X_{3,t} = \omega_2 X_{4,t}\,\mathrm{d}t, \tag{31}$$
$$\mathrm{d}X_{4,t} = \big(-\gamma X_{4,t} - \omega_2\big(1+\varepsilon X_{3,t}^2\big)X_{3,t} + \omega_{21}X_{1,t}\big)\,\mathrm{d}t + \sigma_2\,\mathrm{d}W_{1,t}. \tag{32}$$

Step 1 As before, we consider the linearized system in order to obtain an approximative solution of the form

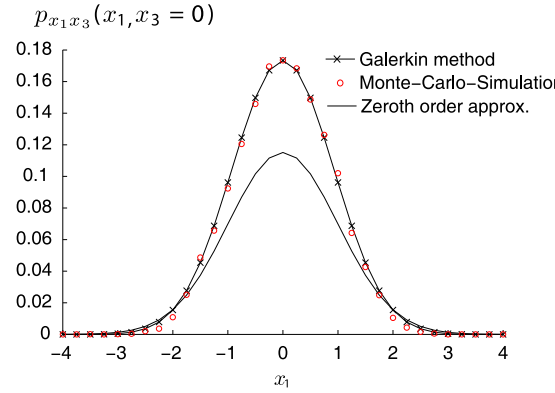
$$p(x_1,\ldots,x_4) = c\exp\left(\sum_{i=1}^{4}\sum_{j=1}^{4}\alpha_{ij}x_i x_j\right). \tag{33}$$

The mechanical coupling between the two bodies leads to strong coupling between the displacements x_1 and x_3. In fact, for symmetrical system parameters, the two displacements are the only coupled variables, so that

$$\alpha_{ij} = 0, \quad \text{for all } i \neq j \text{ except } \alpha_{13}. \tag{34}$$



(a) Weighting function for x_1,x_3



(b) Marginal pdf compared with MCS results (cut at $x_3 = 0$)

Fig. 6 4D problem with coupled weighting functions

In contrast to the previous chapter we will not compute marginal probability density functions but, instead, choose coupled weighting functions in the two displacements x_1 and x_3. As the velocities are not coupled with any of the other state variables, we use decoupled weighting functions in the velocities x_2 and x_4. We thus obtain the polynomial expansion

$$\tilde{p}(x_1,x_2,x_3,x_4) = \sum_{i_1=0}^{n_1}\sum_{i_2=0}^{n_2}\sum_{i_3=0}^{n_3}\sum_{i_4=0}^{n_4} C_{i_1i_2i_3i_4}P_{i_1i_3}^{(13)}(x_1,x_3)P_{i_2}^{(2)}(x_2) \times P_{i_4}^{(4)}(x_4)G^{(13)}(x_1,x_3)G^{(2)}(x_2)G^{(4)}(x_4). \tag{35}$$

Figure 6(a) shows the weighting function in x_1 and x_3, which emphasizes the strong coupling between the two state variables.

Step 2 For the case of bivariate weighting functions $G_{12}(x_1, x_2)$, the orthogonality postulations for the bivariate expansion polynomials

$$P_{mn}^{12}(x_1, x_2) = \sum_{i=0}^{m} \sum_{j=0}^{n} a_{mn,ij} x_1^i x_2^j \tag{36}$$

are similar to those in the decoupled case:

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} P_{mn}^{12}(x_1, x_2) P_{op}^{12}(x_1, x_2) G_{12}(x_1, x_2) \, \mathrm{d}x_1 \, \mathrm{d}x_2 = \begin{cases} 0, & m, n \neq o, p, \\ f_{mn}, & m, n = o, p. \end{cases} \tag{37}$$

Again, we obtain a linear system of equations to be solved for $a_{mn,ij}$, shown here for the case $m = n = 1$:

$$\begin{pmatrix} m_{00} & m_{01} & m_{10} & m_{11} \\ m_{01} & m_{02} & m_{11} & m_{12} \\ m_{10} & m_{11} & m_{20} & m_{21} \\ m_{11} & m_{12} & m_{21} & m_{22} \end{pmatrix} \begin{pmatrix} a_{11,00} \\ a_{11,01} \\ a_{11,10} \\ a_{11,11} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{f_{11}}{a_{11,11}} \end{pmatrix}. \tag{38}$$

The moments m_{mn} are then

$$m_{mn} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x_1^m x_2^n G_{12}(x_1, x_2) \, \mathrm{d}x_1 \, \mathrm{d}x_2. \tag{39}$$

Step 3 Just as for decoupled weighting functions, the Galerkin method leads to a linear system of equations for the expansion coefficients $C_{i_1 i_2 i_3 i_4}$.

In the nonlinear case we calculated the approximative pdf for expansion orders $n_j = 3$ for the coupled as well as the non-coupled variables. We computed the marginal pdf in x_1 and x_3 and compared it with results from Monte Carlo simulation.

Figure 6(b) shows a cut of $p_{x_1,x_3}(x_1, x_3)$ at $x_3 = 0$.

5 Multiple DOF-system with non-polynomial nonlinearities

Finally, we present results for a system with non-polynomial nonlinear elements, such as they are typically found in automotive vehicle dynamics. We have a damping element of the form

$$d_y(\dot{y}) = d\big(1 + \kappa \ \mathrm{sign}(\dot{y})\big), \tag{40}$$

leading to a damping characteristic as shown in Fig. 7.

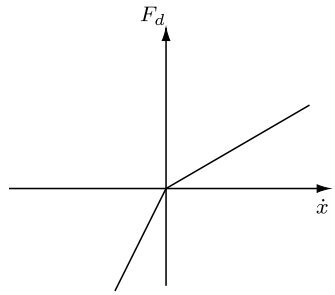


Fig. 7 Nonlinear characteristic for automotive damping devices

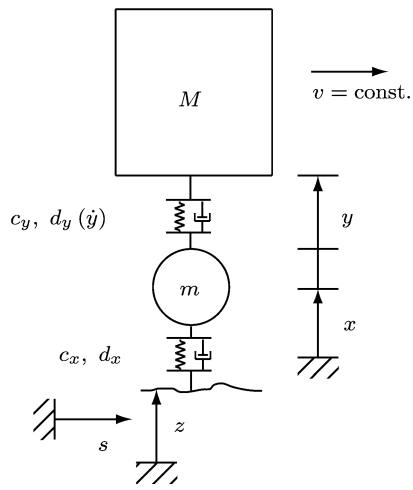


Fig. 8 Quarter car model, excited by road roughness

In [12] pdfs for simple quarter car models with such characteristics have been calculated, resulting in non-zero-mean pdfs. We consider a similar problem with two degrees of freedom, which is excited at one of the two bodies according to Fig. 8.

Enhancing the problem compared with [12], the excitation from road surface roughness is generated by a general second-order filter, which increases the problem dimension from four to six. The roughness of road surfaces Z_s relevant to the dynamics of a car can be generated from a white noise process with the parameters Ω_a, Ω_0, Ω_c and S_0 [15], which define the roughness properties of the road surface:

$$\frac{Z_s''}{\Omega_a} + \left(1 + \frac{\Omega_c}{\Omega_a}\right) Z_s' + \omega_c Z_s = \sqrt{S_0} \Omega_0 \xi_s. \tag{41}$$

The superscript $'$ indicates the derivative with respect to s. For constant velocities v, (41) is transformed into an equivalent filter in the time domain:

$$\frac{\ddot{Z}_t}{v^2\Omega_a} + \left(1 + \frac{\Omega_c}{\Omega_a}\right)\frac{\dot{Z}_t}{v} + \omega_c Z_t = \sqrt{S_0}\Omega_0\xi_t. \tag{42}$$

With given Ω_a, Ω_0, Ω_c, S_0 and constant velocity v, we obtain a filter equation of the form

$$\ddot{Z}_t + \gamma_z\dot{Z}_t + \omega_z^2 Z_t = \omega_z\sigma\xi_t. \tag{43}$$

Substituting

$$\omega_x^2 = \frac{c_x}{m}, \qquad \gamma_x = \frac{d_x}{m},$$
$$\omega_{xy}^2 = \frac{c_y}{m}, \qquad \gamma_{xy} = \frac{d_y}{m},$$
$$\omega_y^2 = \frac{c_y}{m} + \frac{c_y}{M}, \qquad \gamma_x = \frac{d_y}{m} + \frac{d_y}{M},$$

and

$$X_{1,t} = X_t, \qquad X_{2,t} = \dot{X}_t/\omega_x,$$
$$X_{3,t} = Y_t, \qquad X_{4,t} = \dot{Y}_t/\omega_y,$$
$$X_{5,t} = Z_t, \qquad X_{6,t} = \dot{Z}_t/\omega_z,$$

the equations of motion for the quarter car model yield a system of six nonlinear stochastic differential equations:

$$\mathrm{d}X_{1,t} = \omega_x X_{2,t}\,\mathrm{d}t, \tag{44}$$

$$\mathrm{d}X_{2,t} = \Big(-\omega_x X_{1,t} - \gamma_x X_{2,t} + \frac{\omega_{xy}^2}{\omega_x}X_{3,t} + \gamma_{xy}\frac{\omega_y}{\omega_x}\big(1 + \kappa\ \mathrm{sign}(X_{4,t})\big)X_{4,t} + \omega_x X_{5,t} + \gamma_x\frac{\omega_z}{\omega_x}X_{6,t}\Big)\,\mathrm{d}t, \tag{45}$$

$$\mathrm{d}X_{3,t} = \omega_y X_{4,t}\,\mathrm{d}t, \tag{46}$$

$$\mathrm{d}X_{4,t} = \Big(-\omega_y X_{3,t} - \gamma_y\big(1 + \kappa\ \mathrm{sign}(X_{4,t})\big)X_{4,t} + \frac{\omega_x^2}{\omega_y}X_{1,t} + \gamma_x\frac{\omega_x}{\omega_y}X_{2,t} - \frac{\omega_x^2}{\omega_y}X_{5,t} - \gamma_x\frac{\omega_z}{\omega_y}X_{6,t}\Big)\,\mathrm{d}t, \tag{47}$$

$$\mathrm{d}X_{5,t} = \omega_z X_{6,t}\,\mathrm{d}t, \tag{48}$$

$$\mathrm{d}X_{6,t} = (-\omega_z X_{5,t} - \gamma_z X_{6,t})\,\mathrm{d}t + \sigma\,\mathrm{d}W. \tag{49}$$

Again, for $\kappa \to 0$ we obtain a linear system of equations, for which the stationary solution can be found. Using its marginal pdfs as weighting functions $G^{(j)}(x_j)$ for all the state variables, we generate the orthogonal polynomials $P_{i_j}^{(j)}(x_j)$ and use the expansion

$$\tilde{p}(x_1,\ldots,x_6) = \sum_{i_1=0}^{n_1}\cdots\sum_{i_6=0}^{n_6} C_{i_1\cdots i_6} P_{i_1}^{(1)}(x_1)\cdots P_{i_6}^{(6)}(x_6) \times G^{(1)}(x_1)\cdots G^{(6)}(x_6). \tag{50}$$

Inserting (50) into the FPE for the nonlinear problem and computing the Galerkin procedure leads to systems of equations with block-wise dense matrices, because the expansion of non-polynomial nonlinearities requires basis polynomials not only of adjacent order.

Solving the system of equations leads to the expansion coefficients $C_{i_1\cdots i_6}$ and thus the pdf approximation, as well as marginal pdfs via integration.

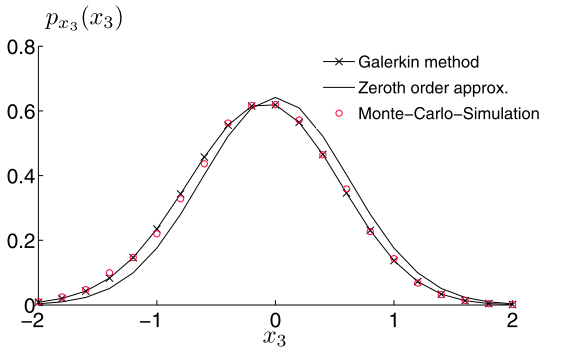
In the following we present results for $\kappa = 0.5$, which is a typical value for nonlinear damping elements in road vehicle dynamics.

The results in Figs. 9(a) and 9(b) clearly show the effect of the nonlinearity κ. The asymmetrical damping characteristic leads to a shift in the mean value of the relative coordinate y, or x_3, respectively. The effect that κ has on x (x_1) is small.

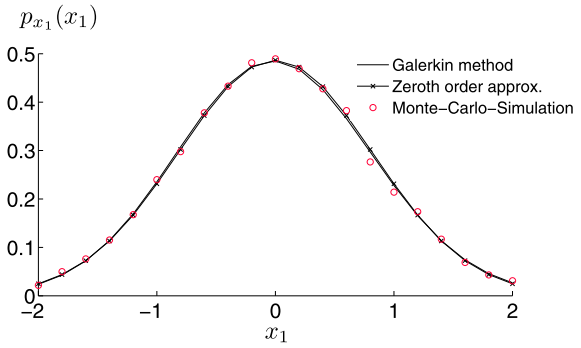
We see that the bend in the damping characteristic leads to an asymmetry in the system dynamics that has not been included in the weighting functions. Still, even for considerably low expansion order ($n = 3$), these non-symmetric results are sufficiently well reproduced by the results. In [12], instead of symmetrical weighting functions, approximations are developed that include the shift in the mean value. Using such approximations will allow the solution of even more complicated problems in the future.

6 Summary

This paper presents a Galerkin method for solving high-dimensional Fokker–Planck-Equations by orthogonal polynomial expansion of weighting functions. The use of global weighting functions, which include a priori knowledge about the shape of the probability density function, allows the solution of mechanical problems of considerably higher dimension



(a) Marginal pdf $p_{x_3}(x_3)$



(b) Marginal pdf $p_{x_1}(x_1)$

Fig. 9 Marginal pdfs for quarter car model (Fig. 8), $\kappa = 0.5$

than most other numerical methods. After demonstrating the general procedure on the example of a single-degree-of-freedom nonlinear oscillator the method is extended to a problem with five degrees of freedom, leading to a 10-dimensional Fokker–Planck Equation. The use of coupled weighting functions and polynomials is introduced as a means of solving mechanical systems with strong coupling between some of the system states. In a last example we present results for a quarter car model with typical nonlinear damping element. The comparison with results from Monte Carlo simulations and exact solutions, where available, shows good coincidence and the numerical efforts even for the high-dimensional case are manageable. Depending on the expansion orders the Galerkin method yields results within seconds, in case of the 10D problem still within less than a minute. The generation of pdf estimates via Monte Carlo simulation has generally proven far more time-consuming. Of course, it needs to be said that the MCS routines used for the results presented in this paper were not pushed to maximum efficiency, as the main objective was to generate reliable reference solutions, without specific focus on optimizing MCS techniques.

The problems that have been investigated so far suggest that there is a wide range of problems that can be dealt with in a very efficient manner with the presented method. Problems with stationary pdfs that are not too far from Gaussian can be solved for significantly higher dimensions than with comparable numerical methods. In the case of nonlinear systems with stationary pdfs that diverge strongly from Gaussian distributions (such as considered in [3], for example), there are obvious limitations to the method as outlined in this paper. However, the use of appropriate nonlinear coordinate transformations, as described in [10] or [11], as well as the use of well-adapted weighting functions also allows the efficient solution of problems with distributions far from Gaussian. These include systems with limit cycle behavior or systems under combined stochastic and harmonic excitation.

Future work will concentrate on extending the possibilities of the method toward yet higher dimensions and more complicated classes of nonlinearities (see for example [10]), with special focus on the mathematical methods of solving large systems of linear equations, systematically taking into account the characteristics of the system structure.

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References

1. Iourtchenko, D.V., Mo, E., Naess, A.: Response probability density functions of strongly nonlinear systems by the path integration method. Int. J. Non-Linear Mech. **41**, 693–705 (2006)
2. Bergman, L.A., Masud, A.: Application of multi-scale finite element methods to the solution of the Fokker–Planck equation. Comput. Methods Appl. Mech. Eng. **194**, 1513–1526 (2005)
3. Kumar, P., Narayanan, S.: Solution of the Fokker–Planck equation by finite element and finite difference methods for nonlinear systems. Sadhana **31**(4), 445–461 (2006)
4. Naess, A., Moe, V.: Efficient path integration methods for nonlinear dynamic systems. Probab. Eng. Mech. **15**, 221–231 (2000)
5. Yu, J.S., Lin, Y.K.: Numerical path integration of a non-homogeneous Markov process. Int. J. Non-Linear Mech. **39**, 1493–1500 (2004)
6. Tombuyses, B., Aldemir, T.: Continuous cell-to-cell mapping. J. Sound Vib. **202**(3), 395–415 (1997)

7. Feuersänger, C.: Sparse grid methods for higher dimensional approximation, pp. 63–101. Dissertation, Institut für Numerische Simulation, Universität Bonn (2010)
8. Kumar, P., Chakravorty, S., Singla, P., Junkins, J.: The partition of unity finite element approach with HP-refinement for the stationary Fokker–Planck equation. J. Sound Vib. **327**, 144–162 (2009)
9. von Wagner, U.: Zur Berechnung stationärer Verteilungsdichten nichtlinearer stochastisch erregter Systeme, pp. 20–31. VDI, Düsseldorf (1999)
10. von Wagner, U., Wedig, W.V.: On the calculation of stationary solutions of multi-dimensional Fokker–Planck equations by orthogonal functions. Nonlinear Dyn. **21**, 289–306 (2000)
11. von Wagner, U.: On double crater-like probability density functions of a duffing oscillator subjected to harmonic and stochastic excitation. Nonlinear Dyn. **28**, 343–355 (2002)
12. von Wagner, U.: On nonlinear stochastic dynamics of quarter car models. Int. J. Non-Linear Mech. **39**, 753–765 (2004)
13. Arnold, L.: Stochastische Differentialgleichungen, pp. 141–146. Oldenbourg, München (1973)
14. Golub, G.H., Van Loan, C.F.: Matrix Computations, 3rd edn. Johns Hopkins University Press, Baltimore (1996)
15. Ammon, D.: Modellbildung und Systementwicklung in der Fahrzeugdynamik, pp. 74–80. Teubner, Stuttgart (1997)